Mathematical Modeling, Numerical Methods and Software Complexes

Stochastic models of *just-in-time* systems and *windows of vulnerability* in terms of the processes of birth and death

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Abstract

The paper proposes a method for constructing models based on the analysis of birth and death processes with linear growth in semimartingale terms. Based on this method, stochastic models of simple *just-in-time* systems (analyzed in the theory of productive systems) and *windows of vulnerability* (widely discussed in risk theory) are considered. The main results obtained in the work are presented in terms of the average values of the time during which the processes reach zero values. At the same time, they are considered and used in the study of assessment models for local times of the processes.

Here, simple Markov processes with a linear growth of intensities (perhaps, depending on time) are analyzed. At the same time, the obtained and used estimates are of theoretical interest. Thus, for example, the average value of the stopping time, at which the process reaches zero, depends on functions such as the harmonic number and the remainder term for the logarithmic function in the Taylor theorem.

As the main result, the method of mathematical modeling of *just-in-time* systems and *windows of vulnerability* is proposed. The semimartingale description method used here should be considered as the first step of such a modeling, since, being a trajectory method, it allows diffusion (including

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non-Markov processes) generalizations when constructing stochastic models of *windows of vulnerability* and *just-in-time*. In the theoretical part of the article, we formulate statements for the average values of the local time and the stopping times when the birth and death processes reach a given value. This allows us to uniformly present estimates for the models of the *just-in-time* system and for *windows of vulnerability*, the result for which is given in the form of a limit theorem. The main results are formulated as theorems and lemmas. The proofs use semimartingale methods.

Keywords: modeling, process of birth and death, stopping time, compensator, intensity, counting process, martingale, trajectory, local time, just-in-time, window of vulnerability.

Introduction

In the paper, we propose a method for constructing models based on an analysis of the processes of birth and death with linear growth in semimartingale terms. As the main examples of stochastic models, simple *just-in-time* systems and *windows of vulnerability* are considered. *Just-in-time* systems (abbreviated as *JIT* are analyzed in the theory of productive systems, and *windows of vulnerability* (abbreviated as *WoV* are widely discussed in risk theory. The main results obtained in the work are presented in terms of the average values of the time during which the processes reach zero values. The work also considers (and uses when studying models) estimates of the local times of the studied processes.

We note that here, simple Markov processes with a linear growth of intensities (perhaps, depending on time) are analyzed. Nevertheless, the obtained and used estimates are of a certain theoretical interest. Thus, for example, the average value of the stopping time, at which the process reaches zero, depends on functions such as the harmonic number and the remainder term for the logarithmic function in the Taylor theorem.

However, as one of the main results, it is proposed to consider the presented method of mathematical modeling of *JIT* and *WoV* systems. The semimartingale description approach used here should be considered as the first step of such a modeling, since, being a trajectory method, it allows diffusion (including non-Markov processes) generalizations when constructing stochastic models of *WoV* and *JIT*.

At present, when constructing probabilistic models of *WoV*, in the theory of risk, researches have appeared that analyzes the average values of the time during which the trajectories are in a certain area and therefore the system is vulnerable (see, for example [1–3] and references in them). Studies of average values of the local time for such models are interesting.

And if the *WoV* descriptions in the risk theory are predominantly stochastic, then the models based on *JIT* are mostly deterministic. Although it is obvious that a probabilistic approach to the study of such processes is in demand. We also note that the stochastic description of *JIT* systems proposed here is in a certain sense a logical (and analytic) continuation of [4], in which a semimartingale description is given for problems of optimal control of such systems.

In this article, statements for the average values of the local time and the stopping times when the birth and death processes reach a given value are obtained. This allows us to consistently present estimates for the models of the *JIT* system and for *WoV*, the result for which is given in the form of a limit theorem.

The principle of *JIT* is well known and used in many areas. However, its emergence and development is associated primarily with the analysis of productive systems (see, for example, [5–9], as well as [4] and references therein). Here, as in cite [4], we touch on the time reversal method. A number of papers are devoted to the study of this method for stochastic systems represented in the semimartingale description (see, for example, [10–12] and references therein). Also close to this method are studies of stochastic bridges (see, for example, [13–15] and references therein). Descriptions in terms of the processes of birth and death are devoted primarily to models in biology (see, for example, [16]). However, the semimartingale approach allows for significant generalizations. So, it allows one to describe and investigate models of random walk processes for a wide class of objects (see, e.g., [17–20] and [4]).

All the main results in the paper are formulated as theorems and lemmas. The proofs of the results use semimartingale methods.

1. Preliminaries and necessary theoretical results

1.1. Notations, definitions and assumptions

Let $\mathbf{B} = (\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}, \mathbf{P})$ denote a stochastic basis, that is, a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with a nondecreasing right-continuous family of σ-algebras $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}$, complete with respect to $\mathbf{P}$ (see e.g., the conditions of [21]).

NOTATION 1. *For any special locally square-integrable semimartingale* $U = (U_t)_{t \geqslant 0}$ *on the basis* $\mathbf{B}$, *we denote by* $\widetilde{U} = (\widetilde{U}_t)_{t \geqslant 0}$ *and* $m^U = (m_t^U)_{t \geqslant 0}$ *the compensator of* U, *and the local martingale, respectively:* $U_t = \widetilde{U}_t + m_t^U$. *For definiteness, we also assume that* $\widetilde{U}_0 = U_0$ *and* $m_0^U = 0$. *The predictable process of locally bounded variation* $\widetilde{U}$, *and the martingale* m^U *are locally square-integrable. We denote by* $\langle m^U \rangle = (\langle m^U \rangle_t)_{t \geqslant 0}$ *the predictable quadratic characteristic of* m^U.

NOTATION 2. *For any process of birth and death* U *on* $\mathbf{B}$ *with strictly positive initial value* $U_0 \geqslant 1$ *we denote by* $\tau^U = \tau^U(\omega)$, $\omega \in \Omega$, *the Markov time at which* U *reaches zero:* $\tau^U = \inf\{t > 0 : U_t = 0\}$ *(where* $\inf\{\varnothing\} = +\infty$*).*

NOTATION 3. *We denote by* $L^U(K)$ *the function of* $K \in \mathbb{N} = \{1, 2, \ldots\}$, *which is the average value of the stopping time* τ^U *for the process* U *with the initial value* $U_0 = K$: $L^U(K) = \mathbf{E}\tau^U$.

DEFINITION 1. *Let* $l_t^U(n)$ *be the local time of any process of birth and death* U *on* $\mathbf{B}$: *for* $n \in \mathbb{N}_0$ *and* $t \in \mathbb{R}_+ = [0, +\infty)$:

$$l_t^U(n) = \int_0^t \mathbb{I}\{U_s = n\}\, ds,$$

where $\mathbb{I}\{\cdot\}$ *is an indicator function (i.e.,* $\mathbb{I}\{true\} = 1$, $\mathbb{I}\{false\} = 0$*).*

Consider on the basis $\mathbf{B}$ a locally square-integrable birth and death process $X = (X_t)_{t \geqslant 0}$ with trajectories in the Skorokhod space, $X_t(\omega) \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,

with $X_0(\omega) = K \in \mathbb{N}$ and $\Delta X_t(\omega) = X_t(\omega) - X_{t-}(\omega) \in \{-1, 0, 1\}$ **P**-a.s. We can represent X as a difference $X = X_0 + A - B$ (see, for example, [18] and [19]), with

$$X_t = X_0 + A_t - B_t, \qquad (1)$$

where $A = (A_t)_{t \geqslant 0}$ and $B = (B_t)_{t \geqslant 0}$ are the counting processes of the number of positive and negative jumps of X, respectively:

$$A_t = \sum_{0 < s \leqslant t} \mathbb{I}\{\Delta X_s = 1\}, B_t = \sum_{0 < s \leqslant t} \mathbb{I}\{\Delta X_s = -1\} \text{ for all } t \geqslant 0 \ \mathbf{P}\text{-a.s.}$$

with the initial values $A_0 = B_0 = 0$ and $X_0 = K \geqslant 1$.

ASSUMPTION 1. *Suppose that X is a process of birth and death with linear growth, that is, the compensators of the submartingales A and B on $\mathbf{B}$ are*

$$\widetilde{A}_t = \int_0^t \alpha \cdot X_s \, ds \quad and \quad \widetilde{B}_t = \int_0^t \beta \cdot X_s \, ds \quad for \ all \ t \geqslant 0. \qquad (2)$$

Then the quadratic characteristics of the locally square-integrable martingales $m^A = (m^A{}_t)_{t \geqslant 0}$ and $m^B = (m^B{}_t)_{t \geqslant 0}$ are

$$\langle m^A \rangle_t = \widetilde{A}_t, \quad \langle m^B \rangle_t = \widetilde{B}_t \quad and \quad \langle m^A, m^B \rangle_t = 0 \quad for \ all \ t \geqslant 0. \qquad (3)$$

ASSUMPTION 2. *We suppose that*

$$\alpha \geqslant 0, \beta > 0 \quad and \quad \beta > \alpha.$$

It is clear that under the representation (2) and (3) the process X with the definition (1) is a square-integrable birth and death process with linear growth. For such a process, it easily follows from Assumption 2 that

$$\lim_{t \to \infty} \boldsymbol{E} \, X_t = \lim_{t \to \infty} \boldsymbol{E} \, X_t^2 = 0.$$

Therefore, from Chebyshev's inequality we obtain

$$\lim_{t \to \infty} \boldsymbol{E} \, \mathbb{I}\{X_t \geqslant i\} = \lim_{t \to \infty} \mathbf{P}\{X_t \geqslant i\} = 0 \text{ for all } i \in \mathbb{N}. \qquad (4)$$

Thus,

$$\mathbf{P}\{\tau^X < \infty\} = 1 - \lim_{t \to \infty} \mathbf{P}\{X_t > 0\} = 1,$$

and τ^X is the stopping time on $\mathbf{B}$.

NOTATION 4. *For any process of birth and death U on $\mathbf{B}$ we denote by $\lambda^U(n)$ the average value of the local time on $\mathbb{R}_+$ for $n \in \mathbb{N}$:*

$$\lambda^U(n) = \boldsymbol{E} \, l_\infty^U(n),$$

where

$$l_\infty^U(n) = \lim_{t \to \infty} l_t^U(n)$$

and $l_t^U(n)$ *is set in Definition* 1.

From Definition 1 it follows that

$$\lambda^X(n) = \int_0^\infty \mathbf{P}\{X_s = n\}\, ds. \tag{5}$$

Since for X under consideration, $\tau^X(\omega) < \infty$ **P**-a.s., then we have

$$\tau^X = \int_0^\infty \mathbb{I}\{X_s \geqslant 1\}\, ds = \lim_{t\to\infty} \sum_{n=1}^\infty l_t^X(n) = \sum_{n=1}^\infty l_\infty^X(n), \tag{6}$$

where

$$l_\infty^X(n) = \lim_{t\to\infty} l_t^X(n).$$

From (5), Notation 4 and (6) it follows that

$$L^X(K) = \boldsymbol{E}\,\tau^X = \sum_{n=1}^\infty \lambda^X(n). \tag{7}$$

We define the following auxiliary functions.

DEFINITION 2. *Let $H(n)$ be the harmonic number (i.e. the partial sum of the harmonic series) for $n \in \mathbb{N}$:*

$$H(n) = \sum_{i=1}^n \frac{1}{i}.$$

DEFINITION 3. *Let $R_n(x)$ be the remainder term in Taylor theorem of the function $\log(1-x)$ for n-th order Taylor polynomial $\mathscr{P}_n(x)$ of $x \in (0,1)$, $n \in \mathbb{N}$:*

$$R_n(x) = \log(1-x) - \mathscr{P}_n(x) = \log(1-x) + \sum_{i=1}^n \frac{x^i}{i} = -\sum_{i=n+1}^\infty \frac{x^i}{i}.$$

DEFINITION 4. *We define the following auxiliary constants:*

$$\eta = \beta - \alpha, \quad \gamma = \alpha/\beta \ (\textit{where } 0 < \eta \leqslant \beta,\ 0 \leqslant \gamma < 1).$$

1.2. Theoretical results necessary for the study of models

THEOREM 1. *For the average value of the stopping time τ^X,*

$$L^X(K) = \frac{1}{\beta - \alpha} \cdot \left\{ H(K) + \log\left(1 - \frac{\alpha}{\beta}\right) - \left(\frac{\beta}{\alpha}\right)^K \cdot R_K\left(\frac{\alpha}{\beta}\right) \right\} \quad \textit{if} \quad \alpha > 0 \tag{8}$$

and

$$L^X(K) = \frac{H(K)}{\beta} \quad \textit{if} \quad \alpha = 0. \tag{9}$$

It should be noted that the expression (8) in the statement of the theorem is rather unusual. As for (9), it can also be regarded as the limiting case of (8) for $\alpha \to 0$.

COROLLARY 1. *If $K = 1$, then for the average value of the stopping time τ^X,*

$$L^X(K) = -\frac{1}{\alpha} \cdot \log\left(1 - \frac{\alpha}{\beta}\right) \qquad if \quad \alpha > 0 \tag{10}$$

and

$$L^X(K) = \frac{1}{\beta} \qquad if \quad \alpha = 0. \tag{11}$$

The proof of the theorem is based on the equality (7) and on the following lemma, which is also of independent interest.

LEMMA 1. *For the average value of the local time on $\mathbb{R}_+$ for $n \in \mathbb{N}$,*

$$\lambda^X(n) = \begin{cases} \dfrac{1}{\eta \cdot n} - \dfrac{\gamma^n}{\eta \cdot n} & if \quad n \leqslant K, \\ \dfrac{\gamma^{(n-K)}}{\eta \cdot n} - \dfrac{\gamma^n}{\eta \cdot n} & if \quad n > K. \end{cases} \tag{12}$$

COROLLARY 2. *If $\alpha = 0$, then for the average value of the local time on $\mathbb{R}_+$ for $n \in \mathbb{N}$,*

$$\lambda^X(n) = \begin{cases} 1/(\beta \cdot n) & if \quad n \leqslant K, \\ 0 & if \quad n > K. \end{cases} \tag{13}$$

2. The problem of *JIT*

The term *just-in-time* arose in connection with the need for a description of the so-called production systems (see, for example, [6, 9] and references in [4]).

Suppose that a fixed strictly positive finite time moment $T > 0$ is given. Suppose also that the process of birth and death $Z = (Z_t)_{t \in [0,T]}$ (or an integer-valued random walk for the general case) has the initial value $Z_0 = K \in \mathbb{N}$.

DEFINITION 5. *The process of birth and death Z is* just-in-time T *(abbreviated as JIT T), if $Z_T = 0$ $\mathbf{P}$-a.s. and $\mathbf{P}\{Z_t > 0\} > 0$ for any $t \in [0, T)$ (that is, $\mathbf{P}\{\tau^Z \leqslant T\} = 1$ and $\mathbf{P}\{\tau^Z > t\} > 0$ for any $t \in [0, T)$).*

The process of birth and death Z is just-in-time *(abbreviated as JIT) if there exists a strictly positive finite number $T > 0$ such that the process Z is JIT T.*

As a simple example, consider the linear pure death *JIT T* process $V = (V_t)_{t \in [0,T]}$ with death rates that are properly time-dependent. Suppose that in a certain production system it is necessary to perform a set of homogeneous operations with successively decreasing numbers $\mathbb{K}_0 = \{K, K-1, \dots, 1, 0\}$ exactly for a period of time T, where $K \in \mathbb{N}$ and $T \in (0, +\infty)$. Usually, operation number 0 is considered terminal and unproductive (for example, warehousing or packaging). For systems of this type, it is of interest problem to estimate the time of the beginning of the last operation (with the number 0), since the control possibilities of the system after this moment vanish.

Let the nonincreasing process $V = (V_t)_{t \in [0,T]}$ with values in $\mathbb{K}_0$ "indicates" the number of the operation that is executed at each moment $t \in [0, T]$. We denote by $\xi(n)$ the execution time of each operation with number $n \in \mathbb{K}_0$. Then $\xi(n) = l_T^V(n)$, where l_T^V is the local time defined in Definition 1, for the process V. In these notations, the problem mentioned above is formulated in the same way as in the first section: find $\boldsymbol{E}\tau^V$, where the moment τ^V is defined in Notation 2 and is equal to $\xi(K) + \ldots + \xi(1) = T - \xi(0)$.

Note also that the process with reversed time $\mathfrak{V} = (\mathfrak{V}_u)_{u \in [0,T]}$ with values $\mathfrak{V}_u = V_{(T-u)}$ is a point counting process provided $\mathfrak{V}_T = K$. If the times $\{\xi(0), \xi(1), \ldots, \xi(K)\}$ were independent and identically exponentially distributed random variables, then $\mathfrak{V}$ would be a Poisson process. However, in this case the condition $\mathfrak{V}_T = K$ would not hold. The trajectories of the Poisson process for which this condition is fulfilled form a well-known Poisson bridge, see, for example, [12, 13] and the literature cited therein. The compensator of the process $\mathfrak{V}$ has the form

$$\widetilde{\mathfrak{V}}_u = \int_0^u \frac{K - \mathfrak{V}_r}{T - r} \cdot \mathbb{I}\{r < T\}\, dr.$$

Then the process V can be considered as a Poisson process with reversed time with an initial value $V_0 = K$ and with the condition $V(T) = 0$. Its compensator is equal to

$$\widetilde{V}_t = K - \int_0^t \frac{V_s}{T - s} \cdot \mathbb{I}\{s < T\}\, ds.$$

Note that the condition $V(T) = 0$ can also be achieved with other dependencies of the death rate on time. As an example, consider the process, which is a very simple generalization of the process with the specified compensator $\widetilde{V} = (\widetilde{V}_t)_{t \in [0,T]}$.

Suppose that the linear pure death process Z can be represented as the difference $Z = Z_0 - B$ with $Z_t = Z_0 - B_t$ and $t \in [0, T]$, where $B = (B_t)_{t \in [0,T]}$ is the counting process of numbers of negative jumps of Z:

$$B_t = \sum_{0 < s \leqslant t} \mathbb{I}\{\Delta Z_s = -1\}$$

with the compensator

$$\widetilde{B}_t = \int_0^t \beta \cdot \frac{K - N_s}{T - s} \cdot \mathbb{I}\{s < T\}\, ds, \quad \beta > 0.$$

Consequently, the supermartingale Z has the representation

$$Z_t = K - \int_0^t Z_s \cdot \frac{\beta}{T - s} \cdot \mathbb{I}\{s < T\}\, ds + m_t^Z, \tag{14}$$

For the average value of the stopping time $L^Z(K) = \boldsymbol{E}\,\tau^Z$ of the linear pure death process Z, with the intensity depending on time in accordance with the representation (14), the following statement holds.

THEOREM 2. *Let $\beta > 0$. Then for the average value of the stopping time τ^Z,*

$$L^Z(K) = \frac{T}{\beta} \cdot \sum_{i=1}^{K} \frac{(i-1)!}{\prod_{j=1}^{i}\left(j + \frac{1}{\beta}\right)}. \tag{15}$$

REMARK 1. *Note that* (15) *can be represented in the form, which is to some extent close to* (9) *of Theorem* 1:

$$\boldsymbol{E}\,\tau^Z = \frac{1}{\beta} \cdot \sum_{n=1}^{K} \frac{1}{n} \cdot \chi\{n\},$$

where

$$\chi\{n\} = T / \prod_{i=1}^{n}\left(1 + \frac{1}{\beta \cdot i}\right).$$

COROLLARY 3. *If $\beta = 1$, then for the average value of the local time on $\mathbb{R}_+$ for $K \in \mathbb{N}$,*

$$L^Z(K) = T \cdot \frac{K}{K+1}.$$

REMARK 2. *This result coincides with a value that we can easily obtain for a description in reverse time. Thus, for the stopping time $\xi(1)$ of the first jump of the process in reverse time $\mathfrak{Z} = (\mathfrak{Z}_u)_{u \in [0,T]} = (Z_{(T-u)})_{u \in [0,T]}$ (i.e., $\xi(0) = \inf\{u\colon u > 0, \mathfrak{z}_u = 1\}$), we have the distribution function $F_{\xi(0)}(t) = 1 - (\frac{T-t}{T})^K$. Therefore, for the average value of $\xi(0)$, we have*

$$\boldsymbol{E}\,\xi(0) = \int_0^T t\,dF_{\xi(0)}(t) = \frac{T}{K+1},$$

and

$$L^Z(K) = \boldsymbol{E}\,\tau^Z = T - \boldsymbol{E}\,\xi(0) = T \cdot \frac{K}{K+1}.$$

To prove Theorem 2, we define on a stochastic basis **B** the sequence of stopping times $\mathfrak{d}(n) = \inf\{t \in [0,T]\colon Z(t) = n\}$ and the sequence of mean values $\delta(n) = \boldsymbol{E}\,\mathfrak{d}(n), n \in \mathbb{K}_0$, and note that for $n \in \mathbb{K} = \{K, K-1, \ldots, 1\}$, we have $\mathfrak{d}(n-1) - \mathfrak{d}(n) = l_T^Z(n)$ and

$$\delta(n-1) - \delta(n) = \lambda^Z(n), \tag{16}$$

where $\lambda^Z(n) = \boldsymbol{E}\,l_T^Z(n)$. We also formulate the lemma, which is somewhat more general than Theorem 2, and is of independent interest.

LEMMA 2. *Let $\beta > 0$. Then $\delta(K) = 0$ and*

$$\delta(n) = \frac{T}{\beta} \cdot \frac{1}{n!} \sum_{i=n+1}^{K} \frac{(i-1)!}{\prod_{j=n+1}^{i}\left(j + \frac{1}{\beta}\right)} \quad \textit{for } n \in \{K-1, \ldots, 1, 0\}. \tag{17}$$

3. The problem of *WoV*

Consider the multiple analog of the process of birth and death with linear growth, discussed in the previous section. As is known, birth and death processes can serve as primary stochastic models for describing changes in the size of a simple population of individuals, cells, bacteria or viruses. They are also models of simple queueing systems with possible call replication in queues. In some biological and computer systems, multiple cycles are observed, each of which consists of the following consecutive events and stages: the event of infection, the stage of the disease, the event of recovery, the stage of health. This model assumes that additional infections do not occur during the stage of the disease (as, for example, in the case of infections with the same strain of influenza virus). The initial number of viruses in each case of infection is considered random. In the case of a queueing system, the cycle includes an event of receiving a call packet into the system, a service stage with possible call replication, an event of reaching the zeroth queue, an idle system. It is also assumed that the system is immune to new call packages until the current service package is fully serviced. This model also corresponds to production systems with stochastic execution of homogeneous operations. The initial number of production operations in the plan for each cycle is also considered random. Such a multistage process can be interpreted as follows.

Consider a cycle with number $n \in \mathbb{N}$. The system receives $\nu(n)$ viruses (or calls, in the case of a queuing system) at the stopping time $\varsigma(n-1)$. In this case, the first cycle starts with zero time: $\varsigma(0) = 0$. After the expiration of the time $\varphi(n)$, the number of viruses reaches zero. That is, there are viruses in the system on the time interval $[\varsigma(n-1), \sigma(n))$, where $\sigma(n) = \varsigma(n-1) + \varphi(n)$. This interval is called the vulnerability window (or *the window of vulnerability*, abbreviated to *WoV*), because the system is infected and therefore vulnerable. For example, for a living system, susceptibility to other diseases or threats increases, and the computer system becomes vulnerable, since there is a threat to its information security. During the time $\psi(n)$, that is, on the interval $[\sigma(n), \varsigma(n))$, the system is free of viruses. At the stopping time $\varsigma(n) = \sigma(n) + \psi(n)$ another infection occurs, that is, the next cycle begins.

Thus, the difference between this scheme and the known process of birth and death with linear growth and immigration (see e.g., [20]) is that immigration is suspended for a time of non-zero process values. Apparently, it makes sense to call this process of immigration an episodic immigration. Below we present a formal mathematical description of this process.

Let $Y = (Y_t)_{t \geqslant 0}$ be the process of birth and death with linear growth and episodic immigration, defined on the stochastic basis **B**. Suppose that Y can be represented in the form $Y = G + A - B$, where $A = (A_t)_{t \geqslant 0}$ and $B = (B_t)_{t \geqslant 0}$ are the counting processes of the number of positive and negative jumps, $\Delta Y_t = Y_t - Y_{t-}$, provided that $Y(t-) \geqslant 1$ for $t > 0$:

$$A_t = \sum_{0 < s \leqslant t} \mathbb{I}\{\Delta Y_s = 1, Y_{s-} \geqslant 1\}, \quad B_t = \sum_{0 < s \leqslant t} \mathbb{I}\{\Delta Y_s = -1, Y_{s-} \geqslant 1\},$$

respectively. The process of episodic immigration $G = (G_t)_{t \geqslant 0}$ is defined as follows.

Let $\mathscr{N} = \{\nu(n)\}_{n=0,1,\dots}$ be the set of independent identically distributed square-integrable random variables with values $\nu(n) \in \mathbb{N}$ and the distribution

$$\mathscr{P} = \big\{p(k) = \mathbf{P}\big(\nu(n) = k\big), k \in \mathbb{N}\big\} \text{ for all } n \in \mathbb{N}_0.$$

Suppose also that $\nu(0)$ is $\mathscr{F}_0$-measurable. Consider on **B** a Poisson process $\pi = (\pi_t)_{t \geqslant 0}$ with a parameter $\rho > 0$. Suppose that π and $\mathscr{N}$ are independent. Let $g = (g_t)_{t \geqslant 0}$ be the counting process of the number of jumps of the process of episodic immigration, and $G = (G_t)_{t \geqslant 0}$ is the process of episodic immigration:

$$g_t = \int_0^t \mathbb{I}\{Y_{s-} = 0\}\, d\pi_s \text{ and } G_t = \nu(0) + \sum_{n=1}^{g_t} \nu(n).$$

For t in *WoV*, $Y = (Y_t)_{t \geqslant 0}$ behaves similarly to the process X discussed in the Section 1. Thus, we assume that the compensators of the processes A and B on the stochastic basis **B** are the same as in (2), but with the corresponding replacement of X by Y:

$$\widetilde{A}_t = \int_0^t \alpha \cdot Y_s\, ds \quad \text{and} \quad \widetilde{B}_t = \int_0^t \beta \cdot Y_s\, ds.$$

DEFINITION 6. *We define the window of vulnerability* (*abbreviated as WoV*) *on the interval* $[0, t]$ *for any trajectory* $\big(Y_t(\omega)\big)_{t \geqslant 0}$ *of the process* Y *as the random set*

$$\mathscr{W}_t^Y(\omega) = \{s \in [0, t] : Y_s(\omega) > 0\}, \quad \omega \in \Omega, \quad t \geqslant 0.$$

We also define the function $\mathscr{L}_t^Y = \boldsymbol{E}\,\mu\{\mathscr{W}_t^Y\}$ *for its Lebesgue measure* $\mu\{\cdot\}$.

Note that

$$\mu\{\mathscr{W}_t^Y(\omega)\} = \int_0^t \mathbb{I}\{Y_s \geqslant 1\}\, ds = \sum_{n=1}^\infty l_t^Y(n).$$

ASSUMPTION 3. *Suppose that* $\rho > 0$ *and* $\sum_{k=1}^\infty L^X(k) \cdot p(k) < \infty$.

From this assumption it follows that $\boldsymbol{E}\,\psi(1) < \infty$ and $\boldsymbol{E}\,\varphi(1) < \infty$ and for all $n \in \mathbb{N}$

$$\boldsymbol{E}\,\psi(n) = 1/\rho \text{ and } \boldsymbol{E}\,\varphi(n) = \sum_{k=1}^\infty L^X(k) \cdot p(k). \tag{18}$$

THEOREM 3. *For the WoV of* Y, *under Assumptions 2 and 3,*

$$\lim_{t \to +\infty} \frac{\mu\{\mathscr{W}_t^Y(\omega)\}}{t} = \lim_{t \to +\infty} \frac{\mathscr{L}_t^Y}{t} \quad \boldsymbol{P}\text{-}a.s.$$

and

$$\lim_{t \to +\infty} \frac{\mathscr{L}_t^Y}{t} = \frac{\sum\limits_{k=1}^\infty L^X(k) \cdot p(k)}{1/\rho + \sum\limits_{k=1}^\infty L^X(k) \cdot p(k)},$$

where $L^X(k)$ *is given in Theorem 1.*

4. Proof of the results
4.1. Proof of Lemma 1

From (1) it follows that for any $X_0 = K \in \mathbb{N}$, for all $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$ it holds

$$\mathbb{I}\{X_t \geqslant i\} = \mathbb{I}\{X_0 \geqslant i\} + \int_0^t \mathbb{I}\{X_{s-} = i-1\}\, dA_s - \int_0^t \mathbb{I}\{X_{s-} = i\}\, dB_s. \quad (19)$$

Since each integrable semimartingale (19) is the difference of submartingales, then

$$\widetilde{\mathbb{I}}\{X_t \geqslant i\} = \mathbb{I}\{K \geqslant i\} + \int_0^t \mathbb{I}\{X_{s-} = i-1\}\, d\widetilde{A}_s - \int_0^t \mathbb{I}\{X_{s-} = i\}\, d\widetilde{B}_s.$$

Therefore, for the compensators of the processes in (19), we obtain from (2) that

$$\widetilde{\mathbb{I}}\{X_t \geqslant i\} = \begin{cases} -i{\cdot}\beta{\cdot}l_t^X(i) + 1 & \text{if} \quad i = 1, \\ (i-1){\cdot}\alpha{\cdot}l_t^X(i-1) - i{\cdot}\beta{\cdot}l_t^X(i) + 1 & \text{if} \quad 1 < i \leqslant K, \\ (i-1){\cdot}\alpha{\cdot}l_t^X(i-1) - i{\cdot}\beta{\cdot}l_t^X(i) & \text{if} \quad i > K. \end{cases} \quad (20)$$

From (4), (5) and (20) we receive the following equalities:

$$\begin{cases} \text{if} \quad i = 1, & -i{\cdot}\beta{\cdot}\lambda^X(i) + 1 = 0, \\ \text{if} \quad 1 < i \leqslant K, & (i-1){\cdot}\alpha{\cdot}\lambda^X(i-1) - i{\cdot}\beta{\cdot}\lambda^X(i) + 1 = 0, \\ \text{if} \quad i > K, & (i-1){\cdot}\alpha{\cdot}\lambda^X(i-1) - i{\cdot}\beta{\cdot}\lambda^X(i) = 0. \end{cases} \quad (21)$$

For $\alpha = 0$, the statement of Lemma 1 (and also Corollary 2) follows directly from (21). If $\alpha > 0$, then multiplying (21) by $(\alpha/\beta)^{(n-i)}$ and summing over i from 1 to n, we obtain for $n \in \mathbb{N}$:

$$\begin{cases} \text{if} \quad n \leqslant K, & \sum_{i=1}^n (\alpha/\beta)^{(n-i)} - n \cdot \beta \cdot \lambda^X(n) = 0, \\ \text{if} \quad n > K, & \sum_{i=1}^K (\alpha/\beta)^{(n-i)} - n \cdot \beta \cdot \lambda^X(n) = 0. \end{cases} \quad (22)$$

The statement (12) of Lemma 1 follows from (22) with Notations 4. $\square$

4.2. Proof of Corollary 2

If $\alpha = 0$, then $\gamma = 0$. Hence, $\eta = \beta$ and (13) coincides with (12). $\square$

4.3. Proof of Theorem 1

From (7) and Lemma 1 we receive

$$L^X(K) = \sum_{n=1}^K \left(\frac{1}{\eta{\cdot}n} - \frac{\gamma^n}{\eta{\cdot}n} \right) + \sum_{n=K+1}^\infty \left(\frac{\gamma^{(n-K)}}{\eta{\cdot}n} - \frac{\gamma^n}{\eta{\cdot}n} \right) =$$

$$\frac{1}{\eta} \cdot \sum_{n=1}^K \frac{1}{n} - \frac{1}{\eta} \cdot \sum_{n=1}^\infty \frac{\gamma^n}{n} + \frac{1}{\eta} \cdot \sum_{n=K+1}^\infty \frac{\gamma^{(n-K)}}{n}. \quad (23)$$

If $\alpha = 0$, then $\gamma = 0$ and $\eta = \beta$. Therefore, from (23) we receive (9). If $\alpha > 0$, then $\gamma > 0$, and from (23) it follows that

$$L^X(K) = \frac{1}{\eta} \cdot \left\{ \sum_{n=1}^K \frac{1}{n} + \sum_{n=1}^\infty \left(-\frac{\gamma^n}{n} \right) + \frac{1}{\gamma^K} \cdot \sum_{n=K+1}^\infty \frac{\gamma^n}{n} \right\}. \quad (24)$$

The first term in braces in (24) is $H(K)$, the second is $\log(1 - \gamma)$, and the third $-R_K(\gamma)$. Thus, (8) is proved. $\square$

4.4. Proof of Corollary 1

If $K = 1$, then $H(K) = 1$. Therefore (11) obviously follows from (9). If $\alpha > 0$, then (10) follows from (8), since $\frac{1}{\beta-\alpha}\cdot(1 - \frac{\beta}{\alpha}) = -\frac{1}{\alpha}$, and for $K = 1$ we have $(\beta/\alpha)^K\cdot R_K(\alpha/\beta) = 1 + (\beta/\alpha)\cdot \log\left(1 - (\alpha/\beta)\right)$. $\square$

4.5. Proof of Corollary 3

If $\beta = 1$, then $L^Z(K) = \boldsymbol{E}\,\tau^Z = T \cdot \sum_{n=1}^K \frac{(n-1)!}{(n+1)!} = T\cdot\frac{K}{K+1}$. $\square$

4.6. Proof of Lemma 2

The equality $\delta(K) = 0$ follows from $Z_0 = K$. From (14) we obtain the relation for the conditional distribution function $F^n(x) = \boldsymbol{P}\{l_T^Z(n) \leqslant x|\mathscr{F}_{\mathfrak{d}(n)}\}$, which is a simple generalization of the well-known Dellacherie theorem, [21]:

$$\frac{dF^n(x)}{1 - F^n(x-)} = n \cdot \frac{\beta}{T - \mathfrak{d}(n) - x} \cdot \mathbb{I}\{x < T - \mathfrak{d}(n)\}\,dx.$$

From this relation it follows that

$$F^n(x) = 1 - \left(\frac{T - \mathfrak{d}(n) - x}{T - \mathfrak{d}(n)}\right)^{n\beta} \cdot \mathbb{I}\{x < T - \mathfrak{d}(n)\}.$$

Therefore, $\boldsymbol{E}\{l_T^Z(n)|\mathscr{F}_{\mathfrak{d}(n)}\} = \int_0^T x\,dF^n(x) = \frac{T - \mathfrak{d}(n)}{n\beta + 1}$, and

$$\lambda^Z(n) = \boldsymbol{E}\{\boldsymbol{E}(l_T^Z(n)|\mathscr{F}_{\mathfrak{d}(n)})\} = \frac{T - \delta(n)}{n\beta + 1}. \tag{25}$$

Thus, from (16) and (25) we have the equality $\delta(n-1) = \frac{T}{n\beta+1} + \delta(n)\cdot\frac{n\beta}{n\beta+1}$, from which it easily follows that

$$\left\{\frac{\beta}{T}\delta(n - 1)\right\} = \frac{n}{n + 1/\beta} \cdot \left(\frac{1}{n} + \left\{\frac{\beta}{T}\delta(n)\right\}\right) \text{ for all } n \in \mathbb{K}. \tag{26}$$

Then we denote $x(n) = \left\{\frac{\beta}{T}\delta(n)\right\}\cdot q(n)$, where the auxiliary function $q(n)$ satisfies the recurrence relation

$$\frac{q(n)}{q(n - 1)} = \frac{n}{n + 1/\beta} \text{ with } q(K) = 1. \tag{27}$$

In this notation,

$$\delta(n) = \frac{T}{\beta} \cdot \frac{x(n)}{q(n)} \text{ for all } n \in \mathbb{K}_0. \tag{28}$$

Thus, from (26) and (27) we obtain the recurrence relation

$$x(n - 1) = q(n)/n + x(n) \text{ for all } n \in \mathbb{K}, \text{ with } x(K) = 0.$$

Hence,

$$x(n) = \sum_{i=n+1}^K q(i)/i \text{ and } q(n) = \frac{n!}{K!} \prod_{j=n+1}^K (j + 1/\beta) \text{ for all } n < K \tag{29}$$

with $x(K) = 0, q(K) = 1$. Therefore, (17) follows from (28) and (29). $\square$

4.7. Proof of Theorem 2

Since $\boldsymbol{E}\,\tau^Z = \delta(0)$, (15) follows from (17) for $n = 0$. $\square$

4.8. Proof of Theorem 3

We define the auxiliary constant

$$r = \boldsymbol{E}\,\psi(n) + \boldsymbol{E}\,\varphi(n) = 1/\rho + \sum_{k=1}^{\infty} L^X(k)\cdot p(k) \tag{30}$$

and the auxiliary function $\theta(t) = \lfloor t/r \rfloor \cdot r$, where $\lfloor \cdot \rfloor$ is the floor function. We note that $\lim_{t \to \infty} \theta(t)/t = 1$ and for $t \geqslant r$

$$\frac{\mu\{\mathscr{W}_t^Y(\omega)\}}{t} = \frac{\mu\{\mathscr{W}_t^Y(\omega)\} - \mu\{\mathscr{W}_{\theta(t)}^Y(\omega)\}}{\theta(t)} \cdot \frac{\theta(t)}{t} + \frac{\mu\{\mathscr{W}_{\theta(t)}^Y(\omega)\}}{\theta(t)} \cdot \frac{\theta(t)}{t}. \tag{31}$$

For the first summand in (31) we have

$$\left(\mu\{\mathscr{W}_t^Y(\omega)\} - \mu\{\mathscr{W}_{\theta(t)}^Y(\omega)\}\right)/\theta(t) \in \left[0, \varphi(\theta(t) + 1)/\theta(t)\right),$$

and $\lim_{t \to \infty} \varphi(\theta(t) + 1)/\theta(t) = 0$ **P**-a.s. For the second summand we receive from the strong law of large numbers,

$$\lim_{t \to \infty} \frac{\mu\{\mathscr{W}_{\theta(t)}^Y(\omega)\}}{\theta(t)} = \frac{1}{r} \cdot \lim_{t \to \infty} \frac{1}{\lfloor t/r \rfloor} \cdot \sum_{i=1}^{\lfloor t/r \rfloor} \varphi(i) = \frac{\boldsymbol{E}\,\varphi(1)}{r} \quad \textbf{P}\text{-a.s.,}$$

and therefore the proof of the proposition follows from (18) and (30). $\square$

5. Discussion

This article is devoted to the development of modeling methods for discrete stochastic systems based on semimartingale descriptions in terms of local time processes. For this it was necessary to formulate (and prove) a number of theoretical results for fairly simple systems. The proposed method is interesting not only because it allows obvious generalizations, but also by the possibilities of simple computer simulation. This is due primarily to the fact that all the descriptions and methods under consideration are trajectory. We also note that the definitions of *JIT* and *WoV* given in the paper are quite general and provide for generalizations of the results considered in the article.

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Стохастические модели систем *точно-в-срок* и *окон уязвимости* в терминах процессов размножения и гибели

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Аннотация

В работе предлагается метод построения моделей на основе анализа процессов размножения и гибели с линейным ростом в семимартингальных терминах. На основе этого метода рассматриваются стохастические модели простых систем *точно-в-срок* (анализируемые в теории продуктивных систем) и *окна уязвимости* (широко обсуждаемые в теории риска). Основные результаты, полученные в работе, представлены в терминах средних значений времени, за которое процессы достигают нулевых значений. При этом рассматриваются и используются при исследовании моделей оценки для локальных времен процессов.

Здесь анализируются простые марковские процессы с линейным ростом интенсивностей (скорость которого, быть может, зависит от времени). При этом полученные и используемые оценки представляют теоретический интерес. Так, например, среднее значение момента, в который процесс достигает нулевого значения, зависит от таких функций параметров модели, как гармоническое число и остаточный член логарифмической функции в разложении Тейлора.

В качестве основного результата предлагается метод математического моделирования систем *точно-в-срок* и *окон уязвимости*. Используемый здесь семимартингальный метод описания следует рассматривать как первый шаг такого моделирования, поскольку, являясь траекторным, он допускает диффузионные (в том числе немарковские) обобщения при построении стохастических моделей *окон уязвимости* и систем *точно-в-срок*. В настоящей работе получены утверждения для средних значений локального времени и моментов достижения процессами размножения и гибели заданного значения. Это позволило единообразно

Научная статья

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представить оценки для моделей системы *точно-в-срок* и для *окон уязвимости* (результат для которых представлен в форме предельной теоремы). Основные результаты сформулированы в виде теорем и лемм. Доказательства используют семимартингальные методы.

Ключевые слова: моделирование, процесс размножения и гибели, момент остановки, компенсатор, интенсивность, считающий процесс, мартингал, траектория, локальное время, точно-в-срок, окно уязвимости.