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Exact solutions to generalized plane Beltrami–Trkal and Ballabh flows

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Abstract

Nonstationary plane flows of a viscous incompressible fluid in a potential field of external forces are considered. An elliptic partial differential equation is obtained, with each solution being a vortex flow stream function described by an exact solution to the Navier–Stokes equations. The obtained solutions generalize the Beltrami–Trkal and Ballabh flows. Examples of such new solutions are given. They are intended to verify numerical algorithms and computer programs.

Keywords: exact solutions to the Navier–Stokes equations, Beltrami–Trkal flow, Ballabh flow.


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Introduction

Starting from the studies by Gromeka and Lamb [1, 2] proposing a new method of writing the Euler equations, a method for integrating the fluid motion equations began to be developed. The essence of this method is the rearrangement of the initial equations to the form convenient for integration. As applied to the Navier–Stokes equations, this line of research is discussed in studies where new forms of writing the equations make it possible to obtain previously unknown invariants and hidden symmetries of the constitutive equations [3–10]. One

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of the methods for representing the motion equation (the Aristov–Pukhnachev method [6, 8, 9]) has been introduced to computational fluid dynamics [8, 9]. The numerical solutions of the Navier–Stokes axisymmetric equations were tested by conventional procedures [8, 9], which took no account of the invariant Helmholtz theorems and their extensions [10].

Few nontrivial exact solutions to the Navier–Stokes equations have been known so far [11–25]. The very notion of exact solution is unsettled and expanding [11]. It seems obvious that the exact solutions to the Navier–Stokes equations, which offer new problem statements in terms of different areas of mathematics, mechanics, and physics [11–25], are of the greatest interest.

The main difficulty in the analytical and numerical integration of the fluid motion equations lies in the absence of a clear relation of pressure to the velocity vector components. The evolutionary equation relating pressure to the velocity components has yet to be known [26]. Let us now illustrate the ensuing difficulties by plane flow. If an arbitrary function is given on a plane and viewed as a stream function, the velocity components calculated via the partial derivatives of this function will satisfy the continuity equation. We then can substitute these components into the Navier–Stokes equation, thus arriving at an equation for determining the gradient of pressure p . However, the rotor of this “gradient” may prove to be nonzero, and it will be impossible to select p .

The above-mentioned difficulties hold true not only for nonstationary flows, but also for stationary ones. Only two examples of formulas relating pressure to velocities are an exception, namely the Bernoulli equation (for an ideal fluid) and the Grad–Shafranov equation [26]. A method for integrating the stationary Euler equations for a very wide class of flows was proposed in [26]. The proposed integration method offered a constitutive equation relating pressure to velocity components. Consequently, the hypothesis of the existence of a universal equation establishing a relation between the hydrodynamic fields must not be rejected.

The attempt to relate velocity to pressure resulted in the development of classes of exact solutions to the Navier–Stokes equations. In [11] there is a summary of known classes of exact solutions to equations of continuum mechanics, which were obtained before the mid-1950s. The Couette [27], Poiseuille [28, 29], Stokes [30], von Karman [31], Hiemenz [32] flows have proved to be so efficient that they have been studied up to now [11, 33–35]. These flow motions have in common that they fall within the class of solutions where velocities depend linearly on a part of coordinates [11]. Linearly increasing velocities described by a complex profile depending, as a rule, on the transverse coordinate is successfully used in various applications [11, 19, 25]. After publication of [11], a survey that discussed and studied this class of solutions, the solubility of the overdetermined nonlinear system of partial differential equations for laminar vertical vortex flows was demonstrated [33–35]. Those studies discussed the extension of the Lin class for magnetic fluid dynamics [12] to the case of convective [36–38] and thermal diffusion [39, 40] flows of a viscous incompressible fluid. Potential flow motions, the Beltrami–Trkal flows [41, 42] and their modifications remain significant in theoretical and experimental fluid dynamics. Note that the Beltrami–Trkal flow had been first studied eight years earlier by Gromeka [43].

Different requirements are imposed on exact solutions, depending on the purpose of use. For example, when the correspondence between a real process and its

mathematical model is verified, an exact solution with “real” boundary and initial conditions is required, i.e. with conditions observable in real circumstances or with conditions technically implementable in a natural experiment. The requirement of “reality” of boundary and initial conditions can be cancelled if one deals with the verification of a numerical algorithm, i.e. with testing its accuracy. In doing so, one checks the difference of the numerical solution of a boundary value problem from the exact one rather than the difference of the numerical solution from the parameters of the real process, whereas it is not necessary for such problems to have the technical implementability of initial and boundary conditions in a natural experiment. The majority of numerical algorithms work with any initial and boundary conditions; therefore, the search for corresponding boundary value problems with a known exact solution can start with a search for the flow parameters satisfying the Navier–Stokes equations, without consideration of any boundary and initial conditions. Then, having chosen a spatial region, we can specify initial and boundary conditions in it and on its boundaries, which are taken from the exact solution. The thus-obtained boundary value problem with a known exact solution is well suited for the verification of numerical algorithms.

This paper proposes an elliptic partial differential equation, each solution of which is the stream function of a vortex flow described by an exact solution of the Navier–Stokes equations. Besides, a method for computing the pressure field for each of such stream functions is proposed.

1. The basic notations and equations of motion

We will now consider the flow of a viscous incompressible fluid in a potential field of mass forces. The notations are as follows (the wave sign above the symbol denotes a dimensional quantity or a vector): $\tilde{\mathbf{V}}$ – velocity, $\tilde{\boldsymbol{\Omega}} = \text{rot}\tilde{\mathbf{V}}$ is vorticity, \tilde{p} is pressure, $\tilde{\rho} = \text{const}$ is density, $\tilde{\Pi}$ is the potential of mass forces, $\tilde{\mu}$ is the coefficient of dynamic viscosity. The fluid motion is described by the Navier–Stokes equations [1, 44] as

$$\frac{\partial}{\partial t} \tilde{\mathbf{V}} + (\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{V}} = \frac{\tilde{\mu}}{\tilde{\rho}} \Delta \tilde{\mathbf{V}} - \nabla \left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{\Pi} \right), \tag{1}$$

$$\text{div} \tilde{\mathbf{V}} = 0. \tag{2}$$

Since $\tilde{\rho} = \text{const}$, it is the sum $(\tilde{p}/\tilde{\rho} + \tilde{\Pi})$ rather than the pressure \tilde{p} and the potential $\tilde{\Pi}$ taken separately that is of interest in the exact solution. This explains the convenience of using the following dimensionless variables: $x = \tilde{x}/\tilde{L}$, $y = \tilde{y}/\tilde{L}$, $t = \tilde{t}\tilde{U}/\tilde{L}$, $\mathbf{V} = \tilde{\mathbf{V}}/\tilde{U}$, $\boldsymbol{\Omega} = \tilde{\boldsymbol{\Omega}}\tilde{L}/\tilde{U}$, $\text{Re} = \tilde{\rho}\tilde{U}\tilde{L}/\tilde{\mu}$, $p = (\tilde{p}/\tilde{\rho} + \tilde{\Pi})/\tilde{U}^2$, where \tilde{L} and \tilde{U} are the characteristic length and velocity values in the flow under study.

2. Plane flows

We denote the velocity components in a rectangular Cartesian coordinate system Oxy by u, v , i.e. $\mathbf{V} = (u, v)$. Then, equations (1), (2) are written as

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = - \frac{\partial}{\partial x} p + \left\{ \frac{1}{\text{Re}} \Delta u - \frac{\partial}{\partial t} u \right\}, \tag{3}$$

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = - \frac{\partial}{\partial y} p + \left\{ \frac{1}{\text{Re}} \Delta v - \frac{\partial}{\partial t} v \right\}, \quad (4)$$

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0. \quad (5)$$

We describe the method of obtaining a family of exact solutions to the system (3)–(5). Consider the Beltrami elliptic differential equation [41] with respect to the function $\psi = \psi(x, y)$:

$$\Delta \psi = \lambda \psi \quad (6)$$

where λ is an arbitrary constant, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

For any solution $\psi = \psi(x, y)$ of equation (6), we assume that

$$u = \left(\frac{\partial}{\partial y} \psi \right) \exp \frac{t\lambda}{\text{Re}}, \quad v = - \left(\frac{\partial}{\partial x} \psi \right) \exp \frac{t\lambda}{\text{Re}}. \quad (7)$$

This representation of the velocity components ensures that the expressions between the curly brackets in the right-hand parts of (3) and (4) are equal to zero. Indeed, according to (6), we have

$$\begin{aligned} \left\{ \frac{1}{\text{Re}} \Delta u - \frac{\partial}{\partial t} u \right\} &= \frac{1}{\text{Re}} \Delta \left[\left(\frac{\partial}{\partial y} \psi \right) \exp \frac{t\lambda}{\text{Re}} \right] - \frac{\partial}{\partial t} \left[\left(\frac{\partial}{\partial y} \psi \right) \exp \frac{t\lambda}{\text{Re}} \right] = \\ &= \frac{1}{\text{Re}} \exp \frac{t\lambda}{\text{Re}} \frac{\partial}{\partial y} (\Delta \psi - \lambda \psi) = \frac{1}{\text{Re}} \exp \frac{t\lambda}{\text{Re}} \frac{\partial}{\partial y} 0 = 0. \end{aligned}$$

Similarly, $\left\{ \frac{1}{\text{Re}} \Delta v - \frac{\partial}{\partial t} v \right\} = 0$. Next, we substitute the expressions from (7) into the left-hand part of (3) and transform it in view of (6) as follows:

$$\begin{aligned} u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u &= \left[\frac{\partial}{\partial y} \psi \frac{\partial^2}{\partial x \partial y} \psi - \frac{\partial}{\partial x} \psi \frac{\partial^2}{\partial y^2} \psi \right] \exp \frac{2t\lambda}{\text{Re}} = \\ &= \left[\frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \psi \right)^2 - \frac{\partial}{\partial x} \psi \left(- \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial x^2} \psi + \frac{\partial^2}{\partial y^2} \psi \right) \right] \exp \frac{2t\lambda}{\text{Re}} = \\ &= \left[\frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \psi \right)^2 + \frac{\partial}{\partial x} \psi \frac{\partial^2}{\partial x^2} \psi - \lambda \psi \frac{\partial}{\partial x} \psi \right] \exp \frac{2t\lambda}{\text{Re}} = \\ &= \frac{1}{2} \frac{\partial}{\partial x} \left[\left(\frac{\partial}{\partial y} \psi \right)^2 + \left(\frac{\partial}{\partial x} \psi \right)^2 - \lambda \psi^2 \right] \exp \frac{2t\lambda}{\text{Re}}. \quad (8) \end{aligned}$$

Similarly, for the left-hand part of (4) we obtain

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = \frac{1}{2} \frac{\partial}{\partial y} \left[\left(\frac{\partial}{\partial y} \psi \right)^2 + \left(\frac{\partial}{\partial x} \psi \right)^2 - \lambda \psi^2 \right] \exp \frac{2t\lambda}{\text{Re}}. \quad (9)$$

Assume that

$$p = p_0 + \frac{1}{2} \left\{ \lambda \psi^2 - \left(\frac{\partial}{\partial y} \psi \right)^2 - \left(\frac{\partial}{\partial x} \psi \right)^2 \right\} \exp \frac{2t\lambda}{\text{Re}}, \quad (10)$$

where p_0 is an arbitrary constant. Then it follows from (8) and (9) that u , v and p satisfy equations (3) and (4). The continuity equation (5) will also be fulfilled, this following directly from (7).

Thus, any solution of any (6) type equation gives birth to the exact solution (7), (10) of the Navier–Stokes equations. Note that $\psi = \psi(x, y)$ is the stream function of such solutions and that the equations $\psi(x, y) = \text{const}$ define the streamlines. Also note that the solutions of equation (6) for the case $\lambda = 0$ correspond to stationary vortex-free motion, which was studied in detail in the complex variable function theory [45]; therefore, flows for $\lambda \neq 0$ are presented in what follows.

The solutions of equations of the form (6) for $\lambda \neq 0$ can be exemplified by the following functions $\psi = \psi(x, y)$:

$$x \cos \beta y, \quad A \cos \alpha x + B \cos \alpha y, \quad \cos \alpha x \sin \beta y, \quad \cos \alpha x \exp \beta y, \\ \cosh \alpha x \cosh \beta y, \quad A \cosh \alpha x + B \cosh \alpha y, \quad \sinh \alpha x \cosh \beta y,$$

where A, B, α, β are arbitrary constants.

This list can be easily continued. Various solutions are obtained, particularly, by the variable separation method. All these solutions of equation (6) offer exact solutions to the Navier–Stokes equations by formulas (7) and (10). The obtained solutions will be nonstationary, but with fixed streamlines. These streamlines coincide with the streamlines of another flow, namely the stationary flow of an ideal incompressible fluid. Indeed, if the non-stationary multiplier $\exp(t\lambda/\text{Re})$ is discarded, the velocity components $u = (\frac{\partial}{\partial y}\psi)$, $v = -(\frac{\partial}{\partial x}\psi)$ and the pressure

$$p = p_0 + \frac{1}{2} \left\{ \lambda \psi^2 - \left(\frac{\partial}{\partial y} \psi \right)^2 - \left(\frac{\partial}{\partial x} \psi \right)^2 \right\}$$

will satisfy the stationary Euler equations [1, 44] for incompressible fluids

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} p, \quad u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = -\frac{\partial}{\partial y} p, \quad \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0.$$

Such (vortex) solutions for the stationary flows of an ideal fluid are a partial case of the solutions obtained in [46]. In that paper, instead of equation (6), for finding the stream function, an equation of a more general form was used, $\Delta\psi = f(\psi)$, where f is an arbitrary function.

3. Exact solution examples

EXAMPLE 1. Consider the function $\psi = \cos x \sin y$. It satisfies equation (6) when $\lambda = -2$. The corresponding exact solution of the Navier–Stokes equations is as follows:

$$\mathbf{V} = (\mathbf{i} \cos x \cos y + \mathbf{j} \sin x \sin y) \cdot \exp\left(-\frac{2t}{\text{Re}}\right), \\ p = p_0 - \frac{1}{2} (\cos x^2 + \sin y^2) \cdot \exp\left(-\frac{4t}{\text{Re}}\right),$$

where \mathbf{i} and \mathbf{j} are the directional vectors of the coordinate axes. The fluid velocity field and the streamlines in the square $[0; \pi] \times [0; \pi]$ are shown in Fig. 1. In this

and all the other figures, the Ox axis is positioned horizontally and the Oy axis is directed vertically. The velocity field \mathbf{V} is shown in the left part of Fig. 1, the length of the arrows being proportional to $|\mathbf{V}|$.

EXAMPLE 2. If we take a stream function with smaller periods along x and y , we will have a flow with a cellular structure. The smaller the stream function periods, the finer the cells. The streamlines for the case $\psi = \cos 4x \sin 4y$ are shown in Fig. 2.

EXAMPLE 3. Consider the function $\psi = A \sin \sqrt{8}y + \cos 2x \sin 2y$. It satisfies equation (6) when $\lambda = -8$ for any value of the constant A . The corresponding exact solution of the Navier–Stokes equations is as follows:

$$\mathbf{V} = \left(\mathbf{i}(A\sqrt{8} \cos \sqrt{8}y + 2 \cos 2x \cos 2y) + \mathbf{j} 2 \sin 2x \sin 2y \right) \cdot \exp\left(-\frac{8t}{\text{Re}}\right), \quad (11)$$

$$p = p_0 - \frac{1}{2} \left\{ 8(A \sin \sqrt{8}y + \cos 2x \sin 2y)^2 + \right.$$

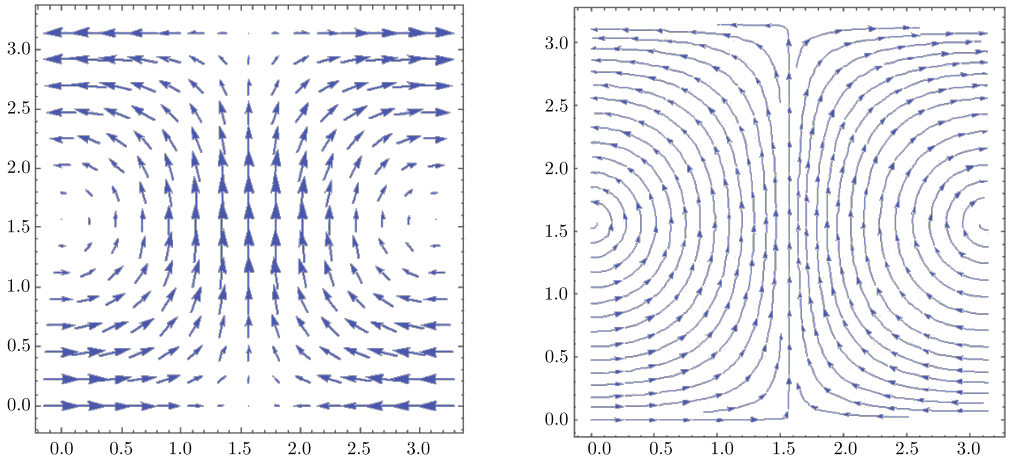


Figure 1. The velocity field and the fixed streamlines of a decaying flow

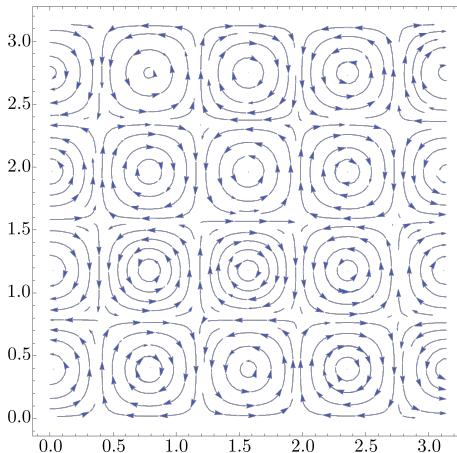


Figure 2. Cellular structure

$$+(A\sqrt{8} \cos \sqrt{8}y + 2 \cos 2x \cos 2y)^2 + (2 \cos 2x \cos 2y)^2 \} \cdot \exp\left(-\frac{16t}{\text{Re}}\right).$$

The velocity field of this flow results from the addition of the flow fields corresponding to the stream functions $\psi_1 = A \sin \sqrt{8}y$ and $\psi_2 = \cos 2x \sin 2y$, each of which satisfies equation (6) when $\lambda = -8$. The function ψ_1 defines the horizontal ($v = 0$) flow; the function ψ_2 defines the flow with square cells ($A = 0$ in Fig. 3). The larger the constant A , the greater the contribution of the horizontal flow. The streamlines of the total flow (11) for four values of A ($A = 0; 0.5; 1/\sqrt{2}; 1$) are shown in Fig. 3.

For three values $A = 0.5; 1/\sqrt{2}; 1$ in Fig. 3, fluid streams with recirculation zones in between are clearly visible. As the contribution of the horizontal flow increases (i.e. with increasing A), the recirculation zones change their shape, with some zones expanding and the other ones shrinking.

Note that all the discussed solutions of the form (7) are valid at any Reynolds

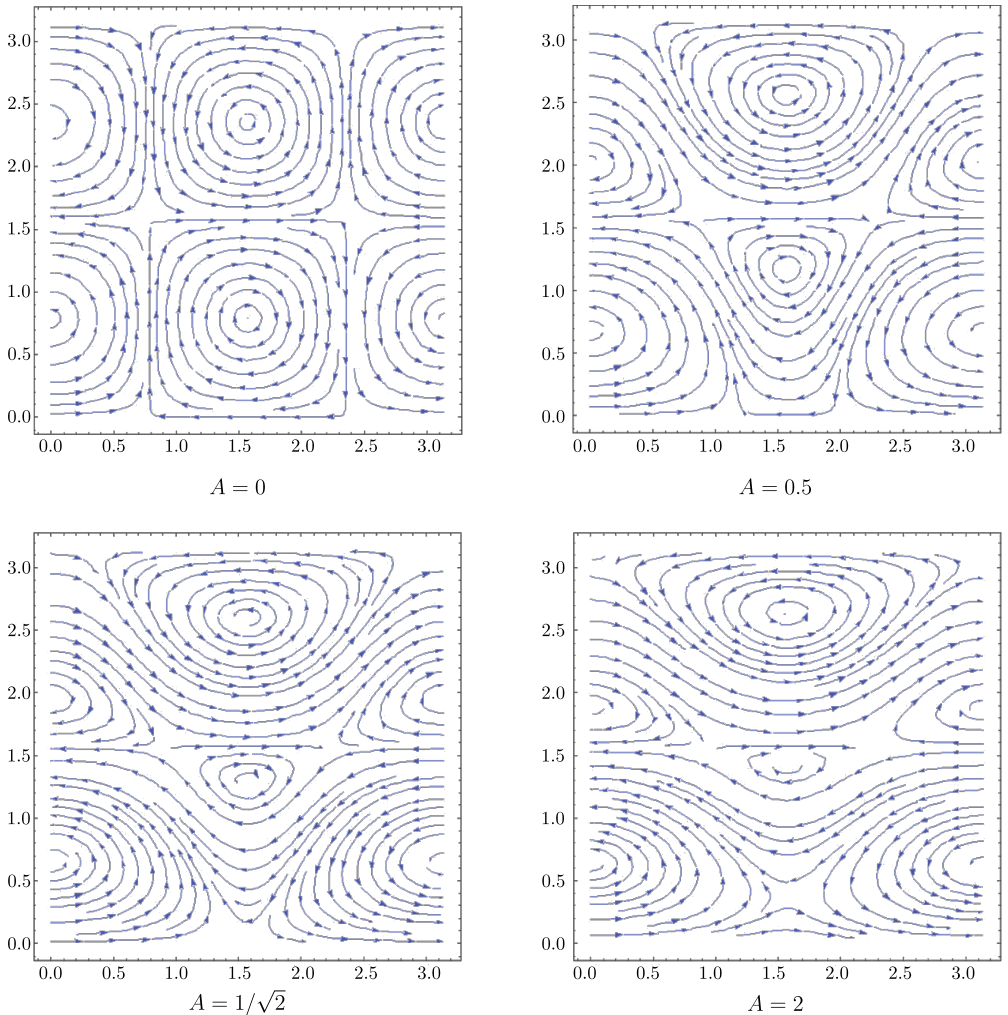


Figure 3. The streamlines of the total flow for four values of the constant A

number, and this makes them advantageous over many previously known exact solutions [11, 19, 25].

4. Summation of the solutions

By virtue of the linearity of equation (6), the velocity fields corresponding to identical λ can be added up to yield a velocity field of another exact solution of the Navier–Stokes equations. And although the pressure field in the obtained “new” flow is not equal to the sum of the “initial” pressure fields, the fact of the possibility of summing the velocities is somewhat unexpected since the nonlinear terms of the Navier–Stokes equations are nonzero in all the flows under study.

The summation of the velocity fields was demonstrated in the previous section (example 3), the streamline patterns were shown for different linear combinations of the stream functions, each satisfying equation (6) when $\lambda = -8$.

If an exact solution is obtained by the here-proposed method, then there is a flow with the same number λ for shear, rotation, and axial symmetry. This is a “source” for obtaining various flow patterns.

Note that the above-mentioned property of the superposition of two flows of a viscous incompressible fluid, which leads to the formation of a new velocity field, was discussed by Ballabh in [47–49]. The condition enabling the superposition of the here-obtained flows is determined by the linearity of equation (6) and the heat conduction type equations

$$\frac{\partial}{\partial t}u = \frac{1}{\text{Re}}\Delta u, \quad \text{and} \quad \frac{\partial}{\partial t}v = \frac{1}{\text{Re}}\Delta v.$$

The solution of these equations is given by (7). The presented condition for obtaining solutions by the superposition method differs from the constraints reported in [47–49].

Conclusion

Plane and nonstationary flows of a viscous incompressible fluid in potential fields of external forces have been considered. These flows are described by the Navier–Stokes equations. A method for constructing boundary value problems with a known exact solution has been proposed and exemplified. The exact solution (7) is special in that the streamlines of the nonstationary flow coincide with the trajectories of the fluid particles and that they also coincide with the streamlines of another flow – the stationary flow of an ideal incompressible fluid. In the solutions proposed in the paper the sum of the nonstationary and viscous terms in the Navier–Stokes vector equation is zero; consequently, the rotor of this sum is zero. In this sense, the solution family (7) extends the Beltrami–Trkal flows to the nonlinear Navier–Stokes equations since in the Beltrami–Trkal flows the rotor of the above-mentioned sum is also zero; this has enabled us to simplify the study of these flows.

The solution class (7) gives a new example of the Ballabh flow, for which the addition of the velocity fields is possible.

The obtained method of integrating the nonstationary Navier–Stokes equations can be applied in computational fluid dynamics to verifying numerical algorithms and computer programs.

Competing interests. I have no competing interests.

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Точные решения обобщенных плоских течений Бельтрами–Тркала и Беллаба

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Аннотация

Рассмотрены плоские нестационарные течения вязкой несжимаемой жидкости в потенциальном поле внешних сил. Получено уравнение в частных производных эллиптического типа, каждое решение которого является функцией тока вихревого течения, описываемого некоторым точным решением уравнений Навье–Стокса. Полученные решения обобщают течения Бельтрами–Тркала и Беллаба. Даны примеры таких новых решений. Они предназначены для верификации численных алгоритмов и компьютерных программ.


Ключевые слова: точные решения уравнений Навье–Стокса, течение Бельтрами–Тркала, течение Беллаба.

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