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On a new Lagrangian view on the evolution of vorticity in spatial flows

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Abstract

The purpose of the study is to extend to the spatial case proposed by G. B. Sizykh approach to a two-dimensional vorticity evolution, which is based on the idea of considering a vorticity evolution in the form of such a motion of vortex lines and tubes that the intensity of these tubes changes over time according to a predefined law. **Method.** Thorough analysis is determined by describing the flow velocity field of an ideal incompressible fluid and a viscous gas in the general case, using the idea of the movement of imaginary particles. **Results.** For any given time law of change of velocity circulation (i. e. for an exponential decay) of a real fluid along the contours the method of evaluating the field of velocity of such contours and vortex tubes is proposed (e. g. getting a field of imaginary particles, which transfer them). It is established that for a given time law the velocity of imaginary particles is determined ambiguously, and the method of how to adjust their motion preserving defined law of circulation change is proposed. **Conclusion.** A new Lagrangian approach to the evolution of vorticity in three-dimensional flows is derived, as well as the expressions for the contours' velocity, which imply stated changing over the time of the velocity circulation of a real fluid along any contour. This theoretical result can be

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utilized in spatial modifications of the viscous vortex domain method to limit the number of vector tubes used in calculations.

Keywords: contour velocity, contour intensity, imaginary fluid motion, Zoravski's criterion, Friedmann's theorem, viscous vortex domain method.

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Introduction. In the middle of past century a meshless method for spatial vortex flows of ideal incompressible fluid simulation (the discrete vortex method [1–3]) was developed, which is based on Helmholtz vortex theorems. This approach has been successfully implemented (e. g., [4–6]). Later these theorems were genera-lized to the case of viscous incompressible fluid, however only for two-dimensional (plane-parallel [7] and irrotational axisymmetric [8]) flows, and formulas for veloci-ty $\mathbf{U}$ of vortex tubes transfer while preserving their intensity were derived. Using traditional notation for flow velocity and vorticity vectors as well as for kinematic viscosity coefficient, both expressions from [7, 8] might be formulated in the following way:

$$\mathbf{U} = \mathbf{V} - \nu(\mathbf{\Omega} \times \operatorname{rot}\mathbf{\Omega})/\mathbf{\Omega}^2.$$

A numerical method proposed in [9] for studying two-dimensional viscous flows, the so-called "viscous vortex domain method" (VVD), which uses the concept of the motion of vortex tubes of constant intensity with a velocity of $\mathbf{U}$, obtained in [7, 8]. A comprehensive explanation of this method is given, for example, in [10], and a short explanation in [11]. Being a meshless method, the VVD possesses many advantages, in particular, one may fulfill the boundary conditions for unbounded spatial flows [12] that is essential regarding the modeling of natural phenomena (cyclones, ocean streams, etc.). Nevertheless, the use of the VVD method might cause some troubles such as unlimited growth of total number of considering domains that are generated during each computational step. Nowadays a limitation of the total amount of domains is performed via rearrangement of their position and intensity [13–15]. As one of the possible ways how to "cope" with unlimited growth of their total number a new formula was proposed in [16] for the velocity of vortex tubes transfer under any predefined time law of change of their intensity, which, in the case of an exponential law of decay, allows one to neglect each domain after some finite number of time steps and thereby limit the number of domains taken into account during the calculation. The implementation of above-mentioned velocity into the VVD method is a separate and comprehensive task of computational hydrodynamics to be dealt with, whereas [16] was published recently. Therefore, despite the fact that there are no examples where this velocity is used, the authors of the current paper are sure that this absence is temporal only, and the result of [16] will contribute to the development of the VVD method.

As it was previously said, the VVD method and the corresponding new velocity, which is proposed in [16], are only applicable to two-dimensional streams. Further-more, formulas for the velocity [16] are valid just for those flows, where the vorticity vector and its curl are orthogonal to each other. It is worth to be mentioned that one managed to extend the ideas of [7, 8] to rotational axisymmetric flows because of the independent analysis of vorticity evolution for both radial-axial and transversal components of velocity in [17, 18]. This approach implies that the curls of these vorticities are orthogonal to them. Consequently, one may use formulas from [16] for each of two vorticity fields (in addition, it is also possible to set different time laws of change of vortex tubes' intensity for both of them). In a general spatial case this extension is no longer valid, because (as it was already pointed out) in two-dimensional flows the vorticity vector and its curl are orthogonal to each other, which makes it possible to transform the Navier–Stokes equations (used in [7, 8]) in a way that is impossible to perform in a general spatial case, where the vorticity vector and its curl may not be orthogonal (for details see [7, 8]). Thus, for a long time after [7, 8] had been published there was a persuasion among the scientists known to the authors that in a general spatial case there is no such velocity **U**. This opinion was ensured by [19], which has "proven" the absence of the velocity **U** in a general spatial case. Then, however, a mistake was found in [19]. That happened right after [11] has shown that there is a velocity **U** in a general spatial case, moreover for flows of all kinds: starting from ideal incompressible fluids and ending with viscous gas. This issue of [19] is presented by the fact that a solution of one of the equations (numbered (22) in this paper) is not unique, whilst one proposed by the authors is unique only for the unbounded case, when it is natural to set the value of target function to zero at infinity (the note of the mistake in [19] is published for the first time).

In order to evaluate the velocity **U** in a general spatial case so-called non-local method was proposed, which requires an integration along vortex lines. This makes computation incredibly complex, so for a long time, the theoretical result [11] has not been applied to the development of the VVD method, which remained two-dimensional. Nevertheless, the first spatial version of VVD method [20] appeared recently that is based on the generalization of [11] two-dimensional viscous analogues of the Helmholtz theorems [7, 8] to a general spatial case. A similar situation has occurred here, which was resolved in the two-dimensional case with [16]. There is a problem of unlimited growth of the number of vortex tubes during the simulation process, but in the spatial case now. In this paper, in order to overcome this issue, an attempt is made in a general spatial case to find an analogue of the velocity **U** (proposed in [16]) that provides the change of intensity of vortex tudes during their motion according to a given law.

1. The dynamic equation of motion of fluid and gas. The velocity field of fluid and gas (that are applicable from arbitrary media, from incompressible fluid and to viscous gas) in a general spatial case is governed by the following equation

$$\partial \mathbf{V}/\partial t + \mathbf{\Omega} \times \mathbf{V} = \mathbf{F} - \nabla f, \tag{1}$$

where t is time, $\mathbf{F}$ is net non-conservative mass-specific force, and f is some scalar field that contains mass-specific kinetic energy $\mathbf{V}^2/2$. We assume all flow parameters in below to be sufficiently smooth, so the corresponding fields are differentiable in order to keep the validity of the following equations. Let $\alpha = \alpha(t)$ be an arbitrary smooth function of time.

We will use the idea proposed and implemented by G. B. Sizykh in [11] to prove the existence of the velocity of vortex tubes transfer while keeping their intensity constant (a contribution of the other author of [11] V. V. Markov is that he established the ambiguity of such a velocity). In a spatial open region of rotational flow ($\mathbf{\Omega} \neq \mathbf{0}$) consider a plane region σ, the normal vector of which shares an acute angle with vortex lines crossing σ within some time interval $[t_1, t_2]$. Let us highlight a spatial simply connected fragment G_σ that lies at the intersection of all vortex tubes passing through σ at various timestamps $t \in [t_1, t_2]$, and such that σ belongs to G_σ. Suppose there is an arbitrary independent of time function g_σ with the domain of σ. Using an integration along vortex lines for any t from $[t_1, t_2]$ we continue g_σ from σ to G_σ with the function $g(x, y, z, t)$, which gradient satisfies an equality

$$\mathbf{\Omega} \cdot \nabla g = \mathbf{\Omega} \cdot (\mathbf{F} + \alpha \mathbf{V}). \tag{2}$$

Consider a cross product

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{F} + \alpha \mathbf{V} - \nabla g)) = \mathbf{\Omega}(\mathbf{\Omega} \cdot (\mathbf{F} + \alpha \mathbf{V} - \nabla g)) - (\mathbf{F} + \alpha \mathbf{V} - \nabla g)\, \mathbf{\Omega}^2,$$

which, with the help of (2), results in

$$\mathbf{F} = -\mathbf{\Omega} \times \frac{\mathbf{\Omega} \times (\mathbf{F} + \alpha \mathbf{V} - \nabla g)}{\mathbf{\Omega}^2} - \alpha \mathbf{V} + \nabla g. \tag{3}$$

Substituting (3) into (1) after some rearrangement one has

$$\partial \mathbf{V}/\partial t + \mathbf{\Omega} \times \mathbf{U} = -\alpha \mathbf{V} + \nabla (g - f), \tag{4}$$

where

$$\mathbf{U} = \mathbf{V} + \frac{\mathbf{\Omega} \times (\mathbf{F} + \alpha \mathbf{V} - \nabla g)}{\mathbf{\Omega}^2}. \tag{5}$$

2. Zoravski's criterion. We will then exploit a concept of motion of imaginary fluid particles, which was initially proposed in [11, 21] and is being fruitfully implemented recently [22–28]. For that let us formulate Zoravski's criterion [29, 30], which is also known as Friedmann's theorem [31], in terms of vortex tubes' motion along with imaginary fluid particles.

Suppose G is an open region simultaneously filled with two distinct fluids that do not interact with each other (and do not interfere with the movement of each other). Particles of the first fluid move with the velocity $\mathbf{U}(x, y, z, t)$, whereas particles of the second one — with velocity $\tilde{\mathbf{V}}(x, y, z, t)$. In addition, the flow represented by the second fluid is rotational ($\tilde{\mathbf{\Omega}} = \operatorname{rot} \tilde{\mathbf{V}} \neq \mathbf{0}$) within some time interval (t_1, t_2). Moreover, assume that during $t \in (t_1, t_2)$ in G the vorticity vector of the second fluid $\tilde{\mathbf{\Omega}}$ and velocity field of the first one $\mathbf{U}$ are connected via the equation

$$\partial \tilde{\mathbf{\Omega}}/\partial t + \operatorname{rot}(\tilde{\mathbf{\Omega}} \times \mathbf{U}) = \mathbf{0}. \tag{6}$$

Thus, as it follows from Zoravski's criterion, for $t \in (t_1, t_2)$ vortex lines and vortex tubes $\tilde{\mathbf{\Omega}}$ are transferred with the velocity $\mathbf{U}$, and the intensity of vortex tubes (which is equal to the circulation $\tilde{\Gamma}$ of the velocity $\tilde{\mathbf{V}}$ along any contour that encircles a vortex tube of $\tilde{\mathbf{\Omega}}$ only once) is being preserved as long as these particles are inside G.

This consequence is going to be used below to investigate the link between the vorticity of a real fluid $\mathbf{\Omega}$ and velocity fields of some imaginary fluids.

3. Vortex tubes' motion. In this chapter in order to use Zoravski's criterion, we will regard two imaginary fluids at once and the velocity field of a real fluid $\mathbf{V}$. Assume that particles of the first imaginary fluid move with the velocity $\mathbf{U}$ determined by (5) in terms of $\mathbf{V}$ whilst particles of the second one — with the velocity $\tilde{\mathbf{V}} = \mathbf{V} \exp\left(\int_{t_1}^{t} \alpha(\tau)\, d\tau\right)$. Substituting $\mathbf{V} = \tilde{\mathbf{V}} \exp\left(-\int_{t_1}^{t} \alpha(\tau)\, d\tau\right)$[1] into (4), one obtains

$$\frac{\partial \tilde{\mathbf{V}}}{\partial t} + \tilde{\mathbf{\Omega}} \times \mathbf{U} = \mathbf{\nabla}(g - f) \exp\left(\int_{t_1}^{t} \alpha(\tau)\, d\tau\right). \tag{7}$$

Applying the curl operator both to the left and the right sides of (7) we achieve an equation in the form of (6) $\partial \tilde{\mathbf{\Omega}}/\partial t + \operatorname{rot}(\tilde{\mathbf{\Omega}} \times \mathbf{U}) = \mathbf{0}$. Hence (Zoravski's criterion), vortex tubes $\tilde{\mathbf{\Omega}}$ are transferred by particles of the first imaginary fluid, which move with the velocity (5). And a circulation $\tilde{\Gamma}$ of the velocity $\tilde{\mathbf{V}}$ of the second imaginary fluid along the contours moving together with particles of the first imaginary fluid with the velocity (5) is preserved over the time and is equal to $\tilde{\Gamma}(t) = \tilde{\Gamma}(t_1)$.

Taking into account the fact that the vorticity vector of the second imaginary fluid $\tilde{\mathbf{\Omega}}$ and the vorticity vector of the real fluid $\mathbf{\Omega}$ are connected by $\mathbf{\Omega} = \tilde{\mathbf{\Omega}} \exp\left(-\int_{t_1}^{t} \alpha(\tau)\, d\tau\right)$, we establish the main result. *The vortex lines and vortex tubes of the velocity field of a (real) fluid transport along with particles of an imaginary fluid moving with the velocity* (5), *and besides that the intensity of all the vortex tubes changes according to*

$$\Gamma(t) = \Gamma(t_1) \exp\left(-\int_{t_1}^{t} \alpha(\tau)\, d\tau\right). \tag{8}$$

Consequently, one has determined that in a spatial case there is an analogue of the velocity [16], with which vortex tubes are driven, and the intensity of these tubes changes according to the certain law (8). If function $\alpha = \alpha(t)$ is known, this velocity is defined by (5) with an integration of (2) along the vortex lines. The rational choice of g_σ allows one to adjust the magnitude and direction of the velocity of imaginary fluid particles $\mathbf{U}$ within some range. Various α and g_σ conform to different velocities $\mathbf{U}$ and, therefore, different points of view on the evolution of vorticity, which are all equivalent, according to [11].

As in [16], the proposed new method for calculating the velocity $\mathbf{U}$ is a generalization of the [11] method, since it coincides with the latter at $\alpha = 0$.

4. A viscous incompressible fluid. The Navier–Stokes equation for incompressible fluid has the form of (1), in which the scalar field f might be expressed

[1] Therefore, $\mathbf{\Omega} = \tilde{\mathbf{\Omega}} \exp\left(-\int_{t_1}^{t} \alpha(\tau)\, d\tau\right)$.

as $f = p/\rho + \mathbf{V}^2/2 + \Pi$, where p/ρ is the pressure-to-density ratio, Π is the volume forces potential, and $\mathbf{F} = -\nu \operatorname{rot} \boldsymbol{\Omega}$. Then, according to (5), we have

$$\mathbf{U} = \mathbf{V} - \nu(\boldsymbol{\Omega} \times \operatorname{rot} \boldsymbol{\Omega})/\boldsymbol{\Omega}^2 + \alpha(\boldsymbol{\Omega} \times \mathbf{V})/\boldsymbol{\Omega}^2 + (\boldsymbol{\Omega} \times \boldsymbol{\nabla} g)/\boldsymbol{\Omega}^2.$$

If the contours (domains) move with this velocity, their intensity changes according to the law (8). During the implementation of the VVD method with the use of such velocity, the functions α and g_σ must belong to a certain smoothness class within one time step. Generally speaking, these smoothness requirements are unknown, and their establishment is of current interest in mathematical physics. However, for the validity of the presented equations, as follows from the course of PDE, these functions must be at least continuously differentiable in the studied flow region. Moreover, these functions can change abruptly when passing from one time step to another, since this will correspond to the change of the "old" Lagrangian point of view "new". The words "old" and "new" are in quotation marks because these points of view existed and continue to exist at all time steps, but one of them is applied earlier, and the other later. Possible options for α are suggested, for example, in [16].

5. The ambiguity of the velocity U. From the mathematical point of view (5) reflects not all possible velocities of (decaying) vorticity transfer $\mathbf{U}$, which satisfy (4). Namely, one should add the term $\gamma\boldsymbol{\Omega}$ that is collinear to the vorticity vector $\boldsymbol{\Omega}$ (here γ is an arbitrary smooth function of space and time), since it does not affect the general form of (4).

Despite that, the fundamental ambiguity in the computation of (5) arises because of the term containing the function $g(x, y, z, t)$, which is a result of function g_σ being integrated along the vortex lines $\boldsymbol{\Omega}$ into the open region of vortex fragment at any moment of time, and thus is determined up to some scalar field $W(x, y, z)$ that is constant along these vortex lines: $\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} W = 0$ (similar reasoning was discussed in [11]). Therefore, omitting intermediate results, (5) might be generalized as

$$\mathbf{U} = \mathbf{V} + \frac{\boldsymbol{\Omega} \times (\mathbf{F} + \alpha\mathbf{V} - \boldsymbol{\nabla} g + \boldsymbol{\nabla} W)}{\boldsymbol{\Omega}^2} + \gamma\boldsymbol{\Omega}.$$

Conclusion. A new point of view on the evolution of vorticity in liquid and gas flows, proposed in [16] for two-dimensional flows, is extended to a general spatial case. This point of view consists in representing the vorticity evolution in the form of such a motion of vortex lines and vortex tubes, in which the intensity of the vortex tubes changes according to any predetermined time law, in particular, at $\alpha = 1$ it decreases exponentially. Of course, different laws of intensity variation, i. e., various $\alpha = \alpha(t)$ will correspond to different velocities of vortex lines and vortex tubes. From the point of view of the complexity of the implementation of the proposed approach, associated with the need for integration along vortex lines, no additional issues arise compared to [11], since in both cases an equation of the type (2) is integrated along vortex lines.

The proposed new point of view on the evolution of vorticity in spatial modifications of the VVD method can be used to limit the number of vector tubes taken into account in the calculations.

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О новом лагранжевом взгляде на эволюцию завихренности в пространственных течениях

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Аннотация

Цель исследования состоит в распространении на пространственный случай разработанного Г. Б. Сизых подхода к эволюции завихренности для двумерных течений, базирующегося на представлении эволюции завихренности в виде такого движения вихревых линий и вихревых трубок, при котором интенсивность этих трубок меняется со временем по любому наперед заданному закону. **Метод.** Строгий анализ уравнений, описывающих поле скорости течения идеальной несжимаемой жидкости и вязкого газа в общем пространственном случае с использованием представления о движении воображаемых частиц. **Результаты.** Для любого заданного временного закона изменения циркуляции скорости (например, для экспоненциального убывания) реальной жидкости по контурам предложен способ построения поля скорости движения этих контуров и вихревых трубок (т. е. построение поля скорости переносящих их воображаемых частиц). Установлено, что при заданной функции времени скорость воображаемых частиц определяется неоднозначно, и предложен способ коррекции их движения при сохранении выбранного закона изменения циркуляции. **Заключение.** Предложен новый лагранжев подход к эволюции завихренности в пространственных течениях и получены выражения для скорости движения контуров, обеспечивающие заданное изменение со временем циркуляции скорости реальной жидкости по любому контуру. Данный теоретический результат может быть

Научная статья

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использован в пространственных модификациях метода вязких вихревых доменов для ограничения количества учитываемых в расчетах векторных трубок.

Ключевые слова: скорость движения контуров, интенсивность контуров, движение воображаемой жидкости, критерий Зоравского, теорема Фридмана, метод вязких вихревых доменов.

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