Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki
[J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2023, vol. 27, no. 4, pp. 737–744
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu1998

MSC: 47H10, 54H25

# A new common fixed point theorem on orthogonal metric spaces and an application

***Y. Touail*[1], *A. Jaid*[2], *D. El Moutawakil*[3]**

[1] Université Sidi Mohamed Ben Abdellah, Fez, Morocco.
[2] Université Sultan Moulay Slimane, Beni-Mellal, Morocco.
[3] Université Chouaib Doukkali, El Jadida, Morocco.

**Abstract**

In the present work, a common fixed point result for self-mappings on orthogonal complete metric spaces, which are not necessarily complete, is proved. Furthermore, as an application, we find the existence of solutions to two differential equations.

**Keywords:** common fixed point, orthogonal metric space.

Received: 28th January, 2023 / Revised: 17th November, 2023 / Accepted: 13th December, 2023 / First online: 25th December, 2023

**1. Introduction and Preliminary.** The pioneering mathematician in the area of fixed point theory was Banach, who established and proved the first fixed point theorem is named the Banach contraction theorem [1]. After that, extensions of this theorem have been obtained either by generalizing the distance properties of the underlying metric space or by modifying the contractive condition on the mappings.

In 2017, Eshaghi Gordji et al. [2] defined orthogonal metric spaces as a generalization of metric spaces, as follows:

**Differential Equations and Mathematical Physics**
**Short Communication**



**Please cite this article in press as:**
Touail Y., Jaid A., El Moutawakil D. A new common fixed point theorem on orthogonal metric spaces and an application, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2023, vol. 27, no. 4, pp. 737–744. EDN: KPNGNF. DOI: 10.14498/vsgtu1998.

**Authors' Details:**

*Youssef Touail* https://orcid.org/0000-0003-3593-8253
Département de Mathématique; FSDM, Faculté des Sciences Dhar El Mahraz; e-mail: youssef9touail@gmail.com

*Amine Jaid* https://orcid.org/0000-0001-7322-2008
Equipe de Recherche en Mathématiques Appliquées, Technologies de l'Information et de la Communication; Faculté Polydisciplinaire de Khouribga; e-mail: aminejaid1990@gmail.com

*Driss El Moutawakil* https://orcid.org/0000-0001-7322-2008
Département de Mathématique; Ecole Supérieure de l'Education et de la Formation; e-mail: d.elmoutawakil@gmail.com

Definition 1 [2]. Let $X \neq \emptyset$ and let $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following hypothesis:

$$\exists x_0 : (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y).$$

Then $(X, \perp)$ is called an orthogonal set (briefly $O$-set).

The triplet $(X, \perp, d)$ is called an orthogonal metric space if $(X, d)$ is a metric space and $(X, \perp)$ is an $O$-set, and $x_0$ is said to be an orthogonal element.

Then, an important extension of the Banach fixed point principal is given, as follows:

Theorem 1 [2]. *Let $(X, \perp, d)$ be an $O$-complete metric space and $T$ a self-mapping on $X$ which is $\perp$-preserving and $\perp$-continuous. If there exists $k \in [0, 1)$ such that for all $x, y \in X$:*

$$x \perp y \text{ implies } d(Tx, Ty) \leqslant kd(x, y). \tag{1}$$

*Then, $T$ has a unique fixed point.*[1]

Later, many remarkable works in this area can be found in [3–5].

Motivated by [2] and other works concerning the theory of common fixed points [6–9], in this paper we restrict our studies only to orthogonal elements, to prove a result of common fixed points in a new setting and under weak conditions. In other words, we extend condition (1) to two self-mappings $f$, $g : X \mapsto X$, as follows:

$$x \perp y \text{ implies } d(fx, gy) \leqslant \phi(d(x, y)),$$

where $\phi \in \Phi$, the class of all nondecreasing selfmaps $\phi$ on $[0, +\infty)$ satisfying $\sum_{n=1}^{+\infty} \phi^n(t) < +\infty$ for all $t > 0$.

Moreover, an extension of the Banach fixed point theorem is delivered for a large class of mappings, we call it weakly-$\perp$-preserving.

In addition, we give an example to support the proven theorem and to show the usability of this new direction of research.

At the end of the results, an application to the study of the existence of common solutions for a class of differential equations is presented.

Finally, we assert some definitions that will be needed in the topic:

Definition 2 [2]. Let $(X, \perp)$ be an $O$-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $Tx \perp Ty$ whenever $x \perp y$.

Definition 3 [2]. Let $(X, \perp)$ be an $O$-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, $O$-sequence) if

$$(\forall n, x_n \perp x_{n+1}) \text{ or } (\forall n, x_{n+1} \perp x_n).$$

Definition 4 [2]. Let $(X, \perp, d)$ be an orthogonal metric space. Then, a mapping $T : X \to X$ is said to be orthogonally continuous (briefly $\perp$-continuous) in $x \in X$, if for each $O$-sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$, we obtain $Tx_n \to Tx$ as $n \to \infty$. In addition, $T$ is said to be $\perp$-continuous on $X$ if $T$ is $\perp$-continuous in each $x \in X$.

[1] In the sequel, we will recall the related basic notions of orthogonality.

Definition 5 [2]. Let $(X, \perp, d)$ be an orthogonal metric space. Then, $X$ is said to be orthogonally complete (or $\perp$-complete) if every Cauchy $O$-sequence is convergent.

Remark 1 [2]. Every complete metric space (continuous mapping) is $O$-complete metric space ($\perp$-continuous mapping) and the converse is not true.

Example 1 [2]. Let $X = Z$. Define the binary relation $\perp$ in $X$ by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(X, \perp)$ is an $O$-set.

**2. Main results.** The main result of this article is the following:

Theorem 2. *Let $(X, \perp, d)$ be an $O$-complete metric space and $f$, $g : X \to X$ be $\perp$-continuous mappings such that:*
*1) $x \perp y \implies (fx \perp gy$ or $gy \perp fx)$ and $(gx \perp fy$ or $fy \perp gx)$;*
*2) $x \perp y \implies d(gx, fy) \leqslant \phi(d(x, y))$, for all $x, y \in X$, where $\phi \in \Phi$.*
*Then, $f$, $g$ have a common fixed point.*

*Proof.* Since $X$ is an $O$-set, there exists at least $x_0 \in X$ such that

$$\forall y \in X,\ x_0 \perp y \text{ or } \forall y \in X,\ y \perp x_0.$$

So, in particular we have $x_0 \perp fx_0$ or $fx_0 \perp x_0$. We can choose a sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}^*$. The condition 1) implies

$$\forall n \in \mathbb{N}^*,\ x_n \perp x_{n+1} \text{ or } \forall n \in \mathbb{N}^*,\ x_{n+1} \perp x_n.$$

Then, $\{x_n\}$ is an $O$-sequence.

Hence, we have

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leqslant \phi(d(x_{2n}, x_{2n+1})).$$

Similarly, we have

$$d(x_{2n+2}, x_{2n+3}) = d(fx_{2n+2}, gx_{2n+1}) \leqslant \phi(d(x_{2n+1}, x_{2n+2})).$$

Therefore,

$$d(x_n, x_{n+1}) \leqslant \phi(d(x_{n-1}, x_n)) \leqslant \phi^2(d(x_{n-2}, x_{n-1})) \leqslant \cdots \leqslant \phi^n(d(x_0, x_1)),$$

for all $n \in \mathbb{N}$. Let $n$, $m \in \mathbb{N}^*$, we have

$$d(x_n, x_{n+m}) \leqslant \sum_{k=n}^{k=n+m-1} d(x_k, x_{k+1}) \leqslant \sum_{k=n}^{k=n+m-1} \phi^k(d(x_k, x_{k+1})). \tag{2}$$

Letting $n, m \to \infty$ in (2), we deduce that $\{x_n\}$ is an Cauchy $O$-sequence. Since $X$ is an $O$-complete space there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. On the other side, the orthogonal continuity of $f$, $g$ implies $\lim\limits_{n\to\infty} fx_n = fu$ and $\lim\limits_{n\to\infty} gx_n = gu$, which leads to $u = fu = gu$. □

Example 2. Let $X = \mathbb{Q}$ and $d(x, y) = |x - y|$ for all $x$, $y \in X$ is the usual metric on $X$.

Define a binary relation on $X$ by

$$x \perp y \Longleftrightarrow x = 0 \text{ or } y = 0.$$

Note that $(X, \perp, d)$ is not a complete metric space, but is an $O$-complete metric space.

Consider the mappings $f$, $g : X \to X$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ x/3, & \text{if } x \neq 1, \end{cases}$$

and

$$g(x) = \begin{cases} 1, & \text{if } x = 1, \\ x/2, & \text{if } x \neq 1. \end{cases}$$

Without loss of generality, let $x_n \perp x_{n+1}$ for each $n \in \mathbb{N}$. Then we have $x_n = 0$, which leads to $fx_n = x_n/3 = 0 = f0$ and $gu_n = x_n/2 = 0 = g0$. Therefore $f$, $g$ are $\perp$-continuous mappings.

Clearly, the mappings $f, g$ satisfy the condition 1) of Theorem 2.

On the other hand, let $\phi$ be a function defined by $\phi(t) = 3t/4$, for all $t > 0$. Let $x, y \in X$ such that $x \perp y$, we obtain $x = 0$ or $y = 0$.

Case 1: If $x = 0$, we have

$$d(f0, gy) = \frac{|y|}{2} \leqslant \frac{3}{4} d(0, y) \leqslant \phi(d(0, y)).$$

Case 2: If $y = 0$, we have

$$d(fx, g0) = \frac{|x|}{3} \leqslant \frac{3}{4} d(x, 0) \leqslant \phi(d(x, 0)).$$

Then, all assumptions of Theorem 2 are satisfied and $1 = f1 = g1$ is the common fixed point.

Now, we introduce a new definition named weakly-$\perp$-preserving self-mapping:

Definition 6. Let $(X, \perp)$ be an $O$-set. A mapping $T : X \to X$ is said to be weakly-$\perp$-preserving if $Tx \perp Ty$ or $Ty \perp Tx$ whenever $x \perp y$.

Remark 2. It is clear that a $\perp$-preserving mapping is a weakly-$\perp$-preserving mapping, but in general the converse is not true.

Example 3. Let $X = [0, 1]$, define the function $Tx = 1 - x$, $x \in X$. Define a binary relation $\perp \subset X \times X$ by

$$x \perp y \Longleftrightarrow x \leqslant y.$$

Therefore, $(X, \perp)$ is an $O$-set with the orthogonal element $x_0 = 0$.

We have $0 \perp 1$, $T0 = 1$ and $T1 = 0$, thus $T1 \perp T0$ but $T0 \perp T1$ does not hold.

Thus, the mapping $T$ is weakly-$\perp$-preserving, but not $\perp$-preserving.

By taking $g = f$ in our main theorem, we obtain a new generalization of Theorem 1.

THEOREM 3. *Let $(X, \perp, d)$ be an O-complete metric space and $T$ be a self-mapping on $X$ which is weakly-$\perp$-preserving and $\perp$-continuous. If there exists $\phi \in \Phi$ such that for all $x, y \in X$, we have*

$$x \perp y \text{ implies } d(Tx, Ty) \leqslant \phi(d(x, y)).$$

*Then, $T$ has a unique fixed point.*

**3. Application.** In this section, we will prove the existence of a common solution for the two differential equations:

$$\begin{cases} x'(t) = k(t, x(t)), & t \in I = [0, \theta],\ \theta \in (1, +\infty); \\ x(1) = a, & a \geqslant 2, \end{cases} \tag{3}$$

and

$$\begin{cases} x'(t) = k\left(t, a + \displaystyle\int_1^t k(u, x(u)du\right), & t \in I = [0, \theta],\ \theta \in (1, +\infty); \\ x(1) = a, & a \geqslant 2, \end{cases} \tag{4}$$

where $x \in \mathcal{C}(I)$, the space of all continuous functions from $I$ into $\mathbb{R}$ and $k : I \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

Let $X = \{x \in \mathcal{C}(I)/x(t) \geqslant 1\}$ endowed by the metric

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|.$$

Define the mappings $f$, $g : X \to X$, as follows:

$$fx(t) = a + \int_1^t k(s, x(s))ds, \tag{5}$$

and

$$gx(t) = a + \int_1^t k\left(s, a + \int_1^s k(u, x(u))du\right)ds, \tag{6}$$

for all $t \in I$.

Hence, equations (3) and (4) have a common solution if and only if the mappings $f$ and $g$ have a common fixed point.

THEOREM 4. Let $f$, $g : X \to X$ be the mappings defined by (5) and (6). *Assuming that the following conditions are satisfied:*

1) *$k(t, x) \geqslant 0$ for all $x \geqslant 0$ and $t \in I$;*

2) *there exists $h < 1$ such that for all $x, y \in X$,*

*we have*

$$|k(\,\cdot\,, fx) - k(\,\cdot\,, y)| \leqslant \frac{h}{\theta - 1}|x - y|, \tag{7}$$

*for any $x, y \in \mathcal{C}(I)$, with $xy \geqslant y$ or $xy \geqslant x$.*

*Then, the differential equations (3) and (4) have a positive common solution.*

*Proof.* Let $x, y \in X$. Define an orthogonal relation $\perp$ on $X$ by

$$x \perp y \iff x(t)y(t) \geqslant y(t) \text{ or } x(t)y(t) \geqslant x(t), \text{ for all } t \in I. \tag{8}$$

It is clear that $(X, \perp, d)$ is a $O$-complete metric space.

Let $x, y \in \mathcal{C}(I)$ be such that $x \perp y$, since $fx(t), gy(t) \geqslant 2$, then $fx(t)gy(t) \geqslant gy(t)$ and $gx(t)fy(t) \geqslant fy(t)$, which means that condition 1) of Theorem 2 holds. Also, from the definitions of $f$ and $g$, we see that $f, g$ are $\perp$-continuous.

On the other hand, we will show that the contraction 2) of Theorem 2 is satisfied.

By considering (7) and (8), we obtain

$$|gx(t) - fy(t)| \leqslant \int_1^t |k(s, fx(s)) - k(s, y(s))| ds \leqslant \leqslant \int_1^\theta \frac{h}{\theta - 1} |x(s) - y(s)| ds \leqslant hd(x, y).$$

So

$$d(gx, fy) \leqslant \phi(d(x, y)).$$

where $\phi(t) = ht$, with $h < 1$.

Finally, we conclude by Theorem 2 that the differential equations (3) and (4) have a positive common solution. □

Remark 3. In the above theorem, the function $x_0(t) = 2$ for all $t \in I$ is an orthogonal element.

**Competing interests.** On behalf of all authors, the corresponding author states that there is no conflict of interest.

**Authors' contributions and responsibilities.** Each author has participated in the development of the concept of the article and in the writing of the manuscript. The authors are absolutely responsible for submitting the final manuscript in print. Each author has approved the final version of the manuscript.

**Data availability.** No data were used to support this study.

## References


1. Banach S. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fund. Math.*, 1922, vol. 3, no. 1, pp. 133–181. http://eudml.org/doc/213289.
2. Gordji M. E., Rameani M., De La Sen M., Cho Y.J. On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, 2017, vol. 18, no. 2, pp. 569–578. DOI: https://doi.org/10.24193/fpt-ro.2017.2.45.
3. Baghani H., Eshaghi Gordji M., Ramezani M. Orthogonal sets: The axiom of choice and proof of a fixed point theorem, *J. Fixed Point Theory Appl.*, 2016, vol. 18, pp. 465–477. DOI: https://doi.org/10.1007/s11784-016-0297-9.
4. Khalehoghli S., Rahimi H., Eshaghi Gordji M. Fixed point theorems in $R$-metric spaces with applications, *AIMS Mathematics*, 2020, vol. 5, no. 4, pp. 3125–3137. DOI: https://doi.org/10.3934/math.2020201.
5. Touail Y., El Moutawakil D. Fixed point theorems on orthogonal complete metric spaces with an application, *Int. J. Nonl. Anal. Appl.*, 2021, vol. 12, no. 2, pp. 1801–1809. DOI: https://doi.org/10.22075/ijnaa.2021.23033.2464.
6. Jaid J., Touail Y., El Moutawakil D. On $\mu F$-contraction: New types of fixed points and common fixed points in Banach spaces, *Asian-European J. Math.*, 2023, vol. 16, no. 11, 2350198. DOI: https://doi.org/10.1142/S179355712350198X.

7. Touail Y., El Moutawakil D. ew common fixed point theorems for contractive self mappings and an application to nonlinear differential equations, *Int. J. Nonlinear Anal. Appl.*, 2021, vol. 12, no. 1, pp. 903–911. DOI: https://doi.org/10.22075/IJNAA.2021.21318.2245.
8. Touail Y., El Moutawakil D. Some new common fixed point theorems for contractive selfmappings with applications, *Asian-European J. Math.*, 2022, vol. 15, no. 4, 2250080. DOI: https://doi.org/10.1142/S1793557122500802.
9. Touail Y., Jaid A., El Moutawakil D. A note on common fixed point theorems in a bounded metric space, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2023, vol. 27, no. 2, pp. 241–249. EDN: ZXSBPZ. DOI: https://doi.org/10.14498/vsgtu1940.

Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. 2023. Т. 27, № 4. С. 737–744
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu1998
EDN: KPNGNF

УДК 517.988.523

# Новая общая теорема о неподвижной точке в ортогональных метрических пространствах и ее приложение

***Y. Touail*[1], *A. Jaid*[2], *D. El Moutawakil*[3]**

[1] Университет Сиди Мохамеда бен Абделлы, Фес, Марокко.
[2] Университет Султана Мулая Слимана, Бени-Меллаль, Марокко.
[3] Университет Шуайба Дуккали, Эль-Джадида, Марокко.

**Аннотация**

Доказывается общий результат о неподвижной точке для самоотображений на ортогональных полных метрических пространствах, которые не обязательно полны. В качестве приложения полученного результата найдено существование решений двух дифференциальных уравнений.

**Ключевые слова:** общая неподвижная точка, ортогональное метрическое пространство.

Получение: 28 января 2023 г. / Исправление: 17 ноября 2023 г. / Принятие: 13 декабря 2023 г. / Публикация онлайн: 25 декабря 2023 г.

**Конкурирующие интересы.** От имени всех авторов автор-корреспондент заявляет об отсутствии конфликта интересов.

**Авторский вклад и ответственность.** Все авторы принимали участие в разработке концепции статьи; все авторы сделали эквивалентный вклад в подготовку публикации. Авторы несут полную ответственность за предоставление окончательной рукописи в печать. Окончательная версия рукописи была одобрена всеми авторами.

**Доступность данных.** Никакие данные не использовались в этом исследовании.

**Дифференциальные уравнения и математическая физика**
**Краткое сообщение**



**Образец для цитирования**
Touail Y., Jaid A., El Moutawakil D. A new common fixed point theorem on orthogonal metric spaces and an application, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2023, vol. 27, no. 4, pp. 737–744. EDN: KPNGNF. DOI: 10.14498/vsgtu1998.

**Сведения об авторах**

*Youssef Touail* https://orcid.org/0000-0003-3593-8253
e-mail: youssef9touail@gmail.com

*Amine Jaid* https://orcid.org/0000-0001-7322-2008
e-mail: aminejaid1990@gmail.com

*Driss El Moutawakil* https://orcid.org/0000-0001-7322-2008
e-mail: d.elmoutawakil@gmail.com