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Periodic solutions for an impulsive system of integro-differential equations with maxima

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Abstract

A periodical boundary value problem for a first-order system of ordinary integro-differential equations with impulsive effects and maxima is investigated. A system of nonlinear functional-integral equations is obtained and the existence and uniqueness of the solution of the periodic boundary value problem are reduced to the solvability of the system of nonlinear functional-integral equations. The method of successive approximations in combination with the method of compressing mapping is used in the proof of one-valued solvability of nonlinear functional-integral equations. We define the way with the aid of which we could prove the existence of periodic solutions of the given periodical boundary value problem.

Keywords: impulsive integro-differential equations, periodical boundary value condition, nonlinear kernel, compressing mapping, existence and uniqueness of periodic solution.

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1. Problem Statement

The mathematical models of many problems of modern sciences, technology, and economics are described by differential and integro-differential equations, the solutions of which are functions with first-kind discontinuities at fixed or non-fixed times. Such differential and integro-differential equations are called equations with impulsive effects. Various publications are appearing on the study of differential

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and integro-differential equations with impulsive effects, describing many natural and technical processes (see for instance [1–13]).

As is known, in recent years, interest in the study of differential and integro-differential equations with periodical boundary conditions has increased. In particular, in the works [14–17], periodic solutions of differential equations with impulsive effects are studied.

In this paper, we investigate a periodical boundary value problem for a system of first-order integro-differential equations with impulsive effects, nonlinear kernel depending on construction of maxima. The questions of existence and uniqueness of the solution to the periodical boundary value problem are investigated. We note that differential and integro-differential equations with maxima have singularity in the study of the questions of solvability [18].

On the interval $[0,T]$ for $t \neq t_i$ $(i = 1,2,\dots,p)$ we consider the questions of existence and constructive methods of calculating the periodic solutions of the system of nonlinear ordinary first-order integro-differential equations with impulsive effects and maxima

$$x'(t) = f\left(t, x(t), \int_{-\infty}^{t} K\big(t, s, \max\big\{x(\tau) : \tau \in [\lambda_1(s), \lambda_2(s)]\big\}\big) ds\right). \tag{1}$$

We study the integro-differential equation (1) with periodic conditions

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty, 0], \\ x(0) = x(T), \end{cases} \tag{2}$$

and nonlinear impulsive effect

$$x(t_i^+) - x(t_i^-) = F_i(x(t_i)), \quad i = 1,2,\dots,p, \tag{3}$$

where $0 < \bar{t} < T$, $\bar{t} \neq t_i$, $i = 1,2,\dots,p$; $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$; x, $y \in X$; X is the closed bounded domain in the space $\mathbb{R}^n$, ∂X is its border; $f \in \mathbb{R}^n$, $-\infty < \lambda_1(t) < \lambda_2(t) \leqslant t$, $\varphi(t) \in C((-\infty,0],\mathbb{R}^n)$; $\varphi(0^-) = x(0^+)$; $x(t_i^+) = \lim\limits_{\nu\to 0^+} x(t_i+\nu)$, $x(t_i^-) = \lim\limits_{\nu\to 0^-} x(t_i-\nu)$ are the limits of the function on the right and left sides $x(t)$ at the point $t = t_i$, respectively. The function f is T-periodic $F_i = F_{i+p}$, $t_{i+p} = t_i + T$,

$$\int_{-\infty}^{t} \big|K(t,s,x)\big| ds < \infty.$$

By $C([0,T],\mathbb{R}^n)$ denoted the Banach space, which consists continuous vector function $x(t)$, defined on the segment $[0,T]$, with the norm

$$\|x(t)\| = \sqrt{\sum_{j=1}^{n} \max_{0\leqslant t\leqslant T} |x_j(t)|}.$$

By $PC([0,T],\mathbb{R}^n)$ is denoted the following linear vector space:

$$PC([0,T],\mathbb{R}^n) = \big\{x : [0,T] \to \mathbb{R}^n;\ x(t) \in C((t_i,t_{i+1}],\mathbb{R}^n),\ i = 1,2,\dots,p\big\},$$

where $x(t_i^+)$ and $x(t_i^-)$ $(i=0,1,\ldots,p)$ exist and bounded; $x\left(t_i^-\right)=x\left(t_i\right)$. Note, that the linear vector space $PC([0,T],\mathbb{R}^n)$ is Banach space with the following norm:

$$\|x(t)\|_{PC}=\max\left\{\|x(t)\|_{C((t_i,t_{i+1}])},\ i=1,2,\ldots,p\right\}.$$

Formulation of problem. *Find the T-periodic function $x(t)\in PC([0,T],\mathbb{R}^n)$, which for all $t\in[0,T]$, $t\neq t_i$ $(i=1,2,\ldots,p)$ satisfies the system of differential equations (1), periodic condition (2) and for $t=t_i$ $(i=1,2,\ldots,p)$ $0<t_1<t_2<\cdots<t_p<T$ satisfies the nonlinear limit condition (3) and goes through x_0 at $t=0$.*

2. Reduction to Functional-Integral Equation

Let the function $x(t)\in PC([0,T],\mathbb{R}^n)$ be a solution of the periodic boundary value problem (1)–(3). Then by integration of the equation (1) on the intervals $(0,t_1]$, $(t_1,t_2]$, …, $(t_p,t_{p+1}]$, we obtain the following:

$$\int_0^{t_1} f(s,x,y)\,ds=\int_0^{t_1} x'(s)\,ds=x(t_1^-)-x(0^+),\quad t\in(0,t_1],$$

$$\int_{t_1}^{t_2} f(s,x,y)\,ds=\int_{t_1}^{t_2} x'(s)\,ds=x(t_2^-)-x(t_1^+),\quad t\in(t_1,t_2],$$

$$\vdots$$

$$\int_{t_p}^{t_{p+1}} f(s,x,y)\,ds=\int_{t_p}^{t_{p+1}} x'(s)\,ds=x(t_{p+1}^-)-x(t_p^+),t\in(t_p,t_{p+1}],$$

where

$$f(s,x,y)=f\left(t,x(t),\int_{-\infty}^{t}K\big(t,s,\max\{x(\tau):\tau\in[\lambda_1(s),\lambda_2(s)]\}\big)ds\right).$$

Hence, taking $x(0^+)=x(0)$, $x(t_{k+1}^-)=x(t)$ into account, on the interval $[0,T]$ we have

$$\int_0^t f(s,x,y)\,ds=[x(t_1)-x(0^+)]+[x(t_2)-x(t_1^+)]+\cdots+[x(t)-x(t_p^+)]=$$
$$=-x(0)-[x(t_1^+)-x(t_1)]-[x(t_2^+)-x(t_2)]-\cdots-[x(t_p^+)-x(t_p)]+x(t).$$

Taking into account the condition (3), we rewrite the last equality as

$$x(t)=x(0)+\int_0^t f(s,x,y)\,ds+\sum_{0<t_i<t}F_i(x(t_i)).\tag{4}$$

We subordinate the function $x(t)\in PC([0,T],\mathbb{R}^n)$ in (4) to satisfy the periodic condition (2):

$$x(T)=x(0)+\int_0^T f(s,x,y)\,ds+\sum_{0<t_i<T}F_i(x(t_i)).$$

Hence, taking the condition (2) into account, we obtain the following:

$$\int_0^T f(s,x,y)\,ds + \sum_{0<t_i<T} F_i(x(t_i)) = 0.$$

Consequently, the integro-differential equation (1) can be written as

$$x'(t) = f\Big(t, x(t), \int_{-\infty}^{t} K\big(t,s,\max\{x(\tau) : \tau\in[\lambda_1(s),\lambda_2(s)]\}\big)ds\Big) - \\ - \frac{1}{T}\int_0^T f\Big(t, x(t), \int_{-\infty}^{t} K\big(t,s,\max\{x(\tau) : \tau\in[\lambda_1(s),\lambda_2(s)]\}\big)ds\Big)dt - \\ - \frac{1}{T}\sum_{i=1}^{p} F_i(x(t_i)). \quad (5)$$

Then by integration of equation (5) into the intervals $(0,t_1]$, $(t_1,t_2]$, …, $(t_p,t_{p+1}]$ instead of (4) we obtain the following system of equations:

$$x(t) = x_0 + \int_0^t \Big[f\Big(s, x(s), \int_{-\infty}^{s} K\big(s,\theta,\max\{x(\tau) : \tau\in[\lambda_1(\theta),\lambda_2(\theta)]\}\big)d\theta\Big) - \\ - \frac{1}{T}\int_0^T f\Big(\theta, x(\theta), \int_{-\infty}^{\theta} K\big(\theta,\xi,\max\{x(\tau) : \tau\in[\lambda_1(\xi),\lambda_2(\xi)]\}\big)d\xi\Big)d\theta - \\ - \frac{1}{T}\sum_{i=1}^{p} F_i(x(t_i))\Big]ds + \sum_{0<t_i<t} F_i(x(t_i)). \quad (6)$$

3. Preliminaries

Lemma 1. *For the equation (6) the following estimate is true:*

$$\|x(t)-x_0\|_{PC} \leqslant M_1\frac{T}{2} + 2M_2 p, \quad (7)$$

where

$$M_1 = \|f(t,x(t),y(t))\|, \quad M_2 = \max_{1\leqslant i\leqslant p}\|F_i(t,x(t))\|.$$

Proof. We rewrite the equation (6) as follows:

$$x(t) - x_0 = \int_0^t \Big[f(s,x(s),y(s)) - \\ - \frac{1}{T}\int_0^T f(\theta,x(\theta),y(\theta))d\theta - \frac{1}{T}\sum_{i=1}^{p} F_i(x(t_i))\Big]ds + \sum_{0<t_i<t} F_i(x(t_i)) = \\ = \int_0^t f(s,x(s),y(s))ds - \frac{t}{T}\int_0^t f(s,x(s),y(s))ds -$$

$$-\frac{t}{T}\int_t^T f(s,x(s),y(s))ds-\frac{t}{T}\sum_{i=1}^{p}F_i(x(t_i))+\sum_{0<t_i<t}F_i(x(t_i)).$$

Hence, this implies that the following estimate is true:

$$\|x(t)-x_0\|_{PC}\leqslant \alpha(t)\|f(t,x(t),y(t))\|+2p\max_{1\leqslant i\leqslant p}\|F_i(t,x(t))\|, \tag{8}$$

where $\alpha(t)=2t(1-t/T)$. It is easy to check that from (8) follows (7). Lemma 1 is proved. □

Remark. T-periodic solution $x_{\varphi(t)}=\psi(t)$ of the system (1) with the initial value function $\varphi(t)$ in the initial set $(-\infty,0]$ is defined by the initial value function $\varphi(t)$, which is a periodic continuation of the solution $\psi(t)$ in the initial set $(-\infty,0]$.

Lemma 2. *For the difference of two functions with maxima, we have the following estimate*:

$$\|\max\{x(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}-\max\{y(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}\|\leqslant$$
$$\leqslant\|x(t)-y(t)\|+2h\left\|\frac{\partial}{\partial t}[x(t)-y(t)]\right\|, \tag{9}$$

where

$$h=\sup_{-\infty<t\leqslant T}|\lambda_1(t)-\lambda_2(t)|.$$

Proof. We use obvious true relations:

$$\max\{x(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}=\max\{[x(\tau)-y(\tau)+y(\tau)]:\tau\in[\lambda_1(t),\lambda_2(t)]\}\leqslant$$
$$\leqslant\max\{[x(\tau)-y(\tau)]:\tau\in[\lambda_1(t),\lambda_2(t)]\}+\max\{y(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}.$$

Hence, we obtain the following:

$$\max\{x(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}-\max\{x(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}\leqslant$$
$$\leqslant\max\{[x(\tau)-y(\tau)]:\tau\in[\lambda_1(t),\lambda_2(t)]\}. \tag{10}$$

We denote by t_1 and t_2 the points of the interval $[\lambda_1(t),\lambda_2(t)]$, on which the maximums of the functions $x(t)$ and $y(t)$ are reached:

$$\max\{x(\tau)\,|\tau\in[\lambda_1(t),\lambda_2(t)]\}=x(t_1),\max\{y(\tau)\,|\tau\in[\lambda_1(t),\lambda_2(t)]\}=y(t_1),$$

$$\max\{[x(\tau)-y(\tau)]\,|\tau\in[\lambda_1(t),\lambda_2(t)]\}=x(t_2)-y(t_2).$$

Then, taking (10) and last equalities, we have

$$\|\max\{x(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}-\max\{y(\tau):\tau\in[\lambda_1(t),\lambda_2(t)]\}-x(t)+y(t)\|\leqslant$$
$$\leqslant\|[x(t)-y(t)]-[x(t_1)-y(t_1)]\|+\|[x(t_2)-y(t_2)]-[x(t_1)-y(t_1)]\|. \tag{11}$$

From another side, it is obvious that the estimate is valid:

$$\|[x(\bar{t})-y(\bar{t})]-[x(\bar{\bar{t}})-y(\bar{\bar{t}})]\|\leqslant h\left\|\frac{d}{dt}[x(t^*)-y(t^*)]\right\|\leqslant h\left\|\frac{d}{dt}[x(t)-y(t)]\right\|, \tag{12}$$

where $\bar{t}$, $\bar{\bar{t}} \in [\lambda_1(t), \lambda_2(t)]$, $t^* \in (\bar{t}, \bar{\bar{t}})$. From the estimates (11) and (12) we come to the following estimate:

$$\|\max\{x(\tau) : \tau \in [\lambda_1(t), \lambda_2(t)]\} - \max\{y(\tau) : \tau \in [\lambda_1(t), \lambda_2(t)]\} - x(t) + y(t)\| \leqslant \\ \leqslant 2h \Big\| \frac{d}{dt}[x(t) - y(t)] \Big\|.$$

Therefore, it is easy to check that the inequality (9) and we complete the proof of Lemma 2. □

4. Main Results

THEOREM 1. *Assume that for all $t \in [0, T]$, $t \neq t_i$ $(i = 1, 2, \dots, p)$ the following conditions are satisfied*:

1) $\|f(t, x(t), y(t))\| \leqslant M_1 < \infty$, $\max\limits_{1 \leqslant i \leqslant p} \|F_i(t, x(t))\| \leqslant M_2 < \infty$;
2) $\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqslant L_1\big[\|x_1 - x_2\| + \|y_1 - y_2\|\big]$;
3) $\|K(t, s, x_1) - K(t, s, x_2)\| \leqslant L_2(s)\|x_1 - x_2\|$, $0 < \sup\limits_{t} \int_{-\infty}^{t} L_2(s) ds < \infty$;
4) $\|F_i(t, x_1) - F_i(t, x_2)\| \leqslant L_3\|x_1 - x_2\|$;
5) *the radius of the inscribed ball in X is greater than $M_1 T/2 + 2M_2 p$*;
6) $\rho < 1$, *where*

$$\rho = \max\Big\{ L_1(1 + M_3)\Big(1 + \frac{T}{2}\Big) + pL_3\Big(2 + \frac{1}{T}\Big), 2L_1 M_3 (T + 1) h \Big\}.$$

If the system (1) has a solution for all $t \in [0, T]$, $t \neq t_i$ $(i = 1, 2, \dots, p)$, then this solution can be founded by the system of nonlinear functional-integral equations

$$x(t, x_0) = x_0 + \\ + \int_0^t \Big[f\Big(s, x(s, x_0), \int_{-\infty}^{s} K\big(s, \theta, \max\{x(\tau, x_0) : \tau \in [\lambda_1(\theta), \lambda_2(\theta)]\}\big) d\theta \Big) - \\ - \frac{1}{T} \int_0^T f\Big(\theta, x(\theta, x_0), \int_{-\infty}^{\theta} K\big(\theta, \xi, \max\{x(\tau, x_0) : \tau \in [\lambda_1(\xi), \lambda_2(\xi)]\}\big) d\xi \Big) d\theta - \\ - \frac{1}{T} \sum_{i=1}^{p} F_i(x(t_i, x_0)) \Big] ds + \sum_{0 < t_i < t} F_i(x(t_i, x_0)). \quad (13)$$

Proof. The theorem we proof by the method of successive approximations, defining the iteration process as

$$x_0(t, x_0) = x_0, \qquad x_{k+1}(t, x_0) = x_0 + \\ + \int_0^t \Big[f\Big(s, x_k(s, x_0), \int_{-\infty}^{s} K\big(s, \theta, \max\{x_k(\tau, x_0) : \tau \in [\lambda_1(\theta), \lambda_2(\theta)]\}\big) d\theta \Big) - \\ - \frac{1}{T} \int_0^T f\Big(\theta, x_k(\theta, x_0), \int_{-\infty}^{\theta} K\big(\theta, \xi, \max\{x_k(\tau, x_0) : \tau \in [\lambda_1(\xi), \lambda_2(\xi)]\}\big) d\xi \Big) d\theta - \\ - \frac{1}{T} \sum_{i=1}^{p} F_i(x_k(t_i, x_0)) \Big] ds + \sum_{0 < t_i < t} F_i(x_k(t_i, x_0)). \quad (14)$$

We will show that the right-hand side of the system of equations (13) as an operator maps a ball with radius $M_1T/2+2M_2p$ into itself and is a contraction operator. So, according to Lemma 1, from (7) and (14) we have

$$\|x_{k+1}(t,x_0)-x_0\|_{PC}\leqslant M_1\frac{T}{2}+2M_2p. \tag{15}$$

From the system of integro-differential equations

$$x'(t,x_0)=f\Big(t,x(t,x_0),\int_{-\infty}^{t}K\big(t,s,\max\{x(\tau,x_0):\tau\in[\lambda_1(s),\lambda_2(s)]\}\big)ds\Big)-$$
$$-\frac{1}{T}\int_0^T f\Big(t,x(t,x_0),\int_{-\infty}^{t}K\big(t,s,\max\{x(\tau,x_0):\tau\in[\lambda_1(s),\lambda_2(s)]\}\big)ds\Big)dt-$$
$$-\frac{1}{T}\sum_{i=1}^{p}F_i(x(t_i,x_0)) \tag{16}$$

we obtain the following:

$$\|x'_{k+1}(t,x_0)\|_{PC}\leqslant 2M_1+\frac{p}{T}M_2. \tag{17}$$

We consider a difference $x_{k+1}(t,x_0)-x_k(t,x_0)$ of two approximations, where the functions $x_{k+1}(t,x_0)$ and $x_k(t,x_0)$ are defined from the approximations of the system of equations (14). By the conditions of the theorem, from (14) we have

$$\|x_{k+1}(t,x_0)-x_k(t,x_0)\|\leqslant L_1\int_0^t\Big\{\,\|\,x_k(s,x_0)-x_{k-1}(s,x_0)\,\|+$$
$$+\int_{-\infty}^{s}L_2(\theta)\big\|\max\{x_k(\tau,x_0):\tau\in[\lambda_1(\theta),\lambda_2(\theta)]\}-$$
$$-\max\{x_{k-1}(\tau,x_0):\tau\in[\lambda_1(\theta),\lambda_2(\theta)]\}\big\|d\theta+$$
$$+\frac{1}{T}\int_0^T\Big[\|x_k(\theta,x_0)-x_{k-1}(\theta,x_0)\|+$$
$$+\int_{-\infty}^{\theta}L_2(\xi)\|\max\{x_k(\tau,x_0):\tau\in[\lambda_1(\xi),\lambda_2(\xi)]\}-$$
$$-\max\{x_{k-1}(\tau,x_0):\tau\in[\lambda_1(\xi),\lambda_2(\xi)]\}\big\|d\xi\Big]d\theta\Big\}ds+$$
$$+2\sum_{i=1}^{p}L_3\|x_k(t_i,x_0)-x_{k-1}(t_i,x_0)\|\leqslant$$
$$\leqslant\alpha(t)L_1\big[(1+M_3)\|x_k(t,x_0)-x_{k-1}(t,x_0)\|_{PC}+$$
$$+2hM_3\|x'_k(t,x_0)-x'_{k-1}(t,x_0)\|_{PC}\big]+2pL_3\|x_k(t,x_0)-x_{k-1}(t,x_0)\|_{PC}\leqslant$$
$$\leqslant\Big(L_1(1+M_3)\frac{T}{2}+2pL_3\Big)\|x_k(t,x_0)-x_{k-1}(t,x_0)\|_{PC}+$$
$$+2L_1M_3Th\|x'_k(t,x_0)-x'_{k-1}(t,x_0)\|_{PC}, \tag{18}$$

where

$$M_3 = \sup_t \int_{-\infty}^{t} L_2(s)ds < \infty.$$

Similarly, by the assumptions of Theorem 1, from (16) we have the following:

$$\|x'_{k+1}(t,x_0) - x'_k(t,x_0)\|_{PC} \leqslant \\ \leqslant \Big(L_1(1+M_3) + L_3\frac{p}{T}\Big)\|x_k(t,x_0) - x_{k-1}(t,x_0)\|_{PC} + \\ + 2L_1M_3h\|x'_k(t,x_0) - x'_{k-1}(t,x_0)\|_{PC}. \quad (19)$$

Adding both sides of (19) to (18), we obtain

$$\| y_{k+1}(t,x_0) - y_k(t,x_0) \|_{PC} \leqslant \rho\|y_k(t,x_0) - y_{k-1}(t,x_0)\|_{PC}, \quad (20)$$

where

$$\|y_{k+1}(t,x_0) - y_k(t,x_0)\| = \|x_{k+1}(t,x_0) - x_k(t,x_0)\| + \|x'_{k+1}(t,x_0) - x'_k(t,x_0)\|,$$

$$\rho = \max\Big\{L_1(1+M_3)\Big(1+\frac{T}{2}\Big) + pL_3\Big(2+\frac{1}{T}\Big), 2L_1M_3(T+1)h\Big\}.$$

According to the last condition of Theorem 1, $\rho < 1$. Since

$$\|x_{k+1}(t,x_0) - x_k(t,x_0)\| \leqslant \|y_{k+1}(t,x_0) - y_k(t,x_0)\|,$$

from the estimate (20) we deduce that the operator on right-hand side of (13) is compressing. From the estimates (15), (17) and (20) implies that there exists a unique fixed point $x(t,x_0)$. Theorem 1 is proved. □

From the estimate (20) it is easy to see that for x_0, $\bar{x}_0 \in X$ holds

$$\|x(t,x_0) - x(t,\bar{x}_0)\|_{BD} \leqslant \frac{|x_0 - \bar{x}_0|}{1-\rho}.$$

Now we will show the existence of periodic solutions of the system of impulsive integro-differential equations (1). We introduce the following designations:

$$\Delta(x_0) = \frac{1}{T}\sum_{i=1}^{p} F_i(x_\infty(t_i,x_0)) + \\ + \frac{1}{T}\int_0^T f\bigg(t, x_\infty(t,x_0), \int_{-\infty}^{t} K\big(t,s,\max\{x_\infty(\tau,x_0) : \tau \in [\lambda_1(s),\lambda_2(s)]\}\big)ds\bigg)dt, \quad (21)$$

$$\Delta_k(x_0) = \frac{1}{T}\sum_{i=1}^{p} F_i(x_k(t_i,x_0)) + \\ + \frac{1}{T}\int_0^T f\bigg(t, x_k(t,x_0), \int_{-\infty}^{t} K\big(t,s,\max\{x_k(\tau,x_0) : \tau \in [\lambda_1(s),\lambda_2(s)]\}\big)ds\bigg)dt, \quad (22)$$

where $x(t,x_0) = \lim_{k\to\infty} x_k(t,x_0) = x_\infty(t,x_0)$ is the solution of the non-linear system (13). Therefore, $x_\infty(t,x_0)$ is the solution of the system of impulsive integro-differential equations (1) for $\Delta(x_0) = 0$ through x_0 at $t = 0$. Consequently, the questions of the existence of a solution of the system of impulsive integro-differential equations (1) were reduced to the questions of the existence of zeros of the function $\Delta(x_0)$ and we solve this problem by finding zeros of the function $\Delta_k(x_0)$.

THEOREM 2. *Assume that*

1) *all the conditions of Theorem 1 are fulfilled*;
2) *there is a natural number k such that the function $\Delta_k(x_0)$ has an isolated singular point x_0^0 that $\Delta_k(x_0^0) = 0$ and the index of isolated singular point x_0^0 is nonzero*;
3) *there is a closed convex region $X_0 \subset X$, containing a single singular point such that on its border ∂X_0 is an estimate fulfilled*:

$$\inf_{x \in \partial X_0} \|\Delta_k(x)\| \geqslant \frac{M\rho^{k+1}}{1-\rho}. \tag{23}$$

Then the system of impulsive integro-differential equations (1) has a periodic solution for all $t \in [0, T]$, $t \neq t_i$ $(i = 1, 2, \ldots, p)$ that $x(0) \in X_0$.

Proof. By definition, the index of an isolated singular point x_0^0 of continuous mapping $\Delta_k(x_0)$ is equal to the characteristic of the vector field, generated by mapping $\Delta_k(x_0)$ on a sufficiently small sphere S^n with the center in x_0^0. Since in X_0 there is no other singular point, which will be different from x_0^0 and X_0 is homeomorphic to the unit ball E^n, then the characteristic of the vector field $\Delta_k(x_0)$ on the sphere S^n is equal to the characteristic of this vector field on ∂X_0. The fields $\Delta_k(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X_0. Let us consider families of everywhere continuous on ∂X vector fields

$$V(\sigma, x_0) = \Delta_k(x_0) + \sigma(\Delta(x_0) - \Delta_k(x_0)),$$

which connect the fields

$$V(0, x_0) = \Delta_k(x_0), \quad V(1, x_0) = \Delta(x_0).$$

We note that the estimate is true:

$$\|\Delta(x_0) - \Delta_k(x_0)\| \leqslant \frac{M\rho^{k+1}}{1-\rho}. \tag{24}$$

Therefore, the vector field $V(\sigma, x_0)$ does not vanish anywhere on ∂X_0. Indeed, from (23) and (24) implies that

$$\|V(\sigma, x_0)\| \geqslant \|\Delta_k(x_0)\| - \|\Delta(x_0) - \Delta_k(x_0)\| > 0. \tag{25}$$

The fields $\Delta_k(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X and the rotations of the homotopic fields in the compact are equal. Therefore, taking into account (25), we conclude that the rotation of the field $\Delta(x_0)$ on the ∂X_0 is equal to the index of the singular point x_0 of the field $\Delta_k(x_0)$ and nonzero. Consequently, the vector field $\Delta(x_0)$ on X_0 has a singular point x_0, for which $\Delta(x_0) = 0$. Therefore, the system of impulsive integro-differential equations (1) has a periodic solution for all $t \in [0, T]$, $t \neq t_i$ $(i = 1, 2, \ldots, p)$ that $x(0) \in X_0$. In addition, we note that for x_0, $\bar{x}_0 \in X$ from (21) and (22) we have

$$\|\Delta(x_0)\|_{PC} \leqslant M\Big(1 + \frac{p}{T}\Big), \qquad \|\Delta(x_0) - \Delta(\bar{x}_0)\|_{BD} \leqslant \frac{|x_0 - \bar{x}_0|}{1 - \rho}.$$

Theorem 2 is proved. □

Conclusion

The theory of differential and integro-differential equations plays an important role in solving many applied problems. Especially, local and nonlocal periodical boundary value problems for differential and integro-differential equations with impulsive actions have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology. In this paper, we investigate the system of first-order integro-differential equations (1) with periodical boundary value condition (2), with nonlinear kernel and with nonlinear condition (3) of impulsive effects for $t = t_i$, $i = 1, 2, \ldots, p$, $0 < t_1 < t_2 < \cdots < t_p < T$. The nonlinear right-hand side of this equation consists of the construction of maxima. The questions of the existence and uniqueness of the T-periodic solution of the boundary value problem (1)–(3) are studied. If the system (1) has a solution for all $t \in [0, T]$, $t \neq t_i$, $i = 1, 2, \ldots, p$, then this solution can be proven to be based on the system of nonlinear functional-integral equations (13). The questions of the existence of a solution of the system of impulsive differential equations (1) we reduce to the questions of the existence of zeros of function $\Delta(x_0)$ in (21) and we solve this problem by finding zeros of function $\Delta_k(x_0)$ in (22).

The results obtained in this work will allow us in the future to investigate other kind of periodical boundary value problem for the heat equation and the wave equation with impulsive actions.

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Периодические решения системы интегро-дифференциальных уравнений с импульсными воздействиями и максимумами

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Аннотация

Исследуется краевая задача для системы обыкновенных интегро-дифференциальных уравнений первого порядка с импульсными эффектами и максимумами. Получена система нелинейных функционально-интегральных уравнений и, таким образом, существование и единственность решения периодической краевой задачи сводятся к разрешимости системы нелинейных функционально-интегральных уравнений. Метод последовательных приближений в сочетании с методом сжимающих отображений используется при доказательстве однозначной разрешимости нелинейных функционально-интегральных уравнений. Определим способ, с помощью которого можно будет доказать существование периодических решений данной периодической краевой задачи.

Ключевые слова: интегро-дифференциальные уравнения с импульсными воздействиями, периодическое краевое условие, нелинейное ядро, сжимающее отображение, существование и единственность периодического решения.

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