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On covariant non-constancy of distortion and inversed distortion tensors

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Abstract

The paper deals with covariant constancy problem for tensors and pseudotensors of an arbitrary rank and weight in an Euclidean space. Requisite preliminaries from pseudotensor algebra and analysis are given. The covariant constancy of pseudotensors are separately considered. Important for multidimensional geometry examples of covariant constant tensors and pseudotensors are demonstrated. In particular, integer powers of the fundamental orienting pseudoscalar satisfied the condition of covariant constancy are introduced and discussed. The paper shows that the distortion and inversed distortion tensors are not actually covariant constant, contrary to the statements of those covariant constancy which can be found in literature on continuum mechanics.

Keywords: pseudotensor, fundamental orienting pseudoscalar, distortion, inversed distortion, covariant constant tensors, parallel vector field.

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Preliminary remarks. Variational and dynamic equations describing the mechanical behavior of solids require in general the formalism of pseudotensor analysis [1–4]. In this case, the covariant constant tensor and relative tensor fields play an important role[1]. In the present paper the notions and requisite equations from algebra and analysis of pseudotensors are discussed. An in-depth and complete presentation of the pseudotensor formalism can be found in the books on tensor analysis and continuum mechanics [2–6]. The pseudotensor formalism is inevitable for developing isotropic and hemitropic micropolar elastic continua models (see [7–10]).

In this study, the concept of covariant constancy of absolute tensor and pseudotensor fields is introduced and discussed. Examples of fundamental objects of pseudotensor analysis possessing the properties of covariant constancy are given. An algorithm for obtaining covariant constant tensors and pseudotensors proposed in the monograph [2] is considered.

The paper is due to the fact that a number of authors (see, for example, [11, p. 65]) do state that the distortion and inversed distortion tensors are covariant constants. The latter statement is not generally true and should be considered as erroneous, which can be elucidated by the rational mechanics technique [12, 13]. The paper presents the correct equations being the most similar to the covariant constancy of distortion and inversed distortion.

Before all it should be noted that the paper is aimed at determination of the covariant derivative of pseudotensors, which are widely employed in the mechanics of micropolar elastic solids [7–9].

After the Preliminary remarks in Sec. 1 the definitions of fundamental orienting pseudoscalar, generalized Kronecker deltas, permutation symbols (alternating pseudotensors), and alternating tensors are recalled for N–dimensional space. The covariant differentiation of an arbitrary relative tensor is considered and a number of particular cases are given.

Then in Sec. 2 the definitions of covariant constant tensor and pseudotensor fields are proposed. It will be shown that relative and absolute tensors with constant components are covariant constants. A number of covariant constant tensors (e.g. introduced in Sec. 1 fundamental orienting pseudoscalar and its integer powers, generalized Kronecker deltas, permutation symbols, alternating tensors) is collected in the Table for convenience and further references. A general algorithm for obtaining tensors and pseudotensors with constant components which are simultaneously covariant constant is recalled and discussed following the monograph by B. G. Gurevich [2].

Finally, in Sec. 3 the definitions of distortion (deformation gradient) and inversed distortion tensors are considered according to the rational mechanics scheme. The positive absolute scalar J known from rational mechanics is recalled and calculated in terms of the fundamental orienting pseudoscalars related to referential and actual states. The tensor reformulations of Euler–Piola–Jacobi equations are proposed. Equations being the most similar to the covariant constancy of distortion and inversed distortion are obtained.

[1] For example, following I. S. Sokolnikoff [6], covariant constant (parallel) Euler vector fields can be used in order to formulate the principle of virtual displacements.

As a whole, the present paper should be considered as a contemporary framework for problems of covariant differentiation of tensors and pseudotensors fields that is important for nonlinear continuum mechanics.

1. Notions and requisite equations from algebra and analysis of pseudotensors in Euclidean space. Consider an N-dimensional Euclidean space supplied by the coordinates x^k. The fundamental orienting pseudoscalar e [7–10] and its integer powers play an important role in the geometry of multidimensional spaces. In an N-dimensional space, it is defined as the skew product [1, p. 63–65] of absolute covariant basis vectors

$$\lceil \underset{1}{\boldsymbol{\imath}}, \underset{2}{\boldsymbol{\imath}}, \ldots, \underset{N}{\boldsymbol{\imath}} \rfloor = e. \tag{1}$$

It is easy to demonstrate that in an Euclidean space the following relation holds true

$$e^2 = g,$$

where g is the determinant of the metric tensor g_{ij}: $g = \det(g_{ij})$.

In a three-dimensional space ($N = 3$) the equation (1) is reformulated as

$$e = \lceil \underset{1}{\boldsymbol{\imath}}, \underset{2}{\boldsymbol{\imath}}, \underset{3}{\boldsymbol{\imath}} \rfloor = (\underset{1}{\boldsymbol{\imath}} \times \underset{2}{\boldsymbol{\imath}}) \cdot \underset{3}{\boldsymbol{\imath}}.$$

We proceed to discussion of other fundamental objects of N-dimensional geometry which are the absolute tensors $\delta^{j_1 j_2 \ldots j_M}_{i_1 i_2 \ldots i_M}$, usually called as generalized Kronecker deltas. Objects $\delta^{j_1 j_2 \ldots j_M}_{i_1 i_2 \ldots i_M}$ are defined for each $M \leqslant N$ according to

$$\delta^{j_1 j_2 \ldots j_M}_{i_1 i_2 \ldots i_M} = \begin{cases} +1, & \text{if } j_1, j_2, \ldots, j_M \text{ are distinct integers selected from the range} \\ & 1, 2, \ldots, N \text{ and if, } i_1, i_2, \ldots, i_M \text{ is an even permutation of} \\ & j_1, j_2, \ldots, j_M; \\ -1, & \text{if } j_1, j_2, \ldots, j_M \text{ are distinct integers selected from the range} \\ & 1, 2, \ldots, N \text{ and if, } i_1, i_2, \ldots, i_M \text{ is an odd permutation} \\ & j_1, j_2, \ldots, j_M; \\ 0, & \text{if any two of } j_1, j_2, \ldots j_3 \text{ are equal, or if any two of} \\ & i_1, i_2 \ldots i_M \text{ equal, or if the set of numbers } j_1, j_2 \ldots j_M \\ & \text{differs, apart from order, from the set } i_1, i_2 \ldots i_m. \end{cases}$$

By the aid of deltas $\delta^{j_1 j_2 \ldots j_M}_{i_1 i_2 \ldots i_M}$ the permutation symbols (alternating pseudotensors) can be immediately introduced as:

$$\overset{[-1]}{\epsilon}_{i_1 i_2 \ldots i_N} = \delta^{12 \ldots N}_{i_1 i_2 \ldots i_N}, \qquad \overset{[+1]}{\epsilon}{}^{i_1 i_2 \ldots i_N} = \delta^{i_1 i_2 \ldots i_N}_{12 \ldots N}.$$

It should be noted that pseudotensors can be transformed into absolute tensors by using the fundamental orienting pseudoscalar e (see [7–10]). For an arbitrary pseudotensor of integer weight W we have

$$T^{pqr \ldots s}_{\cdots \ldots \cdot ij \ldots l} = e^{-W} \overset{[W]}{T}{}^{pqr \ldots s}_{\cdots \ldots \cdot ij \ldots l}.$$

For example, the alternating tensors can be obtained from permutation symbols multiplied by the corresponding power (+1 or −1) of fundamental orienting pseudoscalar

$$e_{i_1 i_2 \dots i_N} = e \overset{[-1]}{\epsilon}_{i_1 i_2 \dots i_N}, \quad e_{i_1 i_2 \dots i_N} = \frac{1}{e} \overset{[+1]}{\epsilon}{}^{i_1 i_2 \dots i_N}.$$

The weight index $[W]$ in an upper position will be omitted for fundamental symbols such as the fundamental orienting scalar, permutation symbols, and it is also applicable to zero weight absolute tensors.

The covariant derivative of the pseudotensor $\overset{[W]}{T}{}^{pqr\dots s}_{\cdot\cdot\cdot\dots\cdot ij\dots l}$ of a given valency and weight is determined by the following equation corresponding to an analogous operation for absolute tensors [6]:

$$\nabla_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} = \partial_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} + \overset{[W]}{T}{}^{sm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^l_{sp} + \overset{[W]}{T}{}^{ls\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^m_{sp} + \\ + \dots + \overset{[W]}{T}{}^{lm\dots s}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^n_{sp} - \Gamma^s_{ip}\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot sj\dots k} - \Gamma^s_{jp}\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot is\dots k} - \\ - \dots - \Gamma^s_{kp}\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots s} - W\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^s_{sp}, \quad (2)$$

where $\partial_p = \dfrac{\partial}{\partial x^p}$.

The tensor gradient of an arbitrary valency and weight W tensor is defined by the following direct equation[2]:

$$\boldsymbol{\nabla} \otimes \overset{[W]}{\mathbf{T}} = e^W \overset{k}{\boldsymbol{\imath}} \otimes \partial_k (e^{-W} \overset{[W]}{\mathbf{T}}). \quad (3)$$

Expanding the equation (3) and taking account of

$$\partial_p \overset{m}{\boldsymbol{\imath}} = -\Gamma^m_{sp}\overset{s}{\boldsymbol{\imath}}, \quad \partial_p \underset{m}{\boldsymbol{\imath}} = \Gamma^s_{mp}\underset{s}{\boldsymbol{\imath}}, \quad e^{-1}(\partial_p e) = \Gamma^s_{ps},$$

it can be seen that (2) follows from (3). It is clear that (for the sake of compactness, the lengths of polyads are not explicitly indicated)

$$\boldsymbol{\nabla}\otimes \overset{[W]}{\mathbf{T}} = e^W \overset{p}{\boldsymbol{\imath}} \otimes \partial_p (e^{-W} \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}}) = \\ = \big(\partial_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} - W e^{-1}(\partial_p e) \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\big) \overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} + \\ + \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^s_{lp} \overset{p}{\boldsymbol{\imath}} \otimes \underset{s}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} + \\ + \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^s_{mp} \overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{s}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} + \dots + \\ + \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^s_{np} \overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{s}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} -$$

[2]The Hamilton nabla is conventionally defined according to: $\boldsymbol{\nabla} = \overset{s}{\boldsymbol{\imath}}\,\partial_s$.

$$- \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^i_{sp}\overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{s}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} -$$
$$- \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^j_{sp}\overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{s}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} - \dots -$$
$$- \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^k_{sp}\overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{s}{\boldsymbol{\imath}}.$$

Then the similar polyads can be discriminated and after a number of rearrangements we come to the equation:

$$\boldsymbol{\nabla} \otimes \overset{[W]}{\mathbf{T}} = \big(\partial_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} - W\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^s_{sp} + \overset{[W]}{T}{}^{sm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^l_{sp} + \overset{[W]}{T}{}^{ls\dots n}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^m_{sp} +$$
$$+ \dots + \overset{[W]}{T}{}^{lm\dots s}_{\cdot\cdot\dots\cdot ij\dots k}\Gamma^n_{sp} - \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot sj\dots k}\Gamma^s_{ip} - \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot is\dots k}\Gamma^s_{jp} - \dots -$$
$$- \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots s}\Gamma^s_{kp}\big)\overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}} =$$
$$= (\nabla_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k})\overset{p}{\boldsymbol{\imath}} \otimes \underset{l}{\boldsymbol{\imath}} \otimes \underset{m}{\boldsymbol{\imath}} \otimes \dots \otimes \underset{n}{\boldsymbol{\imath}} \otimes \overset{i}{\boldsymbol{\imath}} \otimes \overset{j}{\boldsymbol{\imath}} \otimes \dots \otimes \overset{k}{\boldsymbol{\imath}}.$$

Thus, the equation (2) can be derived on the ground of the definition (3).

The equation (2) in particular cases furnishes:

1) covariant derivative of a pseudoscalar of weight W:

$$\nabla_p \overset{[W]}{T} = \partial_p \overset{[W]}{T} - W\overset{[W]}{T}\Gamma^s_{sp};$$

2) the covariant derivative of a contravariant pseudovector of weight W:

$$\nabla_p \overset{[W]}{T}{}^k = \partial_p \overset{[W]}{T}{}^k + \overset{[W]}{T}{}^s\Gamma^k_{sp} - W\overset{[W]}{T}{}^k\Gamma^s_{sp};$$

3) covariant derivative of 2-contravariant pseudotensor of weight W:

$$\nabla_p \overset{[W]}{T}{}^{ji} = \partial_p \overset{[W]}{T}{}^{ji} + \overset{[W]}{T}{}^{si}\Gamma^j_{sp} + \overset{[W]}{T}{}^{js}\Gamma^i_{sp} - W\overset{[W]}{T}{}^{ji}\Gamma^s_{sp};$$

4) covariant derivative of 1-contravariant and 1-covariant pseudotensor of weight W:

$$\nabla_p \overset{[W]}{T}{}_{i\cdot}^{\cdot j} = \partial_p \overset{[W]}{T}{}_{i\cdot}^{\cdot j} - \overset{[W]}{T}{}_{s\cdot}^{\cdot j}\Gamma^s_{ip} + \overset{[W]}{T}{}_{i\cdot}^{\cdot s}\Gamma^j_{sp} - W\overset{[W]}{T}{}_{i\cdot}^{\cdot j}\Gamma^s_{sp}.$$

2. Definition and important examples of covariant constant tensor and pseudotensor fields. A given pseudotensor field $\overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k}$ of valency N and weight W is called as covariant constant if it satisfies the pseudotensor equation

$$\nabla_p \overset{[W]}{T}{}^{lm\dots n}_{\cdot\cdot\dots\cdot ij\dots k} = \overset{[W]}{0}. \tag{4}$$

Examples of covariant constant tensors and pseudotensors (see [5, 6]) are presented in Table. Among them, the fundamental orienting pseudoscalar e having been often employed in micropolar theories of continuum mechanics.

Note that the tensor equation (4), involving a pseudotensor, being valid in a given coordinate net remains valid in any other coordinate net [5, 6]. In the right-handed Cartesian coordinates, all of the tensors from Table have constant components equalled to 0 or 1. In this case, their covariant derivatives are the usual partial derivatives. Thus, each covariant derivative will be equalled to zero, that proves the covariant constancy of the absolute tensors and pseudotensors from Table in any curvilinear coordinate net.

In the monograph [2, p. 164–176] a general algorithm for constructing tensors and pseudotensors with constant components is proposed. Those are clearly covariant constant ones, since the differentiation rules for sums and products lead to a zero result.

The algorithm permits obtaining an arbitrary absolute tensor $C^{i_1,i_2\ldots i_r}_{\cdot\cdot\ldots\cdot k_1 k_2\ldots k_r}$ with constant components by using the standard two-index Kronecker deltas in the form of a linear combination (of $r!$ terms) with arbitrary constant coefficients, while each term consists of products of r delta symbols permutated in superscripts. Note that all $C^{i_1,i_2\ldots i_r}_{\cdot\cdot\ldots\cdot k_1 k_2\ldots k_r}$ do not constitute the complete set of covariant constant absolute tensors. An evident example is a parallel vector field, which is a covariant constant vector, but not representable as a vector with constant components.

A pseudotensor (r-covariant, s-contravariant, $s = r + N|W|$, N — space dimension, W — weight) having constant components is also easily constructed by reducing it to an absolute tensor. In particular, if $W > 0$, then the absolute tensor should be formed according to

$$\overset{[W]}{C}{}^{i_1 i_2\ldots i_s}_{\cdot\cdot\ldots\cdot k_1 k_2\ldots k_r} \underbrace{\overset{[-1]}{\epsilon}_{i_{r+1}\ldots i_N} \cdots \overset{[-1]}{\epsilon}_{i_{s-N+1}\ldots i_s}}_{W}.$$

The corresponding pseudotensor representations can be found in the monograph [2, p. 175].

3. Distortion and inversed distortion tensors. We denote as x^i ($i = 1, 2, 3$) the spatial (Euler) coordinates and by X^α ($\alpha = 1, 2, 3$) the referential (Lagrangian) coordinates. Hereafter, the Latin indices are associated to the Euler coordinates, whereas the Greek ones to the Lagrangian[3]. The deformation gradient[4] (or distortion tensor) is defined by the following components, called as distortions:

$$x^{\cdot i}_{\alpha} = \partial_\alpha x^i.$$

The inversed deformation gradient (or inversed distortion)[5] is determined accord-

[3] In the early papers on rational mechanics (see, for example, [12]) the Latin capital letters K, L, M have been used in place of Greek. However, in the later work [13] the letters of the Greek alphabet had been imployed.

[4] More precisely, the transposed deformation gradient $\mathbf{F}^{\mathrm{T}}$.

[5] In contemporary continuum mechanics, along with the direct description $X^\alpha \to x^i$, the "inversed motion description" $x^i \to X^\alpha$ [14] is also searchable in literature. It seems that the "inversed description" was introduced into mechanics by G. Piola.

Table. Covariant constant tensors and pseudotensors in N-dimensional space

Standard terminology	Root notation	Weight	Transformation to absolute tensor
fundamental orienting pseudoscalar	e	$+1$	$\overset{[+1]}{e} = e$
fundamental orienting pseudoscalar	$\frac{1}{e}$	-1	$\overset{[-1]}{e}{}^{-1} = \frac{1}{e}$
sign of fundamental orienting pseudoscalar	$\operatorname{sgn} e$	–	
metric tensor	g_{ij}	0	
fundamental tensor	g^{ij}	0	
metric tensor determinant	g	$+2$	$\overset{[+2]}{g} = e^2$
sign of metric tensor determinant	$\operatorname{sgn} g$	0	
generalized Kronocker deltas ($M \leqslant N$)	$\delta^{j_1 j_2 \ldots j_M}_{i_1 i_2 \ldots i_M}$	0	
alternating pseudotensors	$\epsilon^{i_1 i_2 \ldots i_M}$	$+1$	$e^{i_1 i_2 \ldots i_N} = \frac{1}{e} \overset{[+1]}{\epsilon}{}^{i_1 i_2 \ldots i_N}$
alternating pseudotensors	$\epsilon_{i_1 i_2 \ldots i_M}$	-1	$e_{i_1 i_2 \ldots i_N} = e \overset{[-1]}{\epsilon}{}_{i_1 i_2 \ldots i_N}$
alternating tensor	$e^{i_1 i_2 \ldots i_N}$	0	
alternating tensor	$e_{i_1 i_2 \ldots i_N}$	0	
parallel covariant vector field	λ_i	0	

ing to the equation

$$X_i^{\cdot\alpha} = \partial_i X^\alpha.$$

The following equations are clearly valid

$$x_\alpha^{\cdot i} X_j^{\cdot\alpha} = \delta_j^i, \quad X_i^{\cdot\beta} x_\alpha^{\cdot i} = \delta_\alpha^\beta.$$

Following the rational mechanics scheme, distortion $x_\alpha^{\cdot i}$ and inversed distortion $X_j^{\cdot\alpha}$ are equivalently redefined by the relations

$$x_\alpha^{\cdot i} = {}^{\backprime}\nabla_\alpha^{\text{total}} x^i, \quad X_i^{\cdot\alpha} = \nabla_i^{\text{total}} X^\alpha,$$

where the differential operators ${}^{\backprime}\nabla_\alpha^{\text{total}}$ and ∇_i^{total} denote the total covariant derivatives as of rational mechanics script found in [12, p. 810].

In the book by V. L. Berdichevsky [11, p. 65] it is stated that the distortion and inversed distortion are covariant constant tensors. This statement is not generally true. Following the rational mechanics scenario [12, p. 244, equation (16.5)], we introduce the positive absolute scalar J:

$$J = \frac{e}{{}^{\backprime}e} > 0, \tag{5}$$

wherein the fundamental orienting pseudoscalar e is equalled to the triple product of the convected basis vectors, ${}^{\backprime}e$ is equalled to the triple products of the referential basis vectors. Basis vectors in the referential state are ${}^{\backprime}\underset{1}{\boldsymbol{\imath}}$, ${}^{\backprime}\underset{2}{\boldsymbol{\imath}}$, ${}^{\backprime}\underset{3}{\boldsymbol{\imath}}$. Following the deformation they are transformed into $\underset{1}{\boldsymbol{\imath}}$, $\underset{2}{\boldsymbol{\imath}}$, $\underset{3}{\boldsymbol{\imath}}$. Therefore the fundamental orienting pseudoscalars in eq. (5) are determined as:

$$e = (\underset{1}{\boldsymbol{\imath}} \times \underset{2}{\boldsymbol{\imath}}) \cdot \underset{3}{\boldsymbol{\imath}}, \quad {}^{\backprime}e = ({}^{\backprime}\underset{1}{\boldsymbol{\imath}} \times {}^{\backprime}\underset{2}{\boldsymbol{\imath}}) \cdot {}^{\backprime}\underset{3}{\boldsymbol{\imath}}.$$

Obviously, $J = +e$ in the case when the referential basis is right-handed Cartesian, and $J = -e$ if the referential basis is left-handed.

The Jacobian defined by deformation $\Delta = \det(\partial_\alpha x^i)$ will satisfy the Jacobi identity [12, p. 246, equation (17.8)]

$$\frac{\partial \Delta}{\partial x_\alpha^{\cdot i}} = X_i^{\cdot\alpha} \Delta.$$

By using the latter equation, one can obtain the Euler–Piola–Jacobi equations [12, p. 246, equation (17.9)]:

$$\partial_k^{\text{total}}(\Delta^{-1} x_\alpha^{\cdot k}) = 0, \quad \partial_\alpha^{\text{total}}(\Delta X_k^\alpha) = 0. \tag{6}$$

The tensor reformulation of equations (6) read

$$\nabla_l^{\text{total}}(J^{-1} x_\alpha^{\cdot l}) = 0, \quad {}^{\backprime}\nabla_\alpha^{\text{total}}(J X_k^{\cdot\alpha}) = 0. \tag{7}$$

The equations (7) are valid in any coordinate system, including the case when the Euler and Lagrangian coordinates are Cartesian. In this case, the following relation holds:

$$J = \Delta. \tag{8}$$

Taking account of (8) equation (7) can be derived from (6), as in Cartesian coordinates, we have

$$
\begin{aligned}
{}^{\backprime}\nabla_{\alpha}^{\text{total}}(JX_{k}^{\cdot\alpha}) &= \partial_{\alpha}^{\text{expl}}(\Delta X_{k}^{\cdot\alpha}) + \partial_{s}^{\text{expl}}(\Delta X_{k}^{\cdot\alpha})(\partial_{\alpha}x^{s}) = \partial_{\alpha}^{\text{total}}(\Delta X_{k}^{\cdot\alpha}),\\
\nabla_{k}^{\text{total}}(J^{-1}x_{\alpha}^{\cdot k}) &= \partial_{\beta}^{\text{expl}}(\Delta^{-1}x_{\alpha}^{\cdot k})(\partial_{k}X^{\beta}) + \partial_{k}^{\text{expl}}(\Delta^{-1}x_{\alpha}^{\cdot k}) = \partial_{k}^{\text{total}}(\Delta^{-1}x_{\alpha}^{\cdot k}),
\end{aligned}
$$

and by applying equations (6) we can obtain the following equations:

$$
\nabla_{i}^{\text{total}}(J^{-1}x_{\alpha}^{\cdot i}) = 0, \quad {}^{\backprime}\nabla_{\alpha}^{\text{total}}(JX_{i}^{\cdot\alpha}) = 0.
$$

Besides the equations (7) no other statements regarding the covariant constancy of distortion and inversed distortion tensors are known in nonlinear continuum mechanics.

Results and conclusions. Covariant constancy of absolute tensors and pseudotensors of arbitrary valence and weight has been discussed to correct erroneous statements found in the literature on nonlinear continuum mechanics.

1) The notions and requisite equations from algebra and analysis of pseudotensors have been presented for clear understanding and reference framework.
2) The concept of covariant constancy of tensors and pseudotensors has been proposed and discussed.
3) Examples of covariant constant tensors and pseudotensors interesting for micropolar elasticity have been given in Table for convenience. In particular, the notion of fundamental orienting pseudoscalar that satisfies the condition of covariant constancy has been introduced and applied to the problems of concern.
4) A general algorithm for constructing tensors and pseudotensors with constant components which simultaneously are covariant constant has been recalled and discussed.
5) The distortion and inversed distortion tensors, which are fundamental for nonlinear mechanics of solids, have been shown not actually covariant constant, contrary to the erroneous statements of the covariant constancy of distortion and inversed distortion discovered after a literary search.

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О ковариантном непостоянстве тензоров дисторсии и обратной дисторсии

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Аннотация

Обсуждаются вопросы ковариантного постоянства тензоров и псевдотензоров произвольной валентности и веса в евклидовом пространстве. Приводятся минимально необходимые сведения из алгебры и анализа псевдотензоров. Выясняются условия ковариантного постоянства псевдотензоров. Рассматриваются примеры ковариантно постоянных тензоров и псевдотензоров из многомерной геометрии. Речь, в частности, идет о фундаментальном ориентирующем псевдоскаляре, целые степени которого удовлетворяют условию ковариантного постоянства. В работе продемонстрировано, что тензоры дисторсии и обратной дисторсии на самом деле не являются ковариантно постоянными, в противовес указаниям на ковариантное постоянство дисторсии и обратной дисторсии, которые встречаются в литературных источниках по нелинейной механике континуума.

Ключевые слова: псевдотензор, фундаментальный ориентирующий псевдоскаляр, дисторсия, обратная дисторсия, ковариантно постоянные тензоры, параллельное векторное поле.

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