Differential Equations and Mathematical Physics

Delta-problems for the generalized Euler–Darboux equation

I. N. Rodionova, V. M. Dolgopolov, M. V. Dolgopolov

Samara National Research University,
34, Moskovskoye shosse, Samara, 443086, Russian Federation.

Abstract

Degenerate hyperbolic equations are dealing with many important issues for applied nature. While a variety of degenerate equations and boundary conditions, successfully matched to these differential equation, most in the characteristic coordinates reduced to Euler–Darboux one. Some boundary value problems, in particular Cauchy problem, for the specified equation demanded the introduction of special classes in which formulae are simple and can be used to meet the new challenges, including Delta-problems in squares that contain singularity line for equation coefficients with data on adjacent or parallel sides of the square. In this short communication the generalized Euler–Darboux equation with negative parameters in the rectangular region is considered.

Short Communication

Authors' Details:

Irina N. Rodionova
Cand. Phys. & Math. Sci.; Associate Professor; Dept. of Mathematics & Business Informatics

Vyacheslav M. Dolgopolov http://orcid.org/0000-0002-4638-8800
Cand. Phys. & Math. Sci.; Associate Professor; Lab. of Mathematical Physics;
e-mail: paskal1940@mail.ru

Mikhail V. Dolgopolov http://orcid.org/0000-0002-8725-7831
Cand. Phys. & Math. Sci.; Associate Professor; Dept. of General and Theoretical Physics; Head of Laboratory; Lab. of Mathematical Physics; e-mail: volopoglodahsim@mail.ru

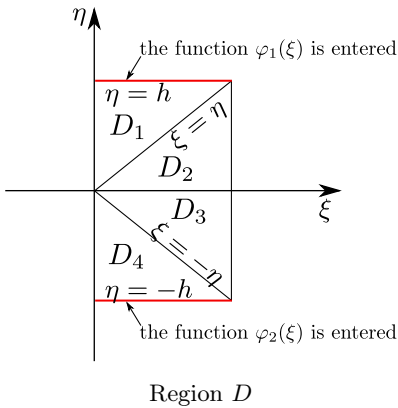**Introduction. Statement of the problem and boundary conditions.** Degenerate hyperbolic equations occur in many important problems of dynamical systems and in questions of applied nature: the theory of infinitesimal bending of surfaces of revolution, the membrane theory of shells, in the plasma magneto-hydrodynamics, gas dynamics. With all the variety of degenerated equations and boundary conditions it is successfully matched to the given differential equation, the latter equation in the characteristic coordinates reduces to the Euler–Darboux equations.

Some boundary value problems (Cauchy problem, in particular) for the specified equations require the introduction of special classes in which the formula for the solution becomes more simple in form and can be used to solve new tasks, including Delta(Δ)-problems in the squares containing the singularity line of the equation coefficients with the data on adjacent or parallel sides of the square (directed by A. M. Nakhushev).

The first works on Delta(Δ)-problems on sets representing the union of two characteristic triangles of hyperbolic equations were works of T. S. Kalmenov [1], V. F. Volkodavov, A. A. Andreev [2], A. M. Nakhushev [3]. The development of Delta-problems was done in the number of works of other authors, from which we should mention [4–7].

Unlike the previous in the present work the formulation of problems Δ_2 are on the set, which includes four of the characteristic triangles. The generalized Euler–Darboux equation with negative parameters is considered:

$$U_{\xi\eta} - \frac{p}{\eta - (\operatorname{sgn}\eta)\cdot\xi}U_\xi + \frac{p}{(\operatorname{sgn}\eta)\cdot\eta - \xi}U_\eta - (\operatorname{sgn}\eta)\cdot\lambda U = 0, \qquad (1)$$

$0 < p < 1/2$, $|\lambda| < \infty$ in the rectangular region D bounded by characteristics of equation (1) $\xi = 0$, $\xi = h$, $\eta = h$, $\eta = -h$ ($h > 0$), containing within itself two lines of singularity of coefficients $\eta = \xi$ and $\eta = -\xi$. For equation (1) in the region D the formulation of boundary value problems Δ_2 with the given values of the sought solution on the parallel sides of the rectangle, with the conditions of conjugation with respect to the solution and its normal derivative as lines of singularity of the coefficients, and the internal characteristic of the line are studied. The unique solvability of problems is proved by integral equations method. Problems are solved in the special class of functions introduced by authors in [8].

the function $\varphi_1(\xi)$ is entered

$\eta = h$

D_1

$\xi = \eta$

D_2

D_3

D_4

$\eta = -h$

$\xi = -\eta$

the function $\varphi_2(\xi)$ is entered

Region D

The solution of the problem. Considered boundary conditions for equation (1) in the rectangular region D at the parallel sides of the rectangle are of the following form:

$$U(\xi, h) = \varphi_1(\xi), \quad U(\xi, -h) = \varphi_2(\xi), \qquad 0 \leqslant \xi \leqslant h.$$

On lines of the singularity of $\eta = \xi$, $\eta = -\xi$ and on characteristic $\eta = 0$ the continuity of the sought solution is gluing on.

Relatively to normal derivatives two cases are considered of the pairing on the lines $\xi = \pm\eta$. In the first case Frankl's condition of occlusion is imposed (Problem Δ_2^*):

$$\nu_1(\xi) = \lim_{\eta\to\xi+0}(\eta - \xi)^{-2p}(U_\xi - U_\eta) = -\lim_{\eta\to\xi-0}(\xi - \eta)^{-2p}(U_\xi - U_\eta) = -\nu_2(\xi),$$

$$\nu_3(\xi) = \lim_{-\eta\to\xi-0}(\eta + \xi)^{-2p}(U_\xi + U_\eta) = -\lim_{-\eta\to\xi+0}(-\xi - \eta)^{-2p}(U_\xi + U_\eta) = -\nu_4(\xi),$$

$$0 < \xi < h.$$

In the second case the gluing is carried out on the continuity of normal derivatives $\nu_1 = \nu_2$; $\nu_3 = \nu_4$ (Problem Δ_2). In both tasks, Δ_2 and Δ_2^*, in characteristic $\eta = 0$ pairing sets:

$$\lim_{\eta\to0+0}\left(\frac{\partial U}{\partial\eta} - \frac{\partial U}{\partial\xi}\right) = \lim_{\eta\to0-0}\left(\frac{\partial U}{\partial\eta} + \frac{\partial U}{\partial\xi}\right).$$

The basis for the decision of tasks in view is taken, obtained by the authors [8–10], the solution of Cauchy problem of the special class R_h, which is in one of four characteristic triangles that make up the region D, has the form ($0 < \xi < \eta < h$):

$$U(\xi,\eta) = \int_\eta^h T_1(s)(s - \xi)^p(s - \eta)^p {}_0F_1\big(1 + p; \lambda(s - \xi)(s - \eta)\big)ds +$$

$$+ \int_\xi^\eta N_1(s)(\eta - s)^p(s - \xi)^p {}_0F_1\big(1 + p; -\lambda(\eta - s)(s - \xi)\big)ds,$$

where $N_1 = k_1 T_1 - k_2\nu_1$; k_1, $k_2 = \mathrm{const}$;

$$_0F_1\big(\alpha; z\big) = \sum_{n=0}^\infty \frac{z^n}{(\alpha)_n n!}.$$

Formulas for Cauchy problem solutions in three other characteristic triangles are not given. They also contain an unknown functions T_k, N_k, $k = 2, 3, 4$, which are searched in the class of continuous in the interval $(0, h)$ and absolutely integrable functions on $[0, h]$.

The solution of the Problem Δ_2^* is reduced to the set of integral equations of the form

$$\int_\xi^h T(s)(\xi - s)^p {}_0F_1\big(1 + p; \lambda s(s - \xi)\big)ds = \Phi(\xi, \lambda),$$

the unique solvability of which takes place when the following conditions are imposed on the given functions:

$$\varphi_i(\xi) \in C^{(2)}[0, h], \qquad \varphi_i(0) = \varphi_i'(0) = 0, \qquad i = 1, 2,$$

$$\varphi_i(\xi) = (h - \xi)^{1+p+\varepsilon}\varphi_i^*(\xi), \qquad \varepsilon > 0, \qquad \int_0^h \varphi_i(s)(h - s)^{-p-2}ds = 0.$$

When you run these conditions the only solution of the problem Δ_2^* is given explicitly.

The complete study of the problem of Δ_2 has managed to get only if $\lambda = 0$. In this case, its solution is reduced to a set of integral equations of the first kind with Cauchy kernel:

$$\int_0^h \frac{\mu_i(s)ds}{s - \xi} = \Phi_i^*(\xi), \quad i = 1,\, 2; \qquad \mu_1 = (h - s)^p s^p [\nu_1 - \nu_3],$$

$$\mu_2 = (h - s)^p s^{p-1} [\nu_1 + \nu_3]. \tag{2}$$

$\Phi_i^*(\xi)$ depends only on the given functions φ_i.

Discussions of solutions. Following the theory of singular integral equations [11], conditions imposed on the given functions φ under which there is the solution of equations (2) (not unique), and solvability conditions of equations (2), and, consequently, of the Problem Δ_2.

1. If $\varphi_i(\xi) \in C[0,1]$, $\varphi_i'' \in C(0,1)$, φ_i'' are absolutely integrable on $[0, h]$, $i = 1, 2$, then solutions of equations (2) have the form [11]:

$$\mu_1 = -\frac{1}{\pi} \sqrt{\frac{\xi}{h - \xi}} \int_0^h \sqrt{\frac{h - y}{y}} \cdot \frac{\Phi_1^*(y)}{y - \xi} dy,$$

$$\mu_2 = -\frac{1}{\sqrt{\xi(h - \xi)}} \left(\frac{1}{\pi} \int_0^h \sqrt{(h - y)y} \cdot \frac{\Phi_2^*(y)}{y - \xi} dy + A_0 \right), \quad A_0 = \text{const.}$$

2. If the existing conditions to add

$$\varphi_1'(h) + \varphi_2'(h) = 0, \quad \text{but} \quad \varphi_i'(h) \neq 0, \quad i = 1,\, 2,$$

then the second of equations (2) is uniquely solvable:

$$\mu_2 = -\frac{1}{\pi} \sqrt{\frac{h - \xi}{\xi}} \int_0^h \sqrt{\frac{y}{h - y}} \cdot \frac{\Phi_2^*(y)}{y - \xi} dy.$$

In conclusion, we note that for the problem definition authors were inspired by results, published in the work of I. V. Volovich, O. V. Groshev, N. A. Gusev, E. A. Kuryanovich [12].

Competing interests. We have no competing interests.

Authors' contributions and responsibilities. Each author has participated in the article concept development and in the manuscript writing. The authors are absolutely responsible for submitting the final manuscript in print. Each author has approved the final version of manuscript.

References

1. Kal'menov T. Sh. The characteristic Cauchy problem for a certain class of degenerate hyperbolic equations, *Differ. Uravn.*, 1973, vol. 9, no. 1, pp. 84–96 (In Russian).

2. Volkodavov V. F., Andreev A. A. Two boundary value problems for a certain hyperbolic equation, Volzh. mat. sb.. Kuibyshev, 1973, pp. 102–112 (In Russian).

3. Nakhushev A. M. On the theory of boundary value problems for degenerate hyperbolic equations, *Soobshch. Akad. Nauk Gruz. SSR*, 1975, vol. 77, pp. 545–548 (In Russian).

4. Kumykova S. K. A boundary value problem for a degenerate hyperbolic equation in a characteristic crescent, *Differ. Uravn.*, 1979, vol. 15, no. 1, pp. 79–91 (In Russian).

5. Volkodavov V. F., Kulikova N. A. The problem Δ_2 for an equation of hyperbolic type with conjugation of the limits of fractional-order derivatives, *Differ. Equ.*, 2003, vol. 39, no. 12, pp. 1797–1801. doi: 10.1023/B:DIEQ.0000023560.02344.15.

6. Zainullina G. N. The Δ_2 problem for the Euler–Poisson–Darboux equation in the class of unbounded functions, *Russ. Math.*, 2003, vol. 47, no. 3, pp. 13–17.

7. Volkodavov V. F., Andreev A. A. *Kraevye zadachi dlia uravneniia Eilera–Darbu–Puassona* [Boundary value problems for the Euler–Darboux–Poisson equations]. Kuibyshev, Kuibyshev. gos. ped. in-t, 1984, 76 pp. (In Russian)

8. Dolgopolov V. M., Dolgopolov M. V., Rodionova I. N. Construction of special classes of solutions for some differential equations of hyperbolic type, *Dokl. Math.*, 2009, vol. 80, no. 3, pp. 860–866. doi: 10.1134/S1064562409060209.

9. Dolgopolov M. V., Rodionova I. N. Problems involving equations of hyperbolic type in the plane or three-dimensional space with conjugation conditions on a characteristic, *Izv. Math.*, 2011, vol. 75, no. 4, pp. 681–689. doi: 10.1070/IM2011v075n04ABEH002549.

10. Dolgopolov M. V., Rodionova I. N., Dolgopolov V. M. On one nonlocal problem for the Euler–Darboux equation, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2016, vol. 20, no. 2, pp. 259–275 (In Russian). doi: 10.14498/vsgtu1487.

11. Gakhov F. D. *Boundary value problems.* New York, Dover Publ., 1990, xxii+561 pp.

12. Volovich I. V., Groshev O. V., Gusev N. A., Kuryanovich E. A. On Solutions to the Wave Equation on a Non-globally Hyperbolic Manifold, *Proc. Steklov Inst. Math.*, 2009, vol. 265, no. 1, pp. 262–275. doi: 10.1134/S0081543809020242.

УДК 517.956.3

Дельта-задачи для обобщенного уравнения Эйлера—Дарбу

И. Н. Родионова, В. М. Долгополов, М. В. Долгополов

Самарский национальный исследовательский университет
имени академика С.П. Королева,
Россия, 443086, Самара, Московское ш., 34.

Аннотация

Рассмотрено обобщенное уравнение Эйлера—Дарбу с отрицательными параметрами в прямоугольной области, содержащей две линии сингулярности коэффициентов уравнения. Поставлена краевая задача с заданными значениями искомого решения на параллельных сторонах прямоугольника с условиями сопряжения относительно решения и его нормальных производных как на линиях сингулярности коэффициентов, так и на внутренней характеристической линии. Методом интегральных уравнений исследована разрешимость поставленных задач. Задачи решаются в специальном классе, введенном авторами.

Ключевые слова: обобщенное уравнение Эйлера–Дарбу, краевая задача.

Получение: 14 июля 2017 г. / Исправление: 8 сентября 2017 г. /
Принятие: 18 сентября 2017 г. / Публикация онлайн: 22 сентября 2017 г.

Краткое сообщение

Сведения об авторах
Ирина Николаевна Родионова
кандидат физико-математических наук, доцент; доцент; каф. математики и бизнес-информатики

Вячеслав Михайлович Долгополов http://orcid.org/0000-0002-4638-8800
кандидат физико-математических наук, доцент; лаб. математической физики;
e-mail: paskal1940@mail.ru

Михаил Вячеславович Долгополов http://orcid.org/0000-0002-8725-7831
кандидат физико-математических наук, доцент; доцент; каф. общей и теоретической физики; заведующий лабораторией; лаб. математической физики;
e-mail: volopoglodahsim@mail.ru