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Continuum approach to high-cycle fatigue. The finite life-time case with stochastic stress history

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Abstract

In this paper, we consider continuum approach for high-cycle fatigue in the case where life-time is finite. The method is based on differential equations and all basic concepts are explained. A stress history is assumed to be a stochastic process and this leads us to the theory of stochastic differential equations. The life-time is a quantity, which tells us when the breakdown of the material happens. In this method, it is naturally a random variable. The basic assumption is, that the distribution of the life-time is log-normal or Weibull. We give a numerical basic example to demonstrate the method.

Keywords: high-cycle fatigue, life-time, evolution equation.

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1. Introduction. Mechanical fatigue phenomena occurs when a material is subjected to repeated application of stresses or strains which produces changes in the material microstructure, initiation, growth and coalescence of microdefects, thus degrading the material properties, see books by V. Bolotin [1], S. Suresh [2], and Y. Murakami [3]. It is customary to distinguish between high-cycle (HCF) and low-cycle fatigue (LCF). In low-cycle fatigue plastic deformations occur in a macroscopic scale while when the loading is in the high-cycle fatigue regime the macroscopic behaviour can be considered primarily as elastic. If the loading consist of well defined cycles, the transition between LCF and HCF regimes is typically considered to occur between 10^3–10^4 cycles.

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In this study, only high-cycle fatigue is considered. Classical methods for HCF-analysis can be broadly classified as stress invariant, critical plane, strain energy and average stress based approaches. Well known examples are the models by G. Sines [4], W. Findley [5], K. Dang Van [6], A. Carpinteri and A. Spagnoli [7] and I. Papadopoulos [8]. These approaches are well defined if the loading consists of well-defined cycles. For arbitrary loading histories they need the definition of an equivalent uniaxial loading cycle. Another deficient is that heuristic damage accumulation rules have to be applied. To remove these shortcomings N. Ottosen et al. [9] proposed a continuum based model where they postulated a moving endurance surface in the stress space where the movement and damage evolution are governed by properly formulated evolution equations. This evolution equation based continuum approach to HCF is also used by R. Brighenti et al. [10, 11]. Extension to transverse isotropy is given in [12] and gradient effects are included in [13].

There is inherently stochastic nature in fatigue phenomenon. The fatigue life has inherent scatter even under constant cyclic loading. Weibull weakest link-theory [14] has been used to describe the statistically distributed flaws and defects in the material that is reflected in the fatigue behavior [15–18]. In many cases the loading which is acting to the structure is random and we can only describe it by statistical distributions. For irregular loading histories, the classical method to predict a life time is the Rainflow method, which is based on a construction of an equivalent cycle. The method is essentially one dimensional, but can be extended to the multiaxial case considering an equivalent stress criteria. It could also be extended to a stochastic case, c.f. [19]. A common process is to estimate the autocorrelation function from the obtained stress data, then the spectral density function can be found by using the fast Fourier transform, and the life time can be approximated with a level crossing formula, usually the so called Rice's formula, see, e.g., [20].

The stochastic Rainflow method works best in one dimensional cases, because the generalization to a multiaxial case is somewhat artificial. Considering only one equivalent stress process is a gross simplification. Another problem is that generalisation of the method is limited. The main reason is of course that it is derived using a minimal amount of methods from "stochastic toolbox". In this paper, a stochastic approach is described for an evolution equation based multiaxial fatigue model applicable for arbitrary loading histories. The stochastic version of an evolution equation based continuum HCF-model is not only a particular method, but a broad concept to handle stochastic fatigue in a new way. The concept is essentially multiaxial and it is easily extensible to take into account all the stochastic properties, of which are of interest. In this paper, we describe the fundamental idea of the method.

2. Continuum model in high-cycle fatigue. In this section, we recall the basic definition of the continuum model for high-cycle fatigue, following [9,21–24]. The starting point is to define a proper endurance surface in the stress space. In isotropic case, we usually use the following representation for the endurance surface

$$\beta = \frac{1}{\sigma_{-1}}(\bar{\sigma} + AI_1 - \sigma_{-1}) = 0, \tag{1}$$

where I_1 is the first invariant of the stress tensor $\boldsymbol{\sigma}$, that is $I_1 = \mathrm{tr}(\boldsymbol{\sigma})$. The effective stress $\bar{\sigma}$ is defined by the second invariant of the reduced stress $\mathbf{s}-\boldsymbol{\alpha}$, that is

$$\bar{\sigma} = \sqrt{3J_2(\mathbf{s}-\boldsymbol{\alpha})} = \sqrt{\tfrac{3}{2}\operatorname{tr}(\mathbf{s}-\boldsymbol{\alpha})^2}.$$

The deviatoric stress tensor is given by $\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I}$, where the identity tensor is denoted by $\mathbf{I}$.

In [9], the authors study a one dimensional cyclical load case, where the stress cycles between values $\sigma_{\mathrm{m}} \pm \sigma_{\mathrm{a}}$. They deduce, that in this case the parameters of equation (1) satisfy the equation $\sigma_{\mathrm{a}} + A\sigma_{\mathrm{m}} - \sigma_{-1} = 0$, which is the linear part of the Haigh's diagram. Hence, σ_{-1} is a fatigue limit for a SN-curve with the mean stress $\sigma_{\mathrm{m}} = 0$ and A is the slope of the Haigh's diagram.

The form of the endurance surface (1) is as simple as possible. Hence, the estimation of the parameters A and σ_{-1} is straightforward. It would be temptating to use more complex representations for the endurance surface, but then the finding of corresponding parameters will be more complicated.

In the endurance surface, the tensor $\boldsymbol{\alpha}$ represents a "centre" of the surface. The position of $\boldsymbol{\alpha}$ determines the position of the endurance surface in the stress space. The fundamental idea of the continuum approach is that the endurance surface moves in the stress space. The motion of the centre is determined by the evolution equation

$$\dot{\boldsymbol{\alpha}} = \begin{cases} C(\mathbf{s}-\boldsymbol{\alpha})\dot{\beta}, & \text{when } \beta, \dot{\beta} \geqslant 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We see that the tensor $\boldsymbol{\alpha}$ moves if and only if the stress $\boldsymbol{\sigma}$ is outside of the surface and moves outwards. In the equation, the material parameter C is positive and may be estimated from SN-curves, also known as Wöhler curves, of the material. A canonical initial value for the evolution equation is $\boldsymbol{\alpha}(0) = \mathbf{0}$, but some cases this can cause a non-physical failure. The evolution equation is the so called differential-algebraic equation (DAE) and we need to solve it by the proper numerical procedures for DEAs (see, e.g., [25]). For this, we can write it in the standard form

$$F(t, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) = \dot{\boldsymbol{\alpha}} - C(\mathbf{s}-\boldsymbol{\alpha})\dot{\beta}H(\beta)H(\dot{\beta}),$$

where H is the classical Heaviside step function.

The another fundamental idea of the continuum approach is to consider the failure of the body as a process, that is, any cycle counting method is not needed. The main postulate is that the failure growth occur at the same time with evolution of the centre tensor $\boldsymbol{\alpha}$. Hence, the failure is an increasing real valued function D and its values are determined by the initial value problem

$$\begin{aligned} \dot{D}(t) &= \begin{cases} g(D(t), \beta(t)), & \text{when } \beta(t), \dot{\beta}(t) \geqslant 0, \\ 0, & \text{otherwise,} \end{cases} \\ D(0) &= 0. \end{aligned} \tag{3}$$

The function $g(D, \beta)$ is called a damage rule and it is normed such that the material failure occur at t_f when $D(t_f) = 1$. We will use the damage rule of the type Lemaitre and Chaboche, that is

$$g(D,\beta) = \frac{K}{(1-D)^k}\exp(L\beta)\dot{\beta}.$$

where $K > 0$, $L > 0$ and $k \geqslant 0$ are material parameters. The estimation of dimensioless parameters $C > 0$, $K > 0$, $L > 0$ and $k \geqslant 0$ is discussed in [9,12]. The idea is to define a function $N = N(C, K, L, k)$ in the case of one dimensional cyclic loading such that the parameters may be found by the least square estimation. The parameters σ_{-1} and A carry information of infinite life time and C, K, L and k of finite life time.

In above, we are considered only one measured stress $\boldsymbol{\sigma}$. In practice, if any measurement is attempted to repeat as accurately as possible, the result is not the same. Repeating measurement many times, we have a collection of stress histories. They can be really identical, but ever not exactly same. Hence, it is appropriate to assume $\boldsymbol{\sigma}$ to be a stochastic process.

3. Stochastic processes. Let us recall basic ideas of stochastic processes. Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space and let the index set T be an interval, for example $T = [0, \infty)$. A stochastic process is a mapping

$$X : \Omega \times T \to \mathbb{R},$$

such that for a fixed $t \in T$ each

$$X(\cdot, t) = \mathbf{x}_t : \Omega \to \mathbb{R}$$

is a random variable. Hence, a stochastic process is a collection of random variables $\{\mathbf{x}_t\}_{t\in T}$. The index set is usually time, but it can also be another continuous or discrete parameter. In the example in the later section of this article, the index set is a number of cycles. If $\omega \in \Omega$ is fixed, then the function

$$X(\omega, \cdot) : T \to \mathbb{R}$$

is called a realization or a sample path of the process. For a stress process $\boldsymbol{\sigma}(t)$, we denote its realizations by $\widetilde{\boldsymbol{\sigma}}_1(t), \widetilde{\boldsymbol{\sigma}}_2(t), \ldots, \widetilde{\boldsymbol{\sigma}}_n(t)$.

A stress process is a matrix valued stochastic process $\boldsymbol{\sigma}(t) = [\sigma_{ij}(t)]$ where $\sigma_{ij}(t)$ are real valued stochastic processes. A methodological consequence is that the evolution equation (2) and the damage equation (3) need to be considered as stochastic differential equations.

Usually an explicit definition for the probability space is not needed. We need to know finite dimensional distributions of a process. More precisely, if $t_1 < t_2 \cdots < t_m$ are arbitrary points of the index set, we have to know the joint distribution of random variables $\mathbf{x}_{t_1}, \mathbf{x}_{t_2}, \ldots, \mathbf{x}_{t_m}$ given by its probability density function or its cumulative distribution function. Conversely, the following famous theorem by Andrej Nikolaevich Kolmogorov holds true: "*If we know a family of finite dimensional distributions, then under some regularity conditions they define a stochastic process uniquely.*"

In practise, the most used technique to find a "best estimate" of a stochastic process at the current moment is the so called stochastic filtering problem. The filtering problem is a mathematical model for a number of state estimation problems in signal processing. Most used filtering methods is the Kalman filter and its extensions, see, e.g., [26].

4. Distributions for life-time. In the preceding section, we deduced the basic idea of the continuum approach and made the assumption, that the stress $\boldsymbol{\sigma}(t)$ and hence $\beta(t)$, $\boldsymbol{\alpha}(t)$ and $D(t)$ are stochastic processes. In the finite life-time case, the failure criteria is

$$D(t_f) = 1.$$

Hence, the life-time t_f is a real valued random variable. Different realizations of the stress process $\widetilde{\boldsymbol{\sigma}}_1(t), \widetilde{\boldsymbol{\sigma}}_2(t), \ldots, \widetilde{\boldsymbol{\sigma}}_n(t)$ produce a corresponding sample $\widetilde{t}_{f,1}$, $\widetilde{t}_{f,2}$, $\ldots$, $\widetilde{t}_{f,n}$ for the life-time which we can use to study its statistics. In general, the stochastics of life-time is well studied (see, e.g., [27]) and the usual choice for the life time distribution is log-normal or Weibull distribution. In [21], we consider the log-normal case and in the paper [24] we assume that the distribution of the life time t_f is Weibull[1]. General information of Weibull distribution can be found, for instance, in [28].

To compare different distributions, W. Nelson [27] writes: *In many applications, the Weibull and log-normal distributions (and others) may fit a set of data equally well, especially over the middle of the distribution. When both are fitted to a data set, the Weibull distribution has an earlier lower tail than the corresponding log-normal distribution. That is, a low Weibull percentile is below the corresponding log-normal one.*

Log-normal distribution. We say that a random variable t_f is log-normal distributed

$$t_f \sim \mathrm{LogN}(\mu, \nu^2)$$

if and only if $\ln(t_f) \sim \mathrm{N}(\mu, \nu^2)$, where $\mathrm{N}(\mu, \nu^2)$ denotes the classical normal distribution. The parameters of the distribution satisfies $-\infty < \mu < \infty$ and $\nu > 0$.

The log-normal distribution can be considered also independently, without considering normal distribution. Then the distribution function and the cumulative distribution function are given by equations

$$f_{LN}(t; \mu, \nu) = \tfrac{1}{t\nu\sqrt{2\pi}} \exp\Big(-\tfrac{(\ln(t)-\mu)^2}{2\nu^2}\Big), \quad t > 0$$

and

$$F_{LN}(t; \mu, \nu) = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{Erf}\Big(\tfrac{\ln(t)-\mu}{\sqrt{2}\nu}\Big), \quad t > 0.$$

The expectation value and the variance are respectively

$$\mathrm{E}_{LN}(t) = \exp\big(\mu + \tfrac{\nu^2}{2}\big)$$

and

$$\mathrm{Var}_{LN}(t) = \big(\exp(\nu^2) - 1\big)\exp(2\mu + \nu^2).$$

[1]Waloddi Weibull, 1887–1979, Swedish engineer, scientist, and mathematician.

The estimators $\widehat{\mu}$ and $\widehat{\nu}$ can be computed from a sample $\widetilde{t}_{f,1}, \widetilde{t}_{f,2}, \dots, \widetilde{t}_{f,n}$ by computing the normal distribution parameter of the sample $\ln(\widetilde{t}_{f,1}), \ln(\widetilde{t}_{f,2}), \dots, \ln(\widetilde{t}_{f,n})$, that is

$$\widehat{\mu} = \tfrac{1}{n}\sum_{j=1}^{n} \ln(\widetilde{t}_{f,j})$$

and

$$\widehat{\nu}^2 = \tfrac{1}{n}\sum_{j=1}^{n}\big(\ln(\widetilde{t}_{f,j}) - \widehat{\mu}\big)^2.$$

Hence,

$$t_f \sim \mathrm{LogN}(\widehat{\mu}, \widehat{\nu}^2).$$

Weibull distribution. Recall that the distribution function and the cumulative distribution function of Weibull distribution are given by equations

$$f_W(t; a, b) = \tfrac{b}{a^b} t^{b-1} \exp\big(-(\tfrac{t}{a})^b\big), \quad t > 0,$$

and

$$F_W(t; a, b) = 1 - \exp\big(-(\tfrac{t}{a})^b\big), \quad t > 0,$$

where $a > 0$ and $b > 0$ are the parameters of the distribution. For a Weibull distributed random variable t, the expectation value is

$$\mathrm{E}_W(t) = a\Gamma\big(1 + \tfrac{1}{b}\big)$$

and the variance

$$\mathrm{Var}_W(t) = a^2\Gamma\big(1 + \tfrac{2}{b}\big) - \mathrm{E}_W(t)^2,$$

where Γ is the classical gamma function. After some simplification, we obtain the log-likelihood function

$$l(a, b; \widetilde{t}_{f,1}, \dots, \widetilde{t}_{f,n}) = n\ln(b) - nb\ln(a) + (b-1)\sum_{j=1}^{n}\ln(\widetilde{t}_{f,j}) - \sum_{j=1}^{n}\big(\tfrac{\widetilde{t}_{f,j}}{a}\big)^b.$$

A straightforward computation gives us a system of equations

$$\begin{cases} 0 = \frac{\partial l(a,b)}{\partial a} = -\frac{nb}{a} + \frac{b}{a}\sum_{j=1}^{n}\big(\frac{\widetilde{t}_{f,j}}{a}\big)^b, \\ 0 = \frac{\partial l(a,b)}{\partial b} = -\frac{n}{b} - n\ln(a) + \sum_{j=1}^{n}\ln(\widetilde{t}_{f,j}) - \sum_{j=1}^{n}\big(\frac{\widetilde{t}_{f,j}}{a}\big)^b \ln\big(\frac{\widetilde{t}_{f,j}}{a}\big). \end{cases}$$

Unfortunately, there is no analytical solution to this system. It follows that we have to solve it numerically, for instance by Newton method. The solution gives us the estimates $\widehat{a}$ and $\widehat{b}$ for parameters and hence

$$t_f \sim \mathrm{Weibull}(\widehat{a}, \widehat{b}).$$

5. One dimensional toy modell (cyclic load with noise). In this example, we consider a one dimensional cyclic load with the amplitude σ_{a}, mean stress σ_{m}, stress intensity τ and noise $W(t)$:

$$\sigma(t) = \sigma_{\mathrm{a}}\sin(2\pi t) + \sigma_{\mathrm{m}} + \tau W(t).$$

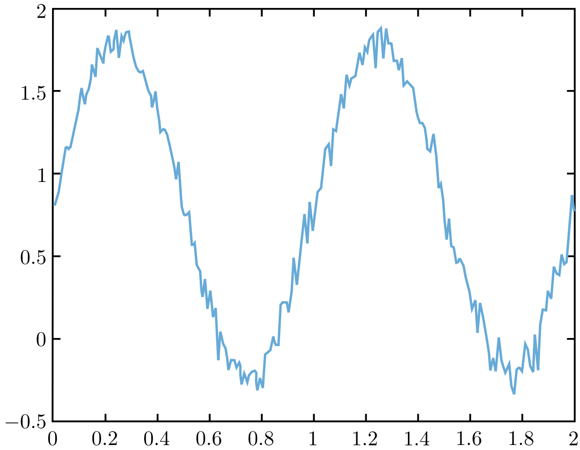

Figure 1. A sample of a stress history

This is a classical sinusoidal cyclic stress history perturbed by a Gaußian noise (see Fig. 1).

In this case, the index set $T = [0, \infty)$ corresponds the number of cycles. The stress intensity τ may be estimated from sinusoidal type noisy data $\widetilde{\sigma}(t)$ as follows. We compute the pure noise by

$$\sigma_n(t) = \sigma(t) - \sigma_a \sin(2\pi t) - \sigma_m,$$

and hence the expectation, in the normal distribution sense, is $\mathrm{E}(N(t)) = 0$. We assume that the noise is a stationary stochastic process and we obtain

$$\sigma_n(t) = \tau W(t) \sim \tau \mathrm{N}(0,1) = \mathrm{N}(0, \tau^2),$$

and thus the estimator $\widehat{\tau}$ may be found as a standard deviation of noise estimates $\widetilde{\sigma}_n(t_i) = \widetilde{\sigma}(t_i) - \sigma_a \sin(2\pi t) - \sigma_m$, $i = 1, 2, \dots, n$.

Now assume that the material is AISI-SAE 4340 alloy steel. From [9] we get the following parameters:

$$A = 0.225, \quad C = 1.25, \quad K = 2.65 \cdot 10^{-5}, \quad L = 14.4, \quad k = 0.$$

In addition, we assume that $\sigma_m = 0.8\sigma_{-1}$, $\sigma_a = \sigma_{-1}$ and $\tau = 0.1\sigma_{-1}$. Computing $n = 50$ realizations for the stress process $\{\widetilde{\sigma}_j(t)\}_{j=1}^{50}$, we obtain a sample $\{\widetilde{t}_{f,j}\}_{j=1}^{50}$ of life-times. The histogram of the sample and the fitted log-normal density function is given in the Fig. 2.

Hence, we may estimate log-normal parameters

$$t_f \sim \mathrm{LogN}(10.7337, 4.9767 \cdot 10^{-7}).$$

We may compute the expectation value for a life-time

$$\mathrm{E}_{LN}(t) = 4.5876 \cdot 10^4,$$

and the variance

$$\mathrm{Var}_{LN}(t) = 1.0474 \cdot 10^3.$$

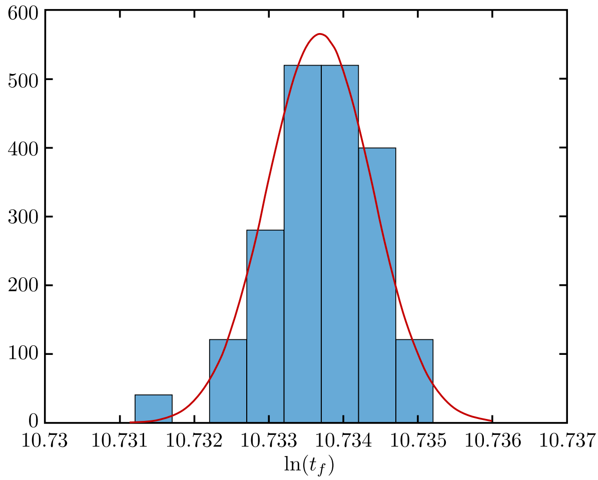

Figure 2. Histogram approximation for a normal distribution of logarithmic life-time

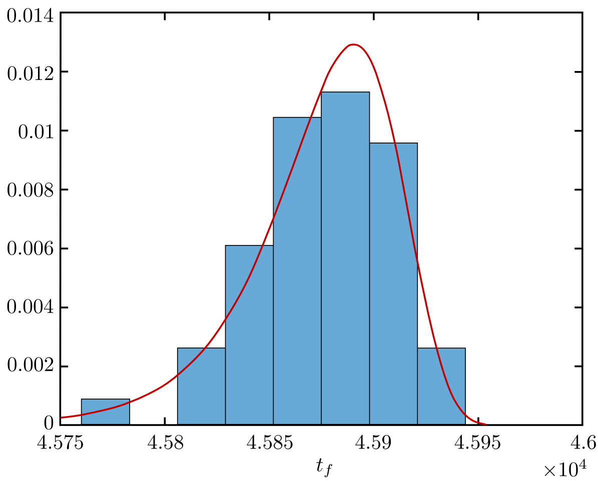

Figure 3. Histogram approximation for a Weibull distribution of life-time

Similarly, using the sample, we may estimate the Weibull parameters and we have

$$t_f \sim \text{Weibull}(4.5895 \cdot 10^4, 1.1952 \cdot 10^3).$$

The histogram of the sample and the fitted Weibull density function is given in the Fig. 3.

We can see that the density function fits good with the histogram approximation of the life-time. The expectation value for a life-time is

$$\text{E}_W(t_f) = 4.5874 \cdot 10^4$$

and the variance

$$\text{Var}_W(t_f) = 1.3309 \cdot 10^2.$$

Now we may pose questions, which are interested from engineering point of view. As an example, we consider the question:

Q : *What is the life-time, what we get with probability* 95 %?
We need to find t such, that

$$\mathbb{P}(t < t_f) = 1 - \mathbb{P}(t_f \leqslant t) = 1 - F(t) = 0.95,$$

where F is a cumulative distribution function of a distribution. Using the quantile functions of distributions to solve $F(t) = 0.05$, we obtain

$$t_{95\%/\mathrm{LogN}} = 4.5875 \cdot 10^4,$$
$$t_{95\%/\mathrm{Weibull}} = 4.5806 \cdot 10^4.$$

6. Conclutions. In this paper, we study the so called continuum approach to high-cycle fatigue. The method is introduced in [9] and recently actively extended. We consider only finite life-time case and complete the method assuming that measured stress $\boldsymbol{\sigma}(t)$ is a realization of a stochastic process. This allow us to estimate the stress process it self and generate its realizations. The assumption is natural, since the nature of measured stress is always stochastic, at least some level. The fundamental consequence is that we can consider all quantities in the theory stochastically. The biggest advantage is that the life-time is a random variable, what is natural. This method allow us to find numerical approximation for the life-time. A distribution of life-time should fulfill certain basic requirements, such as it should contain only positive numbers. The natural two candidates are the log-normal and Weibull distributions. We demonstrate the method by computing the one dimensional example in Sec. 4. In future, more practical cases should be studied and the estimation of stress should be carefully discussed.

Competing interests. I declare that I have no competing interests.

Author's Responsibilities. I take full responsibility for submitting the final manuscript in print. I approved the final version of the manuscript.

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Континуальный подход к многоцикловой усталости. Полный срок службы со случайной историей нагружения

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Аннотация

Рассматривается континуальный подход для описания многоцикловой усталости, когда срок службы изделия конечен. Предложено использовать эволюционную модель накопления усталостных микроповреждений в стохастической постановке. Данный подход позволяет учитывать стохастический разброс параметров повторяющегося нагружения. В рамках этого подхода срок службы изделия (время его жизни) отождествляется с началом разрушения материала, а случайный процесс нагружения описывается с помощью стохастических дифференциальных уравнений. Основное предположение состоит в том, что распределение срока службы изделия подчиняется логнормальному распределению или распределению Вейбулла. Представленная методика позволяет оценить сам процесс нагружения, сформировать реализацию такого нагружения и найти численное приближение по сроку службы изделия. Для демонстрации метода приводится численный пример, в котором рассматривается одномерная начальная задача определения времени жизни образца при его неотнулевом синусоидальном циклическом нагружении растяжением-сжатием, зашумленном винеровским стохастическим процессом. Поставленная задача решена численно для пятидесяти реализаций, в результате чего ответ дан в вероятностной формулировке, позволяющей более осознанно назначать запас прочности.

Ключевые слова: многоцикловая усталость, срок службы, эволюционное уравнение.

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