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Asymmetric tensor representations in micropolar continuum mechanics theories

Y. N. Radayev

A. Ishlinsky Institite for Problems in Mechanics, Russian Academy of Sciences,
101, pr. Vernadskogo, Moscow, 119526, Russian Federation.

Abstract

In this paper, new representations of three-dimensional asymmetric stress tensor and the corresponding form of the differential equilibrium equations are given. Asymmetric theories of solid mechanics continues to attract attention in connection with the necessity of mathematical modelling of the mechanical behaviour of the advanced materials. The study is restricted to such asymmetric second rank tensors, for which it is still possible to keep the notion of real eigenvalues, but not to accept the mutual orthogonality of the directors of the principal trihedron. The exact algebraic formulation of these asymmetry conditions is discussed. The study extends the dyadic tensor representations of the symmetric stress tensor based on the notion of asymptotic directions. The obtained results are a clear evidence in favor of algebraic hyperbolicity both the symmetric and asymmetric second rank tensors in three-dimensional space.

Keywords: micropolar continuum, force stress, couple stress, asymmetric tensor, eigenvalue, eigenvector, asymptotic direction.

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Author's Details:
Yuri N. Radayev https://orcid.org/0000-0002-0866-2151
D.Sc. (Phys. & Math. Sci.), Ph.D., M.Sc., Professor; Leading Researcher; Lab. of Modeling in Solid Mechanics; e-mail: radayev@ipmnet.ru, y.radayev@gmail.com

Preliminary remarks. Asymmetric tensors are rarely used in mechanics and physics in compare to symmetric second rank tensors. The latter are usually required in order to describe strains and stresses in all classical theories of continuum mechanics. However many new models of mechanical behaviour are based on asymmetric stress tensors. It is the case of the micropolar theory of elasticity, where asymmetric stress tensor lies in the foundation of mathematical description of three-dimensional stress states. Asymmetric theories of continuum mechanics at the present time still attract attention of researchers due to necessity of mathematical modelling of the mechanical behaviour of advanced materials (e.g., auxetics by hemitropic micropolar theory of elasticity).

Micropolar theories of continuum mechanics are characterized by the following equilibrium equations, written in terms of the force and couple stresses [1, 2] (see also [3]):

$$\begin{aligned} \nabla_i t^{ik} &= -X^k, \\ \nabla_i \mu^{i\cdot}_{\cdot k} - 2\tau_k &= -Y_k, \end{aligned} \tag{1}$$

wherein X^k are volume forces; Y_k are volume couples; t^{ik} is the asymmetric force stress tensor which is the sum of the symmetric $t^{(ik)}$ and antisymmetric $t^{[ik]}$ parts

$$t^{ik} = t^{(ik)} + t^{[ik]};$$

$\mu^{i\cdot}_{\cdot k}$ is the asymmetric couple stress tensor; τ_j, μ^j are the vectors associate to the force stress tensor and the couple stress tensor, defined according to the formulae

$$\begin{aligned} -\tau_j &= \frac{1}{2} e_{jik} t^{[ik]}, \\ t^{[ik]} &= -e^{ikj}\tau_j, \\ +\mu^i &= \frac{1}{2} e^{iks}\mu_{[ks]}, \\ \mu_{[is]} &= +e_{isj}\mu^j. \end{aligned}$$

In this paper, the asymmetric second rank tensor $\mathbf{t}$ will be of primary interest and we will talk about it, although all of the study can equally be applied to the couple stress tensor $\mu^{i\cdot}_{\cdot k}$.

The paper is arranged as follows. After the Preliminary remarks in Sec. 1 we discuss a special class of second rank asymmetric tensors similar to diagonal tensors. For such asymmetric tensors it is possible to keep the notions of real eigenvalues and eigenvectors. There the algebraic condition providing the tensor similarities is formulated. It involves the characteristic equation discriminant of the second rank asymmetric tensors $\mathbf{t}$. Then in Sec. 2 a dyadic representation of an asymmetric second rank tensor with a multiple real eigenvalue is obtained and discussed. The representation formula involves two spatial directors. The notion of asymptotic direction for the asymmetric second rank tensor $\mathbf{t}$ is introduced. Spatial polarizations of eigenvectors of $\mathbf{t}$ are considered. The analogous considerations but adopted for an asymmetric second rank tensor with all different eigenvalues are given in Sec. 3. Dyadic representation of $\mathbf{t}$ requires four spatial directors. New forms of the equilibrium equations for the asymmetric force stress tensor $\mathbf{t}$ corresponding to the dyadic tensor representations obtained in the previous sections of the paper are obtained in Sec. 4.

1. Asymmetric tensor similar to a diagonal tensor. We restrict ourselves to such second rank asymmetric tensors $\mathbf{t}$, for which it is possible to preserve the notions of the principal axes and real eigenvalues, but not to presume the mutual orthogonality of the directors of the principal trihedron. Therefore, we assume that the tensor $\mathbf{t}$ is similar to a diagonal tensor. The latter means that the tensor $\mathbf{t}$ can be represented (up to the tensor similarity) in the following form

$$\mathbf{S}\mathbf{t}\mathbf{S}^{-1} = \sum_{a=1,2,3} t_a \underset{a}{\mathbf{l}} \otimes \overset{a}{\mathbf{l}},$$

where $\mathbf{S}$ is a nondegenerate second rank tensor; $\underset{a}{\mathbf{l}}$ $(a = 1, 2, 3)$, $\overset{b}{\mathbf{l}}$ $(b = 1, 2, 3)$ are reciprocal vector triples; t_a $(a = 1, 2, 3)$ are real eigenvalues of the tensor $\mathbf{S}\mathbf{t}\mathbf{S}^{-1}$ and also of $\mathbf{t}$.

Recall that the reciprocal triples of linearly independent vectors $\underset{a}{\mathbf{l}}$ $(a = 1, 2, 3)$, $\overset{b}{\mathbf{l}}$ $(b = 1, 2, 3)$ satisfy the fundamental relation

$$\underset{a}{\mathbf{l}} \cdot \overset{b}{\mathbf{l}} = \overset{b}{\underset{a}{\delta}}.$$

As an example (required for the further study) we give a diagonal representation of the unit tensor $\mathbf{I}$:

$$\mathbf{I} = \sum_{a=1,2,3} \underset{a}{\mathbf{l}} \otimes \overset{a}{\mathbf{l}}. \tag{2}$$

Let us find out the conditions when all eigenvalues of the asymmetric tensor $\mathbf{t}$ are to be real. At this aim consider the characteristic equation of tensor $\mathbf{t}$:

$$-\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0, \tag{3}$$

where

$$\begin{aligned}
&J_1 = \operatorname{tr}\mathbf{t},\\
&2J_2 = (\operatorname{tr}\mathbf{t})^2 - \operatorname{tr}(\mathbf{t}^2),\\
&6J_3 = (\operatorname{tr}\mathbf{t})^3 - 3\operatorname{tr}\mathbf{t}\operatorname{tr}(\mathbf{t}^2) + 2\operatorname{tr}(\mathbf{t}^3) = 6\det\mathbf{t}.
\end{aligned}$$

In the case of a general cubic equation

$$e_0\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0 \tag{4}$$

as a result of variable replacement according to

$$\lambda = \lambda' - \frac{e_1}{3e_0}$$

the reduced cubic equation is obtained

$$\begin{aligned}
&\lambda'^3 + e_2'\lambda' + e_3' = 0,\\
&e_2' = \frac{e_2}{e_0} - \frac{e_1^2}{3e_0^2},\\
&e_3' = \frac{2e_1^3}{27e_0^3} - \frac{e_1e_2}{3e_0^2} + \frac{e_3}{e_0}
\end{aligned}$$

with the discriminant

$$d = -27e_3'^2 - 4e_2'^3$$

or

$$d = E_1^2 E_2^2 - 4E_1^3 E_3 - 27E_3^2 - 4E_2^3 + 18E_1 E_2 E_3,$$

wherein

$$E_j = \frac{e_j}{e_0} \qquad (j = 1, 2, 3).$$

The roots of the cubic equation (4) are real if and only if (see [4] for details)

$$d \geqslant 0.$$

As for the tensor $\mathbf{t}$ we have

$$E_1 = -J_1, \quad E_2 = J_2, \quad E_3 = -J_3,$$

then the discriminant of its characteristic equation (3) is obtained in the form

$$d = J_1^2 J_2^2 - 4J_1^3 J_3 - 27J_3^2 - 4J_2^3 + 18J_1 J_2 J_3. \tag{5}$$

2. Dyadic representation of an asymmetric second rank tensor with a multiple eigenvalue. The tensor dyadic representations obtained earlier in [5–7], and valid for symmetric second rank tensors (e.g. for the stress tensor) can be adopted for asymmetric tensors similar to diagonal tensors thus allowing to keep notions of the real eigenvalues and eigenvectors. The most important generalization is related to the notion of asymptotic direction known from [6].

Let the characteristic equation of the tensor $\mathbf{t}$ have a multiple eigenvalue:

$$t_1 = t_2.$$

The third eigenvalue t_3 of $\mathbf{t}$ is to be different from the first one.

It is then possible to give the algebraically exact formulation for the case: the tensor $\mathbf{t}$ characteristic equation discriminant (5) should be equalled to zero, whereas the second coefficient of the reduced equation should be nonzero, that reads

$$d = 0, \qquad 3J_2 - J_1^2 \neq 0.$$

By the aid of the unit tensor representation (2) after a number of transformations we obtain

$$\mathbf{S}\mathbf{t}\mathbf{S}^{-1} = t_1 \mathbf{I} + (t_3 - t_1)\underset{3}{\mathbf{l}} \otimes \overset{3}{\mathbf{l}},$$

and then

$$\mathbf{t} = t_1 \mathbf{I} + (t_3 - t_1)(\mathbf{S}^{-1} \cdot \underset{3}{\mathbf{l}}) \otimes (\mathbf{S}^{\mathrm{T}} \cdot \overset{3}{\mathbf{l}}).$$

By introducing spatial directors

$$\underset{*}{\mathbf{d}} = \mathbf{S}^{-1} \cdot \underset{3}{\mathbf{l}}, \qquad \overset{*}{\mathbf{d}} = \mathbf{S}^{\mathrm{T}} \cdot \overset{3}{\mathbf{l}},$$

the following representation for the asymmetric tensor $\mathbf{t}$ in terms of the two directors is derived

$$\mathbf{t} = t_1\mathbf{I} + (t_3 - t_1)\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}, \tag{6}$$

wherein

$$\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}} = (\underset{3}{\mathbf{l}} \cdot \mathbf{S}^{-\mathrm{T}}) \cdot (\mathbf{S}^{\mathrm{T}} \cdot \overset{3}{\mathbf{l}}) = \underset{3}{\mathbf{l}} \cdot \overset{3}{\mathbf{l}} = 1, \tag{7}$$

that means the directors $\underset{*}{\mathbf{d}}$ and $\overset{*}{\mathbf{d}}$ in the dyadic representation (6) are to satisfy to single scalar relation (7).

The dyadic representation (6) leads us to the following conclusions:

1) $\underset{*}{\mathbf{d}}$ is the eigenvector of $\mathbf{t}$ corresponding to the eigenvalue t_3;
2) any vector which orthogonal to $\overset{*}{\mathbf{d}}$ is the eigenvector of $\mathbf{t}$ corresponding to the multiple eigenvalue $t_1 = t_2$;
3) directions determined by $\underset{*}{\mathbf{d}}$ and $\overset{*}{\mathbf{d}}$ can be treated as asymptotic for the asymmetric tensor $\mathbf{t}$;
4) the cross product $\underset{*}{\mathbf{d}} \times \overset{*}{\mathbf{d}}$ is the eigenvector of $\mathbf{t}$ corresponding to the multiple eigenvalue $t_1 = t_2$;
5) the double cross product $\overset{*}{\mathbf{d}} \times (\underset{*}{\mathbf{d}} \times \overset{*}{\mathbf{d}})$ is the eigenvector of $\mathbf{t}$ corresponding to the multiple eigenvalue $t_1 = t_2$;
6) the triple $\underset{*}{\mathbf{d}}$, $\underset{*}{\mathbf{d}} \times \overset{*}{\mathbf{d}}$, $\overset{*}{\mathbf{d}} \times (\underset{*}{\mathbf{d}} \times \overset{*}{\mathbf{d}})$ forms an eigenbasis of $\mathbf{t}$ in space.

It is worth to note the equations

$$\mathrm{tr}(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}) = \underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}} = 1.$$

The tensor dyadic representation (6) remains valid if the dyad $\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}$ by the scalar product $\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}$, which in view of (7) equals 1. The resulting ratio keeps its value after renorming the asymptotic directors $\underset{*}{\mathbf{d}}$ and $\overset{*}{\mathbf{d}}$ thus setting their lengths to 1. Consequently for the asymmetric tensor $\mathbf{t}$ the following representation is held:

$$\mathbf{t} = t_1\mathbf{I} + (t_3 - t_1)\frac{\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}},$$

wherein

$$\underset{*}{\mathbf{d}} \cdot \underset{*}{\mathbf{d}} = 1, \quad \overset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}} = 1.$$

The symmetric and antisymmetric parts of the asymmetric tensor $\mathbf{t}$ can be easily determined from the dyadic representation (6) resulting in the formulae given below:

$$\mathrm{sym}\,\mathbf{t} = t_1\mathbf{I} + \frac{1}{2}(t_3 - t_1)(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} + \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}),$$

$$\operatorname{asym}\mathbf{t} = \frac{1}{2}(t_3 - t_1)(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} - \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}).$$

When the directors have the unit lengths for the symmetric and antisymmetric parts of the asymmetric tensor $\mathbf{t}$ are expressed as follows

$$\operatorname{sym}\mathbf{t} = t_1\mathbf{I} + \frac{1}{2}\frac{t_3 - t_1}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}}(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} + \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}),$$

$$\operatorname{asym}\mathbf{t} = \frac{1}{2}\frac{t_3 - t_1}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}}(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} - \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}).$$

In the mathematical theory of plasticity operating with the symmetric stress tensor the value given by

$$\frac{|t_3 - t_1|}{2}$$

is the maximum (over all spatial orientations) shear stress at a given point in space.

3. Dyadic representation of an asymmetric second rank tensor with all different eigenvalues. Let all of the tensor $\mathbf{t}$ eigenvalues be different from each other. In this case we can order them (for instance in the decreasing order)

$$t_1 > t_2 > t_3.$$

This situation is described by involving the tensor $\mathbf{t}$ characteristic equation discriminant (5), namely the discriminant should be positive:

$$d > 0.$$

By the aid of the unit tensor representation formula (2) we obtain

$$\mathbf{StS}^{-1} = t_2\mathbf{I} + (t_1 - t_2)\underset{1}{\mathbf{l}} \otimes \overset{1}{\mathbf{l}} + (t_3 - t_2)\underset{3}{\mathbf{l}} \otimes \overset{3}{\mathbf{l}},$$

and therefore

$$\mathbf{t} = t_2\mathbf{I} + (t_1 - t_2)(\mathbf{S}^{-1} \cdot \underset{1}{\mathbf{l}}) \otimes (\mathbf{S}^{\mathrm{T}} \cdot \overset{1}{\mathbf{l}}) + (t_3 - t_2)(\mathbf{S}^{-1} \cdot \underset{3}{\mathbf{l}}) \otimes (\mathbf{S}^{\mathrm{T}} \cdot \overset{3}{\mathbf{l}}).$$

By defining the spatial directors in accordance with

$$\underset{*}{\mathbf{h}} = \mathbf{S}^{-1} \cdot \underset{1}{\mathbf{l}}, \quad \overset{*}{\mathbf{h}} = \mathbf{S}^{\mathrm{T}} \cdot \overset{1}{\mathbf{l}}, \quad \underset{*}{\mathbf{d}} = \mathbf{S}^{-1} \cdot \underset{3}{\mathbf{l}}, \quad \overset{*}{\mathbf{d}} = \mathbf{S}^{\mathrm{T}} \cdot \overset{3}{\mathbf{l}},$$

the following dyadic representation for the asymmetric tensor $\mathbf{t}$ is derived:

$$\mathbf{t} = t_2\mathbf{I} + (t_1 - t_2)\underset{*}{\mathbf{h}} \otimes \overset{*}{\mathbf{h}} + (t_3 - t_2)\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}, \tag{8}$$

wherein the directors are involved in the relations

$$\begin{aligned} &\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}} = 1, \qquad \underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{h}} = 1, \\ &\underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{d}} = 0, \qquad \underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{h}} = 0. \end{aligned} \tag{9}$$

It is easily seen from (8) that:

1) the director $\underset{*}{\mathbf{d}}$ is eigenvector of $\mathbf{t}$, corresponding to the minimum eigenvalue t_3;
2) the director $\underset{*}{\mathbf{h}}$ is eigenvector of $\mathbf{t}$, corresponding to the maximum eigenvalue t_1;
3) any vector orthogonal both the vectors $\overset{*}{\mathbf{d}}$ and $\overset{*}{\mathbf{h}}$ is eigenvector of $\mathbf{t}$, corresponding to the intermediate eigenvalue t_2, thus the third eigenvector of $\mathbf{t}$ can be chosen as the cross product $\overset{*}{\mathbf{d}} \times \overset{*}{\mathbf{h}}$;
4) the triple of vectors $\underset{*}{\mathbf{d}}$, $\overset{*}{\mathbf{d}} \times \overset{*}{\mathbf{h}}$, $\underset{*}{\mathbf{h}}$ constitutes eigenbasis of $\mathbf{t}$ in space.

By renorming the directors (i.e. redicing their lengths to 1) the formula (8) is replaced by

$$\mathbf{t} = t_2\mathbf{I} + (t_1 - t_2)\frac{\underset{*}{\mathbf{h}} \otimes \overset{*}{\mathbf{h}}}{\underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{h}}} + (t_3 - t_2)\frac{\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}}}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}},$$

which should be supplemented by the relations

$$\begin{aligned}
&\underset{*}{\mathbf{h}} \cdot \underset{*}{\mathbf{h}} = 1, \quad \overset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{h}} = 1, \quad \underset{*}{\mathbf{d}} \cdot \underset{*}{\mathbf{d}} = 1, \quad \overset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}} = 1;\\
&\underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{d}} = 0, \quad \underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{h}} = 0.
\end{aligned}$$

After simple considerations the formulae for the symmetric and antisymmetric parts of the asymmetric tensor $\mathbf{t}$ can be found:

$$\begin{aligned}
\operatorname{sym}\mathbf{t} &= t_2\mathbf{I} + \frac{1}{2}\frac{t_1 - t_2}{\underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{h}}}(\underset{*}{\mathbf{h}} \otimes \overset{*}{\mathbf{h}} + \overset{*}{\mathbf{h}} \otimes \underset{*}{\mathbf{h}}) + \frac{1}{2}\frac{t_3 - t_2}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}}(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} + \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}),\\
\operatorname{asym}\mathbf{t} &= \frac{1}{2}\frac{t_1 - t_2}{\underset{*}{\mathbf{h}} \cdot \overset{*}{\mathbf{h}}}(\underset{*}{\mathbf{h}} \otimes \overset{*}{\mathbf{h}} - \overset{*}{\mathbf{h}} \otimes \underset{*}{\mathbf{h}}) + \frac{1}{2}\frac{t_3 - t_2}{\underset{*}{\mathbf{d}} \cdot \overset{*}{\mathbf{d}}}(\underset{*}{\mathbf{d}} \otimes \overset{*}{\mathbf{d}} - \overset{*}{\mathbf{d}} \otimes \underset{*}{\mathbf{d}}).
\end{aligned}$$

The above formulae give the symmetric and antisymmetric parts in terms of the spatial directors and are to be considered simultaneously with the relations (9).

4. New forms of the equilibrium equations for the asymmetric force stress tensor. The tensor dyadic representations of asymmetric $\mathbf{t}$ obtained in the previous Sections of the paper allows us to find out new forms of the equilibrium equations (1) intrinsic to micropolar theories of continuum mechanics.

We start from co-ordinate formulation of the equations (1). In the rectangular co-ordinate net in space these equations read

$$\begin{aligned}
&\partial_j t_{ji} = -X_i,\\
&\partial_j \mu_{ji} + \epsilon_{ijk} t_{jk} = -Y_i.
\end{aligned}$$

The dyadic representations of asymmetric tensor $\mathbf{t}$ given by (6), (8) in the rectangular co-ordinate frame can be rewritten as

$$t_{ji} = t_1\delta_{ji} + (t_3 - t_1)\underset{*}{d_j}\overset{*}{d_i},$$

$$t_{ji} = t_2\delta_{ji} + (t_1 - t_2)\underset{*}{h}_j\overset{*}{h}_i + (t_3 - t_2)\underset{*}{d}_j\overset{*}{d}_i.$$

Then after rather simple calculations with partial differentiations we come to

$$\partial_j t_{ji} = \partial_i t_1 + \overset{*}{d}_i\underset{*}{d}_j\partial_j(t_3 - t_1) + (t_3 - t_1)[\overset{*}{d}_i\partial_j\underset{*}{d}_j + \underset{*}{d}_j\partial_j\overset{*}{d}_i], \tag{10}$$

$$\begin{aligned}\partial_j t_{ji} = \partial_i t_2 + \overset{*}{h}_i\underset{*}{h}_j\partial_j(t_1 - t_2) + (t_1 - t_2)[\overset{*}{h}_i\partial_j\underset{*}{h}_j + \underset{*}{h}_j\partial_j\overset{*}{h}_i]- \\ -\overset{*}{d}_i\underset{*}{d}_j\partial_j(t_2 - t_3) - (t_2 - t_3)[\overset{*}{d}_i\partial_j\underset{*}{d}_j + \underset{*}{d}_j\partial_j\overset{*}{d}_i].\end{aligned} \tag{11}$$

In view of (10) and (11) the equilibrium equations (1) involving the asymmetric force stress tensor $\mathbf{t}$ read as follows:

in the case of a multiple eigenvalue $t_1 = t_2$:

$$\boldsymbol{\nabla} t_1 + \overset{*}{\mathbf{d}}(\underset{*}{\mathbf{d}}\cdot\boldsymbol{\nabla})(t_3 - t_1) + (t_3 - t_1)[\overset{*}{\mathbf{d}}(\boldsymbol{\nabla}\cdot\underset{*}{\mathbf{d}}) + (\underset{*}{\mathbf{d}}\cdot\boldsymbol{\nabla})\overset{*}{\mathbf{d}}] = -\mathbf{X}, \tag{12}$$

in the case of all different eigenvalues ($t_1 > t_2 > t_3$):

$$\begin{aligned}\boldsymbol{\nabla} t_2 - \overset{*}{\mathbf{d}}(\underset{*}{\mathbf{d}}\cdot\boldsymbol{\nabla})(t_2 - t_3) - (t_2 - t_3)[\overset{*}{\mathbf{d}}(\boldsymbol{\nabla}\cdot\underset{*}{\mathbf{d}}) + (\underset{*}{\mathbf{d}}\cdot\boldsymbol{\nabla})\overset{*}{\mathbf{d}}]+ \\ +\overset{*}{\mathbf{h}}(\underset{*}{\mathbf{h}}\cdot\boldsymbol{\nabla})(t_1 - t_2) + (t_1 - t_2)[\overset{*}{\mathbf{h}}(\boldsymbol{\nabla}\cdot\underset{*}{\mathbf{h}}) + (\underset{*}{\mathbf{h}}\cdot\boldsymbol{\nabla})\overset{*}{\mathbf{h}}] = -\mathbf{X}.\end{aligned} \tag{13}$$

The latter equations are to be considered simultaneously with (7) (for equation (12)) and (9) (for equation (13)).

5. Conclusions.

1. Dyadic tensor representations usually used in the theory of perfect plasticity [5–7] and valid for symmetric second rank tensors can be extended to the case of asymmetric tensors similar to diagonal tensors.
2. The generalized dyadic representation of an asymmetric second rank tensor with a multiple eigenvalue is characterized by the greatest formal simplicity, since it includes only two spatial directors related by a single scalar equation.
3. The generalized dyadic representation of an asymmetric second rank tensor with different eigenvalues includes the four spatial directors related by the four scalar equations.
4. The obtained special forms of equilibrium equations are expressed in the terms that are most natural from an algebraic point of view.
5. The obtained results are an additional evidence in favor of algebraic "hyperbolicity" of the second rank symmetric and asymmetric tensors in three-dimensional space discussed earlier in [8].

Competing interests. I declare that I have no competing interests.

Author's Responsibilities. I take full responsibility for submitting the final manuscript in print. I approved the final version of the manuscript.

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Тензорные представления асимметричных тензоров микрополярных теорий механики континуума

Ю. Н. Радаев

Институт проблем механики им. А. Ю. Ишлинского РАН,
Россия, 119526, Москва, просп. Вернадского, 101, корп. 1.

Аннотация

Получены новые представления трехмерного асимметричного тензора напряжений и соответствующие им формы дифференциальных уравнений равновесия. Асимметричные теории механики деформируемого твердого тела по-прежнему привлекают пристальное внимание в связи с необходимостью математического моделирования механического поведения современных материалов (например, ауксетиков с помощью теорий гемитропной микрополярной упругости). Исследование ограничивается только такими асимметричными тензорами второго ранга, для которых удается сохранить понятие о вещественных собственных значениях, но отказаться от взаимной ортогональности направлений главного триэдра. Обсуждается точная алгебраическая формулировка указанных условий асимметричности. В статье обобщаются тензорные представления симметричного тензора напряжений, основанные на естественном репере асимптотических направлений. Полученные результаты являются ярким свидетельством в пользу алгебраической «гиперболичности» симметричных и асимметричных тензоров второго ранга в трехмерном пространстве.

Ключевые слова: микрополярный континуум, силовые напряжения, моментные напряжения, асимметричный тензор, собственное значение, собственный вектор, асимптотическое направление.

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Сведения об авторе

Юрий Николаевич Радаев https://orcid.org/0000-0002-0866-2151
доктор физико-математических наук, профессор; ведущий научный сотрудник; лаб. моделирования в механике деформируемого твердого тела; e-mail: radayev@ipmnet.ru, y.radayev@gmail.com