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Stochastic models of simple controlled systems *just-in-time*

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Abstract

We propose a new and simple approach for the mathematical description of a stochastic system that implements the well-known *just-in-time* principle. This principle (abbreviated *JIT*) is also known as a *just-in-time* manufacturing or Toyota Production System.

The models of simple *JIT* systems are studied in this article in terms of point processes in the reverse time. This approach allows some assumptions about the processes inherent in real systems. Thus, we formulate and solve some, very simple, optimal control problems for a multi-stage *just-in-time* system and for a system with the bounded intensity. Results are obtained for the objective functions calculated as expected linear or quadratic forms of the deviations of the trajectories from the planned values. The proofs of the statements utilize the martingale technique. Often, *just-in-time* systems are considered in logistics tasks, and only (or predominantly) deterministic methods are used to describe them. However, it is obvious that stochastic events in such systems and corresponding processes are observed quite often. And it is in such stochastic cases that it is very important to find methods for the optimal management of processes *just-in-time*. For this description, we propose using martingale methods in this paper. Here, simple approaches for optimal control of stochastic *JIT* processes are demonstrated. As examples, we consider an extremely simple model of rescheduling and a method of

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controlling the intensity of the production process, when the probability of implementing a plan is not necessarily equal to one (with the corresponding quadratic loss functional).

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Introduction. In this paper, we consider some stochastic models of simple *just-in-time* systems. The well-known principle of *just-in-time* system (abbreviated as *JIT* system) is used in many areas. Examples include *just-in-time* production systems (see [1, 2] and references therein), pedagogical strategies of *just-in-time* teaching (often abbreviated to *JiTT*; see, e.g., [3,4]), and *just-in-time* compilation methods in computer programming (see [5, 6]). It should be noted that at present mathematical, especially stochastic, models for *JIT* systems are not sufficiently developed. Such models are necessary for solving optimal control problems, which could allow optimizing the allocation of system resources and implementing optimal planning of a stochastic *JIT* system. The purpose of this article is to present an approach to the stochastic description of *JIT* systems, which would be suitable for both analytical methods and computer simulation. Mathematical models of such systems should allow assuming that the trajectories of processes must take the given values at a fixed time. Such behavior of processes is known in stochastic bridges and stochastic processes in the reverse time. Thus, we should consider models of systems with the requirement of *JIT* in terms of processes with the behavior of trajectories close to stochastic bridges. Models should also allow investigating possible violations of this requirement that are unavoidable for real systems.

The time reversal of stochastic processes has been studied for many years. For example, see [7–10] and references therein. We note that a number of works related to stochastic bridges (for example, the Brownian bridge, the Poisson bridge, also known as the Poisson bridge), is devoted to the investigation of these processes. In addition, some works on reversible Markov processes adjoin process descriptions in reverse time (see, e.g., [11]). In this article, we study models of simple *JIT* systems in semimartingale terms for point processes close to the Poisson bridge mentioned above. Here we allow some assumptions about the processes inherent in real systems. Thus, simple cases of multi-stage *JIT* systems and a system with bounded intensity are investigated. As shown, for these cases simple optimal control problems can be formulated and solved. The proofs of the results utilize the semimartingale technique.

1. Time reversal method for a simple *JIT* system. Consider a *JIT* system that can be described in terms of point (counting) processes. We assume that in the system some integer number K of operations must be performed to a fixed time $T > 0$ (starting from the zero moment). This means that at each time $t \in [0, T]$ the number of remaining operations X_t, is equal to the number K minus the value N_t, of some counting process $N = (N_t)_{t\geqslant 0} : X_t = K - N_t$,. That is, the jumps of this process N are such that their total number at time T is exactly

K. A formal description of such a process X in the simple case corresponds to a Poisson process in reverse time, provided that its value at time T is exactly K (or to the Poisson bridge N, see, e.g., [8, 10, 12]).

We now give a formal description of the mathematical model. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space populated with a nondecreasing right-continuous family of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t\geqslant 0}$, complete with respect to $\mathbf{P}$ (i.e., the conditions of [13] hold). On the stochastic basis $\mathbf{B} = (\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geqslant 0}, \mathbf{P})$ the process $X = (X_t)_{t\geqslant 0}$ is supposed to be the point process with trajectories in the Skorokhod space, $X_t \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\Delta X_t = X_t - X_{t-} \in \{-1, 0\}$ (see, e.g., [14–16]). The process X can be represented as a difference:

$$X = X_0 - N = K - N,$$

where $N = (N_t)_{t\geqslant 0}$ is the counting process of the number of negative jumps of X, with the initial value $X_0 = K > 0$ (i.e., $K \in \mathbb{N} = \{1, 2, \dots\}, N_0 = 0$, and $X_t = K - N_t$, for all $t \geqslant 0$). We suppose that the submartingale N and supermartingale X on $\mathbf{B}$ admit the well-known Doob–Meyer decompositions (see, e.g., [13]):

$$N_t = \tilde{N}_t + m_t^N, \quad X_t = \tilde{X}_t - m_t^N \tag{1}$$

with the compensators $\tilde{N} = (\tilde{N}_t)_{t\geqslant 0}$ and $\tilde{X} = (\tilde{X}_t)_{t\geqslant 0}$, and the square-integrable martingale $m^N = (m_t^N)_{t\geqslant 0}$ with the quadratic characteristic

$$\langle m^N \rangle_t = \tilde{N}_t \text{ for all } t \geqslant 0.$$

We also suppose in this article that

$$\tilde{N}_t = \int_0^t (K - N_s) \cdot \frac{1}{T - s} \cdot \mathbb{I}\{s < t\}\, ds, \tag{2}$$

where $\mathbb{I}\{\cdot\}$ is an indicator function (i.e., $\mathbb{I}\{\text{true}\} = 1$, $\mathbb{I}\{\text{false}\} = 0$). From (1) and (2) it follows that the process X has the decomposition:

$$X_t = K - \int_0^t X_s \cdot \frac{1}{T - s} \cdot \mathbb{I}\{s < t\}\, ds - m_t^N. \tag{3}$$

In the general case, for the basic model we assume that the point process X admits the representation:

$$X_t = K - \int_0^t h_s\, ds + m_t^X. \tag{4}$$

with the intensity of negative jumps $h = h(X) = (h_t(X))_{t\geqslant 0}$ and the martingale $m^X = (m_t^X)_{t\geqslant 0}$. In the particular case (3), the following equality holds:

$$h_t = h_t(X) = X_t \cdot \mathbb{I}\{s < t\}/(T - s), \tag{5}$$

and $m^X = -m^N$, i.e. $m_t^X = -m_t^N$ for all $t \geqslant 0$.

It is well known that the compensator of the point process defined by formula (2) corresponds to the bridge of a Poisson process, [12]. We also should recall the results of [7, 8]. Consider a standard Poisson process $\pi = (\pi_s)_{s\in[0,T]}$ on the stochastic basis $\mathbf{B}$ with the initial value $\pi_0 = 0$ and any positive intensity $\lambda > 0$. Let $\mathscr{F}_t^0 = \sigma\{\pi_s : T - t \leqslant s \leqslant T\}$ for $t \in [0, T]$, $\mathscr{F}_t^0 = \mathscr{F}_T^0$ for $t > T$, and nondecreasing family σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t\geqslant 0}$ be the right continuous completion of $(\mathscr{F}_t^0)_{t\geqslant 0}$. Define the reverse time supermartingale $Y = (Y_t)_{t\geqslant 0}$ as $Y = \pi_{T-t}$ for $t \in [0, T]$ and $Y_t = \pi_0 = 0$ for $t > T$. Then Y is $\mathbb{F}$-adapted and it has the decomposition (as it easily follows, from Theorem 2.6 in [8]):

$$Y_t = \pi_T - \int_0^t \frac{Y_s}{T-s} \cdot \mathbb{I}\{s < t\}\, ds + m_t^Y, \tag{6}$$

where $m^Y = (m_t^Y)_{t\geqslant 0}$ is a square-integrable martingale with the quadratic characteristic

$$\langle m^Y \rangle_t = \int_0^t \frac{Y_s}{T-s} \cdot \mathbb{I}\{s < t\}\, ds.$$

The comparison of (3) and (6) illustrates the fact known for bridge processes: the representation of the process $X = K - N$ (with the initial value K and the Poisson bridge N) coincides with the reverse time representation Y of the Poisson process π (with any strictly positive intensity λ) under the condition for the initial value $Y_0 = \pi_T = K$. Thus, we can consider the behavior of the trajectories of the process X with $X_0 = K$ and $X_t = 0$ for $t \geqslant T$ as the embodiment of the *just-in-time* requirement. Therefore, the main idea of the presented description of *JIT* systems is the realization of the corresponding behavior of trajectories by means of proper control of $h = (h_t)_{t\geqslant 0}$, which is the intensity of the negative jumps of X in the base model (4). This intensity can be regarded as a negative feedback tending to $-\infty$ as $t \to T$ in the case of nonzero X_t. Note that in (6) it does not directly depend on the intensity λ of the initial process π.

The distribution of the main process X in (4) is determined by the intensity of the negative jumps h, which in the particular case of (5) depends on the values of K and $T > 0$. Along with X, we define for the base model (4) the auxiliary functions for $\mathbf{E}X_t$, $\mathbf{E}X_t^2$ and $\mathbf{E}(X_t - R_t)^2 = G_t - R_t^2$ (i.e., for the mean, the second moment, and the variance of X, respectively). For the functional $h = h(X)$ of general form in (4), and the initial value K, it is assumed that

$$R_t = R_t(K; h) = \mathbf{E}X_t, G_t = G_t(K; h) = \mathbf{E}X_t^2, V_t = V_t(K; h) = \mathbf{E}(X_t - R_t)^2. \tag{7}$$

In the particular case (5), these functions depend only on the values of t, K, and T. Therefore, for (5) we use the notations:

$$r_t(K; T) = R_t = \mathbf{E}(X_t | X_0 = K; X_t = 0), \tag{8}$$

$$g_t(K; T) = G_t = \mathbf{E}(X_t^2 | X_0 = K; X_t = 0), \tag{9}$$

$$v_t(K; T) = V_t = \mathbf{E}((X_t - R_t)^2 | X_0 = K; X_t = 0) = g_t(K; T) - r_t(K; T)^2. \tag{10}$$

Lemma 1. *For the functions* (8), (9) *and* (10) *defined for X in* (4) *with the intensity* (5), *we have*

$$r_t(K; T) = K \cdot \frac{T-t}{T} \cdot \mathbb{I}\{t \leqslant T\}, \tag{11}$$

$$g_t(K;T) = \Big(K \cdot \frac{T-t}{T}\Big)^2 \cdot \mathbb{I}\{t \leqslant T\} + K \cdot \frac{(T-t)\cdot t}{T^2} \cdot \mathbb{I}\{t \leqslant T\}, \tag{12}$$

$$v_t(K;T) = K \cdot \frac{(T-t)\cdot t}{T^2} \cdot \mathbb{I}\{t \leqslant T\}. \tag{13}$$

2. Problems of optimal planning for a multi-stage *JIT* process. Consider a model of simple multi-stage *JIT* systems in terms of the proposed description. We assume that it is a set of separate processes in reverse time (or bridges of corresponding processes) with a single aggregate plan. This section presents a simple solution to the problem of the optimal times for changing the stages for the model. In the cases considered here, the mean-square deviations of the trajectories from the planned values are minimized. In addition, we consider the problem of optimal rescheduling for the case of two stages and for its multi-stage generalization.

2.1. Separate processes in reverse time. Let us consider optimal control problem for the following scheduling model. Let the execution of $(K+1)$ operations in time T be subdivided into $n \in \mathbb{N}$ stages: every successive $K(i)$ operations must be performed in stage i, which lasts the time $\varsigma(i)$, for all $i = 1, 2, \ldots, n$. The following conditions for the time and number of operations must be fulfilled:

$$\sum_{i=1}^{n} \varsigma(i) = T, \tag{14}$$

$$\sum_{i=1}^{n} K(i) = K. \tag{15}$$

We also define the condition for the uniformity of the operations:

$$K(i) = K(i) \cdot \varsigma(i)/T \text{ for all } i = 1, 2, \ldots, n. \tag{16}$$

Thus, the model of this *JIT* system is a set of separate processes in reverse time (or of proper bridges). Suppose that we must insure the uniform fulfillment of the plan $\overline{\varsigma} = \{\varsigma(1), \varsigma(2), \ldots, \varsigma(n)\}$ in the sense of (16), minimizing the weighted variance of the deviation from it.

We consider the problem of finding an optimal plan

$$\overline{\varsigma^*} = \{\varsigma^*(1), \varsigma^*(2), \ldots, \varsigma^*(n)\}$$

for which

$$\Phi(\overline{\varsigma^*}) = \inf_{\overline{\varsigma}} \Phi(\overline{\varsigma}), \tag{17}$$

where the objective function $\Phi(\overline{\varsigma})$ is the sum of weighted variances (10) for the processes in (4) with initial values $K(i)$ and times of performance $\varsigma(i)$, $i = 1, 2, \ldots, n$:

$$\Phi(\overline{\varsigma}) = \sum_{i=1}^{n} \alpha(i) \cdot \int_0^{\varsigma(i)} v_t(K(i), \varsigma(i))\, ds \tag{18}$$

under conditions (14) and (15), and for strictly positive weights:

$$\alpha(i) > 0 \text{ for all } i = 1, 2, \ldots, n. \tag{19}$$

THEOREM 1. *For the plan that minimizes the objective function* $\Phi(\overline{\varsigma})$,

$$\varsigma^*(i) = T \cdot \left\{ \alpha(i) \cdot n \cdot \sum_{j=1}^{n} 1/\alpha(j) \right\}^{-1/2} \text{ for all } i = 1, 2, \ldots, n. \tag{20}$$

REMARK 1. *Theorem 1 implies the trivial consequence that for equal weights the equal times are optimal: for* $\alpha(1) = \alpha(2) = \ldots = \alpha(n) > 0$,

$$\varsigma^*(i) = T/n \text{ for all } i = 1, 2, \ldots, n. \tag{21}$$

2.2. The problem of optimal rescheduling for a two-stage JIT process. As it follows from (21), for $n = 2$, in the case of equal weights, it then holds that

$$\varsigma^*(1) = \varsigma^*(2) = T/2. \tag{22}$$

However, in real systems, along with a priori stage planning, a procedure for reviewing the plan during its implementation is encountered – rescheduling. In this case, the operations of the *JIT* system are performed in accordance with the intensity of the process in (3) for planned initial value K and planned time T for $t \in [0, \sigma]$, $\sigma \in [0, T]$, where σ is rescheduling time. Thus in the first stage, for $t \in [0, \sigma]$, the initial plan with the values of K and T is carried out. At time σ, the following re-planning procedure is implemented. The second stage is fulfilled on the time interval $[\sigma, T]$. Here, after rescheduling, the initial value of the number of operations X_σ, and the new execution time $(T - \sigma)$ are set in the interval $[\sigma, T]$ for the new process in the reverse time. For this model of rescheduling for the *JIT* system, the task is to find a time point σ that minimizes the integral standard deviation from the original plan in the first stage and the deviation from the new plan in the second stage. Thus, we consider the problem of finding an optimal value σ^* for which

$$\Psi(\sigma^*) = \inf_{\sigma} \Psi(\sigma), \tag{23}$$

where the objective function $\Psi(\sigma)$ is the integrated variance (7) for the intensity $h = h(X)$ is equal to

$$\Psi(\sigma) = \int_0^T V_t(K; h)\, ds. \tag{24}$$

Here the intensity for the rescheduling is equal to

$$h_t(X) = h_t^{(1)}(X) \cdot \mathbb{I}\{t \in [0, \sigma)\} + h_t^{(2)}(X) \cdot \mathbb{I}\{t \in [\sigma, T)\}, \tag{25}$$

where

$$h_t^{(1)}(X) = X_t/(T - t), \quad h_t^{(2)}(X) = X_t/(T - \sigma - t). \tag{26}$$

LEMMA 2. *For the time* σ *that minimizes the objective function* $\Psi(\sigma)$,

$$\sigma^* = T/3. \tag{27}$$

REMARK 2. *Note that* $\sigma^* \neq \varsigma^*(1)$ *where* $\varsigma^*(1)$ *is defined in problem* (17), *and for which* (22) *holds for* $n = 2$.

2.3. The problem of recurrent rescheduling for a multi-stage JIT process. The procedure for rescheduling presented in Subsection 2.2 is determined a priori – at the time $t = 0$. It is an intensification – an optimal increase in intensity at any stopping time u. If there are no restrictions on intensification in the system, then such a procedure after a stopping time $u(1) = \sigma$ can be repeated at a stopping time $u(2) > u(1)$, etc. The procedure for rescheduling (or re-planning) can be pre-established for all $i \in \mathbb{N}$. However, if for some number $j, X_{u(j)} = 0$, then $h_t(X) = 0$ for all $t \geqslant u(j)$. And therefore, the implementation of rescheduling is meaningless for all numbers $i \geqslant j$, that is, after the process X reaches zero. Let $\tau = \tau(\omega)$, $\omega \in \Omega$, be the he Markov stopping time on the stochastic basis $\mathbf{B}$ at which X reaches zero:

$$\tau = \inf\{t > 0\colon X_t = 0\} \quad (\text{where } \inf\{\varnothing\} = +\infty).$$

Then the number of all possible rescheduling procedures for the process X is equal to

$$J\{X\} = \sum_{i=1}^{\infty} \mathbb{I}\{u(i) < \tau\}. \tag{28}$$

Thus, any set for the sequential stopping times of rescheduling procedures is $\bar{u} = \{u(1), \ldots, u(J(X))\}$, $0 < u(1) < u(2) < \cdots < u(i) < \cdots < u(J(X)) < T$.

Consider the problem of finding such an optimal set $\bar{u^*} = \{u^*(1), \ldots, u^*(J(X))\}$ of recurrent stopping times of re-planning at $u^*(i), i \leqslant J(X)$, for which

$$\Gamma(\bar{u}^*) = \inf_{\bar{u}} \Gamma(\bar{u}). \tag{29}$$

In (29), the objective function $\Gamma(\bar{\sigma})$ is the integrated variance (7) for the intensity $\hat{h} = \hat{h}(X)$:

$$\Gamma(\bar{u}) = \int_0^T V_t(K; \hat{h})\, ds,$$

where the intensity $\hat{h} = \hat{h}(X) = (\hat{h}(X))_{t\geqslant 0}$ is defined by the number $J(X)$ from (28), and by the set $\bar{u}$. In addition, we define auxiliary stopping times $u(0) = 0$, and $u(J(X)+1) = T$. Then, in order to generalize the definition (25)–(26) of the function $h_t(X)$, to the case of successive (multi-stage) re-planning, we obviously use the following expression:

$$\hat{h}_t(X) = X_t \cdot \mathbb{I}\{t < T\} \cdot \sum_{i=0}^{J(X)} \Big(T - u(i) - t\Big)^{-1} \cdot \mathbb{I}\{u(i) < t \leqslant u(i+1)\} \tag{30}$$

For this model of the *JIT* system without restrictions on the intensification, Lemma 2 implies the following result.

Theorem 2. *For the set $\bar{u}^*$ of rescheduling times, which minimizes the objective function $\Gamma(\bar{u})$,*

$$u^*(i) = T \cdot \Big(1 - (2/3)^i\Big) \text{ for all } i = 1, 2, \ldots, J(X). \tag{31}$$

Remark 3. *Note that this optimal plan is finite and has the stochastic time $u^*(J(X))$ of the last rescheduling.*

3. The problem of the optimal level of resources of a simple system with possible violations of the condition. In this section, we consider some assumptions about violations of the *JIT* condition in processes inherent in real systems. Thus, we assume that the intensities of point processes can be bounded. We note that such a representation of the process X in (4) does not correspond to the time reversal procedure for a point process with fixed initial value. Nevertheless, such a representation in terms of point processes is useful for describing a controlled system with a violation of the condition of *JIT*. For such a model, the task of optimal control arises – to find the value of the maximum level of intensity of the point process for each operation under conditions of payment for the value of this boundary, and payment for non-compliance with the *JIT* requirement.

We suppose that the intensity h in (4) can be represented as

$$h_t = h_t(X) = X_t \cdot \min\{\Lambda, \mathbb{I}\{t < T\}/(T-t)\}, \tag{32}$$

where $\Lambda \in [0, \infty)$ is a finite maximum level of intensity for each operation. Under this assumption for h, the *JIT*-condition $X_T = 0$ may not hold, and obviously $\mathbf{P}\{\omega : X_T(\omega) \geqslant 1\} > 0$ and $\mathbf{E}X_T > 0$. We assume that the payment for this violation of the *JIT* condition is proportional to the mean value of the number of uncompleted operation $\mathbf{E}X_T$. The coefficient of proportionality is denoted by α. The greater the upper level Λ, the smaller the value of $\mathbf{E}X_T$ and the closer to the fulfillment of the *JIT* requirement. Since the resources of the real system provide the level Λ, it also has a certain positive cost with a proportionality factor of β. Moreover, Λ can serve as a control parameter in the system (4).

Thus, we consider the problem of optimal control of the process X in (4) for fixed $K \in \mathbb{N}$ and $T > 0$, and under the assumption (32) for h. It is necessary to find an optimal value Λ^* for which the problem is analogous to the problems (17), (23) and (29):

$$\Theta(\Lambda^*) = \inf_{\Lambda \geqslant 0} \Theta(\Lambda), \tag{33}$$

where the objective function $\Theta(\Lambda)$ is equal to

$$\Theta(\Lambda) = \alpha \cdot \mathbf{E}X_T + \beta \cdot \Lambda \tag{34}$$

under the conditions:

$$\alpha > 0, \beta > 0. \tag{35}$$

THEOREM 3. *For the maximum intensity level, which minimizes the objective function* $\Theta(\Lambda)$*,*

$$\Lambda^* = \sqrt{\frac{\alpha \cdot K}{\beta \cdot e \cdot T}} \quad \text{if} \quad \alpha \cdot K \cdot T/\beta \in [e, +\infty), \tag{36}$$

$$\Lambda^* = [\log(\alpha \cdot K \cdot T/\beta)]/T \quad \text{if} \quad \alpha \cdot K \cdot T/\beta \in (1, e), \tag{37}$$

and

$$\Lambda^* = 0 \quad \text{if} \quad \alpha \cdot K \cdot T/\beta \in (0, 1]. \tag{38}$$

REMARK 4. *As can be seen from the proof of the theorem, the coefficients* α *and* β *can be regarded as functions of the time* T*. In this case the statement of the* *theorem remains true with the replacement of* α *by* $\alpha(T)$, *and* β *by* $\beta(T)$. *Thus, for the case of the objective function*

$$\bar{\Theta}(\Lambda) = \alpha \cdot \boldsymbol{E} X_T + \gamma \cdot \Lambda \cdot T \tag{39}$$

under conditions (35) *for* α *and* $\beta = \beta(T) = \gamma \cdot T$, *the problem* (40) (*which is the* (33) *analog*)

$$\bar{\Theta}(\Lambda^*) = \inf_{\Lambda \geqslant 0} \bar{\Theta}(\Lambda) \tag{40}$$

has similar to (36), (37) *and* (38) *solutions. Thus, for the level of intensity that minimizes the objective function* $\bar{\Theta}(\Lambda)$,

$$\bar{\Lambda}^* = \sqrt{\frac{\alpha \cdot K}{\gamma \cdot e}}/T \quad \text{if} \quad \alpha \cdot K/\gamma \in [e, +\infty),$$

$$\bar{\Lambda}^* = [\log(\alpha \cdot K/\gamma)]/T \quad \text{if} \quad \alpha \cdot K/\gamma \in (1, e),$$

and

$$\bar{\Lambda}^* = 0 \quad \text{if} \quad \alpha \cdot K/\gamma \in (0, 1].$$

Note that (39) (*which is the* (34) *analog*) *is interesting for the models of systems with payments for maximum level of resources, proportional to the time of their reservation.*

4. Proof of the results.

4.1. Proof of Lemma 1. From (3) and (8), it follows that for all $K \in \mathbb{N}$, $T > 0$, and $t \geqslant 0$

$$r_t(K;T) = K - \int_0^t r_s(K;T) \cdot \frac{1}{T-s} \cdot \mathbb{I}\{s < t\}\, ds. \tag{41}$$

From (41) we obtain (11). Then, from (3), (9) and the Ito formula, it follows that for all $t \geqslant 0$

$$g_t(K;T) = K^2 - 2 \cdot \int_0^t g_s(K;T) \cdot \frac{1}{T-s} \cdot \mathbb{I}\{s < t\}\, ds + \frac{K}{T} \cdot t.$$

which results in (12). From (11) and (12) and the equality

$$v_t(K,T) = g_t(K,T) - r_t(K,T)^2$$

we obtain (13). □

4.2. Proof of Theorem 1. To prove (20), we consider the sequence of functions $v_t(K(i); \varsigma(i))$, $i = 1, \ldots, n$. From (13) and the equality (16), we obtain

$$\int_0^{\varsigma(i)} v_t(K; \varsigma(i))\, dt = K(i) \cdot \varsigma(i)/6 = \frac{1}{6} \cdot \frac{K}{T} \cdot \varsigma(i)^2,$$

which gives (18) with constants (19):

$$\Phi(\bar{\varsigma}) = \frac{1}{6} \cdot \frac{K}{T} \cdot \sum_{i=1}^{n} \alpha(i) \cdot \varsigma(i)^2. \tag{42}$$

It is clear that $\sum_{i=1}^{n} \alpha(i) \cdot \varsigma(i)^2 = \sum_{i=1}^{n} \left(\sqrt{\alpha(i)} \cdot \varsigma(i)\right)^2$. Hence, from the Cauchy–Bunyakovsky–Schwarz inequality and the equality (14), it follows that

$$\sum_{i=1}^{n} \left(\sqrt{\alpha(i)} \cdot \varsigma(i)\right)^2 \cdot \sum_{i=1}^{n} \left(\frac{1}{\sqrt{\alpha(i)}}\right)^2 \geqslant \left(\sum_{i=1}^{n} \varsigma(i)\right)^2 = T^2,$$

which results in the inequality

$$\Phi(\bar{\varsigma}) \geqslant \frac{K \cdot T}{6 \cdot \sum_{i=1}^{n} 1/\alpha(i)}. \tag{43}$$

Then, from (42) and the representatuin (20), it follows that

$$\Phi(\bar{\varsigma}^*) = \frac{K \cdot T}{6 \cdot \sum_{j=1}^{n} 1/\alpha(j)}. \tag{44}$$

From (43) and (44), we conclude that $\bar{\varsigma}^* = \{\varsigma^*(1), \dots, \varsigma^*(n)\}$ minimizes the objective function $\Phi(\bar{\varsigma})$. □

4.3. Proof of Lemma 2. From (24), (25) and (26), it follows that

$$\Psi(\sigma) = \int_0^{\sigma} v_t(K;T)\,dt + \int_0^{T-\sigma} v_t\big(r_\sigma(K;T), T-\sigma\big)\,dt.$$

Therefore, from (11) and (13) of Lemma 1, we obtain

$$\Psi(\sigma) = \frac{K}{6 \cdot T^2}\big(T^3 - 2 \cdot T^2 \cdot \sigma + 4 \cdot T \cdot \sigma^2 - 2 \cdot \sigma^3\big). \tag{45}$$

Note that $\Psi(\sigma) < \Psi(T) = K \cdot T/6$ for $\sigma \in (0, T)$. The value of σ^* in (23) and (45) is easy to calculate from the requirement $\partial\Psi(\sigma)/\partial\sigma = 0$, which results in the equality (27). For the objective function, we have $\Psi(\sigma^*) = (19/27) \cdot (K \cdot T/6)$. Thus, Lemma 2 is proved. □

4.4. Proof of Theorem 2. It follows from (30) and Lemma 2 that for each successive rescheduling with the number $i \geqslant 1$ the process X evolves in accordance with the model of Subsection 2.2, but with the initial value of $X_{u(i)}$ and in the time interval $[u(i), T]$ with the length $(T - u(i))$. Hence, from (27) we obtain that $u(i+1) - u(i) = \big(T - u(i)\big)/3$, which implies that (31) is true. □

4.5. Proof of Theorem 3. From (4), (7) and (32), it follows that

$$R_t(K;h) = K - \int_0^t R_s(K;h) \cdot \min\{\Lambda, \mathbb{I}\{s < T\}/(T-s)\}\,ds.$$

Denote $U = T - 1/\Lambda$, for which $\Lambda = 1/(T - U)$. Then, from (7) and (11), it follows that for $\Lambda \geqslant 1/T$, $\mathbf{E}X_t = R_t(K;h) = R_{T-U}\big(\mathbf{E}X_U; X \cdot \Lambda\big)$. Hence, $\mathbf{E}X_t = R_{T-U}\big(r_U(K,T); X \cdot \Lambda\big) = R_{1/\Lambda}\big(K/(T \cdot \Lambda); X \cdot \Lambda\big) = \exp\{-1\} \cdot K/(T \cdot \Lambda)$. For $\Lambda \in [0, 1/T)$, we have $\mathbf{E}X_t = K \cdot \exp\{-\Lambda \cdot T\}$. Therefore, for $\Theta(\Lambda)$ from (34), we obtain

$$\Theta(\Lambda) = \alpha \cdot K \cdot \left(\frac{\mathbb{I}\{\Lambda \geqslant 1/T\}}{e \cdot T \cdot \Lambda} + \frac{\mathbb{I}\{\Lambda \in [0, 1/T]\}}{\exp\{\Lambda \cdot T\}}\right) + \beta \cdot \Lambda \tag{46}$$

The value of Λ^* in (33) can be easily calculated from (46) and from the requirement $\partial\Theta(\Lambda)/\partial\Lambda = 0$, taking into account the cases $\alpha \cdot K \cdot T/\beta \in [e, +\infty)$, $\alpha \cdot K \cdot T/\beta \in (1, e)$ and $\alpha \cdot K \cdot T/\beta \in (0, 1]$. □

5. Discussion. The main purpose of this article is to show the possibilities of using of the time reversal approach in problems concerning *just-in-time*. We demonstrate simple methods for optimizing *JIT* systems, for the case of a point (counting) process, represented in semimartingale terms. We also note that the statements of Theorem 1, Lemma 2, and Theorem 2 are valid in the case of a random walk in reverse time (Lemma 1 and Theorem 2 remain true if the coefficients are properly replaced). In this case, the semimartingale representation methods and optimal control problems are close to that of [17]. In the case of nonstationary processes in direct time, the results are also anticipated. Finally, note that the method of representing *JIT* systems discussed in the article in terms of predictable semimartingale characteristics creates opportunities for simple and clear computer modeling. Obviously, the simulation is easy to implement on the basis of the infinitesimal relation for X : $\mathbf{P}\{\Delta X_t = X_{t+\Delta} - X_t = -1|\mathscr{F}_t\} = h_t(X) \cdot \Delta + o(\Delta)$ as $\Delta \to 0$, for all $t \geqslant 0$.

Thus, it follows that the discussed approach can serve as an initial step for the analysis of stochastic *JIT* systems.

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Стохастические модели простых управляемых систем *точно-в-срок*

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Аннотация

Мы предлагаем новый и простой подход для математического описания стохастической системы, которая реализует известный принцип *точно-в-срок*. Этот принцип (сокращенно *JIT*) также известен как *точно-в-срок* мануфактура или *Производственная система Toyota*.

Модели простых *JIT*-систем изучаются в этой статье в терминах точечных процессов в обратном времени. Такой подход позволяет допустить некоторые предположения о процессах, наблюдаемых в реальных системах. Так, в настоящей работе мы формулируем и решаем некоторые очень простые задачи оптимального управления для многостадийной системы *точно-в-срок* и задачи для системы с ограниченной интенсивностью обслуживания. Результаты получены для целевых функций, представляющих собой математические ожидания линейных или квадратичных форм отклонений значений траекторий от запланированных величин. Доказательства утверждений основаны на использовании мартингальных методов. Часто системы *точно-в-срок* рассматриваются в логистических задачах, и для их описания при этом используются только (или преимущественно) детерминистические методы. Однако очевидно, что случайные события в таких системах и соответствующих процессах наблюдаются довольно часто. И именно в таких стохастических случаях очень важно найти методы для оптимального управления процессами *точно-в-срок*. Для такого описания мы предлагаем в этой статье использовать мартингальные методы. Здесь показаны простые подходы к оптимальному управлению стохастическими *JIT*-процессами. В качестве примеров мы рассматриваем чрезвычайно простую модель перепланирования и метод управления интенсивностью производственного процесса, когда вероятность реализации плана необязательно равна единице (с соответствующим квадратичным функционалом потерь).

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