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An efficient method for the analytical study of linear and nonlinear time-fractional partial differential equations with variable coefficients

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Abstract

The residual power series method is effective for obtaining approximate analytical solutions to fractional-order differential equations. This method, however, requires the derivative to compute the coefficients of terms in a series solution. Other well-known methods, such as the homotopy perturbation, the Adomian decomposition, and the variational iteration methods, need integration. We are all aware of how difficult it is to calculate the fractional derivative and integration of a function. As a result, the use of the methods mentioned above is somewhat constrained. In this research work, approximate and exact analytical solutions to time-fractional partial differential equations with variable coefficients are obtained using the Laplace residual power series method in the sense of the Gerasimov–Caputo fractional derivative. This method helped us overcome the limitations of the various methods. The Laplace residual power series method performs exceptionally well in computing the coefficients of terms in a series solution by applying the straightforward limit principle at infinity, and it is also more effective than various series solution methods due to the avoidance of Adomian

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and He polynomials to solve nonlinear problems. The relative, recurrence, and absolute errors of the three problems are investigated in order to evaluate the validity of our method. The results show that the proposed method can be a suitable alternative to the various series solution methods when solving time-fractional partial differential equations.

Keywords: Laplace transform, residual power series method, partial differential equation, Gerasimov–Caputo derivative.

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List of abbreviations

RPSM	residual power series method
HPM	homotopy perturbation method
ADM	Adomian decomposition method
VIM	variational iteration method
TFPDEs	time-fractional partial differential equations
LRPSM	Laplace residual power series method
GCFD	Gerasimov–Caputo fractional derivative
DEs	differential equations
FODEs	fractional order differential equations
Rel-E	relative error
Abs-E	absolute error
Rec-E	recurrence error
LRF	Laplace residual functions

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1. Introduction. Fractional calculus deals with fractional order derivatives and integrations. Fractional calculus was founded by two mathematicians, Leibniz and l'Hôpital, and its official birthday is September 30, 1695. The widespread usage of fractional calculus in fields including image processing, physics, engineering, biology, biochemistry, entropy theory, and fluid mechanics has attracted a lot of researchers in recent years [1–5]. There are numerous definitions for fractional order derivatives; however, not all of them are regularly applied. The Grünwald–Letnikov, Hadamard derivative, Riemann–Liouville, conformable, and Gerasimov–Caputo fractional derivative (GCFD) are the most well known fractional order derivatives [6–9].

The GCFD of order $\beta > 0$ is given by [10]:

$$D^{\beta}_{\omega}\aleph(\omega) = \begin{cases} \dfrac{1}{\Gamma(r-\beta)}\displaystyle\int_{q}^{\omega}(\omega-q)^{r-\beta-1}\dfrac{d^r}{dq^r}\aleph(q)dq, & r-1<\beta<r, \\ \dfrac{d^r}{d\omega^r}\aleph(\omega), & \beta = r \in \mathbb{N}. \end{cases}$$

The following are the important properties of GCFD:

(i) $D^{\beta}_{\omega}\xi = 0$, $\xi \in \mathbb{R}$,

(ii) $D^{\beta}_{\omega}\omega^{\xi} = \dfrac{\Gamma(\xi+1)}{\Gamma(\xi+1-\beta)}\omega^{\xi-\beta}$, $r-1<\beta\leqslant r$, $\xi > r-1$, $r \in \mathbb{N}$, $\xi \in \mathbb{R}$,

(iii) $D^{\beta}_{\omega}(\xi_1\aleph_1(\omega)+\xi_2\aleph_2(\omega)) = \xi_1 D^{\beta}_{\omega}\aleph_1(\omega) + \xi_2 D^{\beta}_{\omega}\aleph_2(\omega)$.

In some cases, fractional order derivatives are preferable to integer-order derivatives for modeling because they can simulate and examine complex structures with complicated nonlinear processes and higher-order behaviors. There are two main causes of this. First, rather than being limited to an integer order, we can choose any order for the fractional order derivatives. Furthermore, when the mechanism has long-term memory, fractional order derivatives are advantageously based on both past and present situations.

In the disciplines of science and engineering, there are natural and physical phenomena that, when described by mathematical relations, turn into differential equations (DEs). Examples of phenomena that are characterized by DEs include equations of motion, simple harmonic motion, beam deflection, and more. The fractional order differential equations (FODEs) have developed a convenient means of expressing naturally occurring phenomena in artificial intelligence, engineering, physics, earth sciences, bioinformatics, finance systems, and biological systems. Consequently, the study of the approximate or exact solution enables us to recognize the mechanism of these FODEs, and their actual physical intention can be perceived from the graphical depiction of the solution. Applications often come with FODEs that are so complex that exact solutions are frequently impractical. For solving FODEs in the initial circumstances given, methods that give approximate solutions present a potent alternative tool. In recent times, numerous approximate methods for solving FODEs have been presented [11–19].

The residula power series method (RPSM) is a very powerful approach in terms of the construction of power series solutions for FODEs. Many FODEs have been successfully solved by RPSM. Y. Zhang *et al.* [20] used RPSM to develop series solutions of the time-fractional Schrödinger equation. A. El-Ajou *et al.* [21] studied the KdV–Burgers equation to find approximate solutions. R. Saadeh *et al.* [22] discussed the fractional Newell–Whitehead–Segal equation by the RPSM.

Approximate analytical solutions of FODEs can be solved effectively using the RPSM. However, to determine the coefficients of the series solution, the $(n-1)\beta$ derivative of the residual function is required. As we all know, it is difficult to compute the fractional order derivatives of a function. This limits the application of the classic RPSM to a certain extent. To overcome this drawback, T. Eriqat *et al.* [23] introduced a new approach, known as the Laplace residual power series method (LRPSM), which is the coupling of the Laplace transforms and the RPSM for approximate series solutions of linear and nonlinear FODEs. The limit idea is used to establish the series coefficients in the LRPSM. The set of guidelines for this novel approach is based on converting the specified equation into the Laplace transforms space, identifying an expansion solution to the transformed equation, and then obtaining the original equation's solution by using the inverse Laplace transform.

Finding solutions to FODEs with variable coefficients is also an interesting area for researchers. E. Hesameddini and A. Rahimi [24] used the variational iterative approach to solve FODEs with variable coefficients. Y. Keskin *et al.* [25] used the generalized Taylor collection technique to find solutions for higher-order linear FODEs with variable coefficients. S. Sarwar *et al.* [26] found approximate solutions to time-fractional wavelike models with variable coefficients using the definition of the GCFD and with the help of optimal homotopy asymptotic method. With the operational matrix technique, D. Rostamy and K. Karimi [27] established approximate solutions of fractional-order wave and heat equations with variable coefficients. H. Bulut *et al.* [28] used the Sumudu transform method to obtain approximate solutions for partial differential equations with variable coefficients. M. Nadeem *et al.* [29] established numerical solutions for the fourth-order parabolic partial differential equation with variable coefficients using the modified Laplace variational iteration method. M. Dehghan and J. Manafian [30] used the homotopy perturbation technique to find a solution for a fourth-order parabolic problem with variable coefficients. T.M. Elzaki and S.M. Ezaki [31] established solutions for ordinary DEs with variable coefficients using the Elzaki transform method. Each of these strategies has particular limitations and drawbacks. These methods require a lot of work and longer running times.

In this research, the LRPSM was used to solve time-fractional partial diffrential equations (TFPDEs) with variable coefficients in the sense of GCFD. This method combines the Laplace transforms with the RPSM, which is based on a revamped version of Taylor's series and yields a convergent series as a solution. By employing the simple limit principle at infinity, the LRPSM excels at calculating the coefficients of terms in a series solution, but other well-known methods such as the variational iteration method (VIM), Adomian decomposition method (ADM), and homotopy perturbation method (HPM) need integration, while the RPSM needs the derivative, both of which are challenging in fractional contexts. LRPSM is also more effective than various series solution methods due to its small processing size and avoidance of Adomian and He polynomials to solve nonlinear problems. Moreover, this method does not require any assumptions about physical parameters, no matter how big or small, for the problem. Therefore, it can be used to handle both mildly and severely nonlinear problems and to circumvent some of the issues that perturbation techniques previously had. To evaluate the efficiency and consistency of the suggested strategy, the relative error (Rel-E), recurrence error (Rec-E), and absolute error (Abs-E) of the three problems are examined. The findings demonstrate that when solving FODEs, the LRPSM can be a viable substitute for the RPSM, VIM, ADM, and HPM.

To emphasize the essential ideas of our suggested method, such as its dependability, capability, and application, we selected the most prevalent forms of TFPDEs with variable coefficients. J. Fourier proposed the heat equation in 1822, which defines how a certain amount of heat diffuses over a region. Consider the TFPDE with variable coefficients shown below [27]:

$$D_{\omega}^{\kappa\beta}\aleph(\theta,\omega)+\Im(\theta)\Psi(\aleph)-\Xi(\theta,\aleph)=0, \tag{1}$$

subject to the initial condition:

$$D_{\omega}^{\eta\beta}\aleph(\theta,0)=\Phi_{\eta},$$

where $\eta=0,1,2,3\ldots,\kappa-1$; $\theta=(\theta_1,\theta_2,\ldots,\theta_e)\in\mathbb{R}^e$; $\beta\in\left(\frac{\kappa-1}{\kappa}\right]$, $\kappa\in\mathbb{N}$, and $\Psi(\aleph)=\Psi(\aleph, D_{\omega}^{\beta}\aleph, D_{\omega}^{2\beta}\aleph,\ldots,D_{\omega}^{(\kappa-1)\beta}\aleph,\ D_{\theta_1}^{\ell_{11}}\aleph,\ D_{\theta_2}^{\ell_{12}}\aleph,\ldots,D_{\theta_e}^{\ell_{1e}}\aleph,\ldots,D_{\theta_1}^{\ell_{p1}}\aleph,$ $D_{\theta_2}^{\ell_{p2}}\aleph,\ldots,D_{\theta_e}^{\ell_{pe}}\aleph)$, with $h-1<\ell_{hu}\leqslant h$, $h=1,2,\ldots,p$; $u=1,2,\ldots,e$.

The GCFD of ω of order $\eta\beta$ and θ_u of order ℓ_{hu} are represented by $D_{\omega}^{\eta\beta}$ and $D_{\theta_u}^{\ell_{hu}}$, respectively. These kinds of DEs give accurate representations of a wide range of physical events in the fields of fluid dynamics, elastic mechanics, and electrodynamics [26–33].

This study is structured as follows. First, in Section 2, we present significant definitions and findings from FC theory. The LRPSM algorithms for solving TFPDEs with variable coefficients are discussed in Section 3. In Section 4, some linear and nonlinear TFPDE problems are solved to demonstrate the accuracy and reliability of LRPSM. The numerical and graphic results obtained using LRPSM are evaluated in Section 5. Finally, we conclude the paper in Section 6.

2. Preliminaries. In this section, basic definitions, Laplace transform characteristics, and LRPSM-related theorems that help establish approximate series solutions are included.

Definition 1 [34]. The Laplace transforms of $\aleph(\theta,\omega)$ is defined as follows:

$$\mathcal{L}[\aleph(\theta,\omega)]=\Omega(\theta,\nu)=\int_0^{\infty}\aleph(\theta,\omega)e^{-\omega\nu}d\omega,$$

and the inverse Laplace transforms is defined by

$$\mathcal{L}^{-1}[\Omega(\theta,\nu)]=\aleph(\theta,\omega)=\int_{b-i\infty}^{b+i\infty}e^{\nu\omega}\Omega(\theta,\nu)d\nu,\quad b=\mathrm{Re}(\nu)>b_0,$$

where $\theta=(\theta_1,\theta_2,\ldots,\theta_e)\in\mathbb{R}^e$ and $e\in\mathbb{N}$ and b_0 lies in the right half plane of the absolute convergence of the Laplace integral.

Lemma 1 [23]. *Consider that $\aleph_1(\theta,\omega)$ and $\aleph_2(\theta,\omega)$ satisfy the axioms of the existence of Laplace transforms. Suppose that $\mathcal{L}[\aleph_1(\theta,\omega)]=\Omega_1(\theta,\nu)$,, $\mathcal{L}[\aleph_2(\theta,\omega)]=\Omega_2(\theta,\nu)$ and the constants ζ_1, ζ_2. When this occurs, the following criteria are met* [23, 24]:

(i) $\mathcal{L}[\zeta_1\aleph_1(\theta,\omega)+\zeta_2\aleph_2(\theta,\omega)]=\zeta_1\Omega_1(\theta,\nu)+\zeta_2\Omega_2(\theta,\nu)$;

(ii) $\mathcal{L}^{-1}[\zeta_1\Omega_1(\theta,\nu)+\zeta_2\Omega_2(\theta,\nu)]=\zeta_1\aleph_1(\theta,\omega)+\zeta_2\aleph_2(\theta,\omega)$;

(iii) $\xi_0(\theta)=\lim\limits_{\nu\to\infty}\nu\Omega(\theta,\nu)=\aleph(\theta,0)$;

(iv) $\mathcal{L}[D_\omega^\beta\aleph(\theta,\omega)]=\nu^\beta\Omega(\theta,\nu)-\sum\limits_{j=0}^{r-1}\dfrac{\aleph^{(j)}(\theta,0)}{\nu^{j-\beta+1}},\quad r-1<\beta\leqslant r,\ r\in\mathbb{N}$;

(v) $\mathcal{L}[D_\omega^{r\beta}\aleph(\theta,\omega)]=\nu^{r\beta}\Omega(\theta,\nu)-\sum\limits_{j=0}^{r-1}\nu^{\beta(r-j)-1}D_\omega^{j\beta}\aleph(\theta,0),\quad 0<\beta\leqslant 1.$

Definition 2 [35]. The multiple fractional power series is defined as follows:

$$\sum_{r=0}^{\infty}\xi_r(\theta)(\omega-\omega_0)^{r\beta}=\xi_0(\omega-\omega_0)^0+\xi_1(\omega-\omega_0)^\beta+\xi_2(\omega-\omega_0)^{2\beta}+\cdots,$$

where $\theta=(\theta_1,\theta_2,\ldots,\theta_e)\in\mathbb{R}^e$ and $e\in\mathbb{N}$, ω represent a variable and $\xi_r(\theta)$ represent the coefficients of the multiple fractional power series.

We present a new type of multiple fractional power series in the sense of Laplace transform space in the following lemma, which is the main pillar for the LRPSM.

Lemma 2 [23]. *Assume that the multiple fractional power series demonstration is in Laplace transforms space for the function* $\mathcal{L}[\aleph(\theta,\omega)]=\Omega(\theta,\nu)$ *as shown follows*:

$$\Omega(\theta,\nu)=\sum_{r=0}^{\infty}\frac{\xi_r(\theta)}{\nu^{r\beta+1}},\quad \nu>0,$$

where, $\theta=(\theta_1,\theta_2,\ldots,\theta_e)\in\mathbb{R}^e$, $e\in\mathbb{N}$.

Theorem 1 [23]. *The coefficients of the multiple fractional power series can be determined as follows*:

$$\xi_r(\theta)=D_\omega^{r\beta}\aleph(\theta,0),$$

where $D_\omega^{r\beta}=D_\omega^\beta\cdot D_\omega^\beta\cdots D_\omega^\beta$ *(r times)*.

The following theorem establishes the prerequisite for the convergence of the new form of multiple fractional power series.

Theorem 2 [23]. *Let* $\mathcal{L}[\aleph(\theta,\omega)]=\Omega(\theta,\nu)$ *can be denoted as the new form of multiple fractional power series given in Lemma 2. If* $|\nu\mathcal{L}[D_\omega^{(j+1)\beta}\aleph(\theta,\omega)]|\leqslant\mathcal{P}$ *on* $0<\nu\leqslant y$ *with* $0<\beta\leqslant 1$, *then the remainder* $R_j(\theta,\nu)$ *of the multiple fractional power series satisfies the following inequality*:

$$|R_j(\theta,\nu)|\leqslant\frac{\mathcal{P}}{\nu^{(j+1)\beta+1}},\quad 0<\nu\leqslant y.$$

3. LRPSM for Solving TFPDEs with Variable Coefficients. In this section, the set of steps for employing the suggested method to acquire approximative analytical solutions to TFPDEs is addressed. First of all, we apply the Laplace transforms to the TFPDEs with variable coefficients and get an algebraic expression as a result. Then, we take into account the multiple fractional power series as the new space solution for the resulting expression obtained in the first step, which is the fundamental idea behind the LRPSM. The way in which the coefficients of this series are obtained by employing the limit idea is the primary difference between the LRPSM and the RPSM. The resulting implications are then translated into actual space using the inverse Laplace transforms. The rules for using the LRPSM to locate solutions are listed below.

The LRPSM Algorithm for Linear and Nonlinear TFPDEs. We describe the LRPSM's set of guidelines for solving Eq. (1).

Step 1. Reformatted Eq. (1):

$$D_\omega^{\kappa\beta}\aleph(\theta,\omega)+\Im(\theta)\Psi(\aleph)-\Xi(\theta,\aleph)=0. \tag{2}$$

Step 2. Utilize $\mathcal{L}$ on both sides of Eq. (2):

$$\mathcal{L}\big[D_\omega^{\kappa\beta}\aleph(\theta,\omega)+\Im(\theta)\Psi(\aleph)-\Xi(\theta,\aleph)\big]=0. \tag{3}$$

Using the Lemma 1(v) and the initial condition, we get the following from Eq. (3):

$$\Omega(\theta,\nu)=\sum_{j=0}^{\kappa-1}\frac{D_\omega^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}-\frac{\Im(\theta)B(\nu)}{\nu^{j\beta}}+\frac{A(\theta,\nu)}{\nu^{j\beta}}, \tag{4}$$

where $\mathcal{L}[\Xi(\theta,\aleph)]=\mathcal{A}(\theta,\nu)$, $\mathcal{L}[\Psi(\aleph)]=\mathcal{B}(\nu)$.

Step 3. Assume that the multiple fractional power series solution of Eq. (4) in Laplace transforms space is below:

$$\Omega(\theta,\nu)=\sum_{r=0}^{\infty}\frac{\xi_r(\theta)}{\nu^{r\beta+1}},\quad \nu>0.$$

Step 4. As a result of applying the Lemma 1(iii) and Theorem 1 we obtained the following results:

$$\begin{gathered}\xi_0(\theta)=\lim_{\nu\to\infty}\nu\Omega(\theta,\nu)=\aleph(\theta,0),\\ \xi_1(\theta)=D_\omega^{\beta}\aleph(\theta,0),\\ \xi_2(\theta)=D_\omega^{2\beta}\aleph(\theta,0),\\ \xi_\eta(\theta)=D_\omega^{\eta\beta}\aleph(\theta,0).\end{gathered}$$

Step 5. Assume that the jth truncated multiple fractional power series solution of Eq. (4) is below:

$$\Omega_j(\theta,\nu)=\sum_{r=0}^{j}\frac{\xi_r(\theta)}{\nu^{r\beta+1}},\quad \nu>0,$$

$$\Omega_j(\theta,\nu)=\frac{\xi_0(\theta)}{\nu}+\frac{\xi_1(\theta)}{\xi^{\beta+1}}+\cdots+\frac{\xi_\eta(\theta)}{\nu^{\eta\beta+1}}+\sum_{r=\eta+1}^{j}\frac{\xi_r(\theta)}{\nu^{r\beta+1}}.$$

Step 6. The Laplace residual functions (LRF) and jth truncated LRF for the Eq. (4) are defined as follows:

$$\mathcal{L}\operatorname{Res}(\theta,\nu)=\Omega(\theta,\theta)-\sum_{j=0}^{\kappa-1}\frac{D_{\omega}^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}+\frac{\Im(\theta)\mathcal{B}(\nu)}{\nu^{j\beta}}-\frac{\mathcal{A}(\theta,\nu)}{\nu^{j\beta}},$$

and

$$\mathcal{L}\operatorname{Res}_j(\theta,\nu)=\Omega_j(\theta,\nu)-\sum_{j=0}^{\kappa-1}\frac{D_{\omega}^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}+\frac{\Im(\theta)\mathcal{B}(\nu)}{\nu^{j\beta}}-\frac{\mathcal{A}(\theta,\nu)}{\nu^{j\beta}}. \tag{5}$$

Step 7. By inserting the jth truncated multiple fractional power series into the Eq. (5) we get the following:

$$\mathcal{L}\operatorname{Res}_j(\theta,\nu)=\Big(\frac{\xi_0(\theta)}{\nu}+\frac{\xi_1(\theta)}{\xi^{\beta+1}}+\cdots+\frac{\xi_\eta(\theta)}{\nu^{\eta\beta+1}}+\sum_{r=\eta+1}^{j}\frac{\xi_r(\theta)}{\nu^{r\beta+1}}\Big)- \\ -\sum_{j=0}^{\kappa-1}\frac{D_{\omega}^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}+\frac{\Im(\theta)\mathcal{B}(\nu)}{\nu^{j\beta}}-\frac{\mathcal{A}(\theta,\nu)}{\nu^{j\beta}}. \tag{6}$$

Step 8. Multiplying the resulting expression $\mathcal{L}\operatorname{Res}_j(\theta,\nu)$ by $\nu^{j\beta+1}$ on both sides of Eq. (6):

$$\nu^{j\beta+1}\mathcal{L}\operatorname{Res}_j(\theta,\nu)=\nu^{j\beta+1}\Big(\frac{\xi_0(\theta)}{\nu}+\frac{\xi_1(x)}{\xi^{\beta+1}}+\cdots+\frac{\xi_\eta(\theta)}{\nu^{\eta\beta+1}}+\sum_{r=\eta+1}^{j}\frac{\xi_r(\theta)}{\nu^{r\beta+1}}- \\ -\sum_{j=0}^{\kappa-1}\frac{D_{\omega}^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}+\frac{\Im(\theta)\mathcal{B}(\nu)}{\nu^{j\beta}}-\frac{\mathcal{A}(\theta,\nu)}{\nu^{j\beta}}\Big). \tag{7}$$

Step 9. Applying $\lim\limits_{\nu\to\infty}$ on both sides of Eq. (7):

$$\lim_{\nu\to\infty}\nu^{j\beta+1}\mathcal{L}\operatorname{Res}_j(\theta,\nu)=\lim_{\nu\to\infty}\nu^{j\beta+1}\Big(\frac{\xi_0(\theta)}{\nu}+\frac{\xi_1(\theta)}{\xi^{\beta+1}}+\cdots+\frac{\xi_\eta(\theta)}{\nu^{\eta\beta+1}}+ \\ +\sum_{r=\eta+1}^{j}\frac{\xi_r(\theta)}{\nu^{r\beta+1}}-\sum_{j=0}^{\kappa-1}\frac{D_{\omega}^{j}\aleph(\theta,0)}{\nu^{j\beta+1}}+\frac{\Im(\theta)\mathcal{B}(\nu)}{\nu^{j\beta}}-\frac{\mathcal{A}(\theta,\nu)}{\nu^{j\beta}}\Big).$$

Step 10. Solve the following expression for $\xi_j(\theta)$:

$$\lim_{\nu\to\infty}\left(\nu^{j\beta+1}\mathcal{L}\operatorname{Res}_j(\theta,\nu)\right)=0,$$

where $j=\eta+1,\eta+2,\ldots$.

Step 11. To obtain the jth step approximate solution of Eq. (4), substitute the obtained values of $\xi_j(\theta)$ into a jth truncated series of $\Omega(\theta,\nu)$.

Step 12. Apply the $\mathcal{L}^{-1}$ on $\Omega_j(\theta,\nu)$ to attain the jth approximate solution $\aleph_j(\theta,\omega)$.

4. Applications to Linear and Nonlinear TFPDEs with Variable Coefficients. In this section, to show the applicability and reliability of the novel algorithm, three linear and nonlinear TFPDEs are solved.

Problem 1. In our first illustration, we take the nonlinear TFPDE provided below [33]:

$$D_\omega^{2\beta}\aleph(\theta_1,\omega)=\aleph^2(\theta_1,\omega)\frac{\partial^2}{\partial\theta_1^2}\big(\aleph_{\theta_1}(\theta_1,\omega)\aleph_{\theta_1\theta_1}(\theta_1,\omega)\aleph_{\theta_1\theta_1\theta_1}(\theta_1,\omega)\big)+ \\ +\big(\aleph_{\theta_1}(\theta_1,\omega)\big)^2\frac{\partial^2}{\partial\theta_1^2}\big(\aleph_{\theta_1\theta_1}(\theta_1,\omega)\big)^3-18\aleph^5(\theta_1,\omega)+\aleph(\theta_1,\omega), \quad (8)$$

where $0<\beta\leqslant 1$, $\omega\geqslant 0$, $\theta_1\in\mathbb{R}$, and with the following initial conditions:

$$\aleph(\theta_1,0)=\mathrm{e}^{\theta_1},\quad D_\omega^{\beta}\aleph(\theta,0)=\mathrm{e}^{\theta_1}.$$

By applying $\mathcal{L}$ to Eq. (8),

$$\mathcal{L}\big[D_\omega^{2\beta}\aleph(\theta_1,\omega)\big]=\mathcal{L}\Big[\aleph^2(\theta_1,\omega)\frac{\partial^2}{\partial\theta_1^2}\big(\aleph_{\theta_1}(\theta_1,\omega)\aleph_{\theta_1\theta_1}(\theta_1,\omega)\aleph_{\theta_1\theta_1\theta_1}(\theta_1,\omega)\big)+ \\ +\big(\aleph_{\theta_1}(\theta_1,\omega)\big)^2\frac{\partial^2}{\partial\theta^2}\big(\aleph_{\theta_1\theta_1}(\theta_1,\omega)\big)^3-18\aleph^5(\theta_1,\omega)+\aleph(\theta_1,\omega)\Big]. \quad (9)$$

By following the steps that are established in Section 3, we get the following result from Eq. (9):

$$\Omega(\theta_1,\nu)=\frac{\mathrm{e}^{\theta_1}}{\nu}+\frac{\mathrm{e}^{\theta_1}}{\nu^{\beta+1}}+ \\ +\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\big(\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\big)^2\frac{\partial^2}{\partial{\theta_1}^2}\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\times \\ \times\frac{\partial^2}{\partial{\theta_1}^2}\mathcal{L}^{-1}[\Omega(\theta,\nu)]\frac{\partial^3}{\partial{\theta_1}^3}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\Big)\Big]+ \\ +\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\omega)]\Big)^2\frac{\partial^2}{\partial\theta^2}\Big(\frac{\partial^2}{\partial\theta_1^2}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\Big)^3- \\ -18\big(\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\big)^5\Big]+\frac{1}{\nu^{2\beta}}\Omega(\theta_1,\nu). \quad (10)$$

Consider the series solutions of Eq. (10), which have the following form:

$$\Omega(\theta_1,\nu)=\sum_{r=0}^{\infty}\frac{\xi_r(\theta_1)}{\nu^{r\beta+1}},\quad \nu>0.$$

The jth truncated expansion is as

$$\Omega_j(\theta_1,\nu)=\sum_{r=0}^{j}\frac{\xi_r(\theta_1)}{\nu^{r\beta+1}},\quad \nu>0.$$

As a result of applying the Lemma 1(iii) and Theorem 1, we obtained the following results:

$$\lim_{\nu\to\infty}\big(\nu\Omega(\theta_1,\nu)\big)=\aleph(\theta_1,0)=\xi_0(\theta_1)=\mathrm{e}^{\theta_1},\quad \xi_1(\theta_1)=D_\omega^\beta\aleph(\theta_1,0)=\mathrm{e}^{\theta_1}.$$

So, Eq. (10) becomes as follows:

$$\Psi_j(\theta_1,\nu)=\frac{\mathrm{e}^{\theta_1}}{\nu}+\frac{\mathrm{e}^{\theta_1}}{\nu^{\beta+1}}+\sum_{r=2}^{j}\frac{\xi_r(\theta_1)}{\nu^{r\beta+1}}. \tag{11}$$

The following is the LRF for Eq. (10):

$$\begin{aligned}\mathcal{L}\,\mathrm{Res}(\theta_1,\nu)&=\Omega(\theta_1,\nu)-\frac{\mathrm{e}^{\theta_1}}{\nu}-\frac{\mathrm{e}^{\theta_1}}{\nu^{\beta+1}}-\\&-\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\big(\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\big)^2\frac{\partial^2}{\partial{\theta_1}^2}\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\times\\&\times\frac{\partial^2}{\partial{\theta_1}^2}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\frac{\partial^3}{\partial{\theta_1}^3}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\Big)\Big]-\\&-\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\omega)]\Big)^2\frac{\partial^2}{\partial\theta_1^2}\Big(\frac{\partial^2}{\partial\theta_1^2}\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\Big)^3-\\&-18\Big(\mathcal{L}^{-1}[\Omega(\theta_1,\nu)]\Big)^5\Big]-\frac{1}{\nu^{2\beta}}\Omega(\theta_1,\nu).\end{aligned}$$

The following is the jth truncated LRF for Eq. (10):

$$\begin{aligned}\mathcal{L}\,\mathrm{Res}_j(\theta_1,\nu)&=\Omega_j(\theta_1,\nu)-\frac{\mathrm{e}^{\theta_1}}{\nu}-\frac{\mathrm{e}^{\theta_1}}{\nu^{\beta+1}}-\\&-\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\big(\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\big)^2\frac{\partial^2}{\partial{\theta_1}^2}\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\times\\&\times\frac{\partial^2}{\partial{\theta_1}^2}\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\frac{\partial^3}{\partial{\theta_1}^3}\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\Big)\Big]-\\&-\frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\Big(\frac{\partial}{\partial\theta_1}\mathcal{L}^{-1}[\Omega_j(\theta_1,\omega)]\Big)^2\frac{\partial^2}{\partial\theta_1^2}\Big(\frac{\partial^2}{\partial\theta_1^2}\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\Big)^3-\\&-18\Big(\mathcal{L}^{-1}[\Omega_j(\theta_1,\nu)]\Big)^5\Big]-\frac{1}{\nu^{2\beta}}\Omega_j(\theta_1,\nu).\end{aligned} \tag{12}$$

In Eq. (11) and Eq. (12), use $j=\overline{1,7}$ to find the undetermined coefficients $\xi_j(\theta_1,\theta_2)$, and then solve the following:

$$\lim_{\nu\to\infty}\big(\nu^{j\beta+1}\mathcal{L}\,\mathrm{Res}_j(\theta_1,\nu)\big)=0.$$

The following outcomes obtained:

$$\xi_2(\theta_1) = \mathrm{e}^{\theta_1}, \quad \xi_3(\theta_1) = \mathrm{e}^{\theta_1}, \quad \xi_4(\theta_1) = \mathrm{e}^{\theta_1},$$

$$\xi_5(\theta_1) = \mathrm{e}^{\theta_1}, \quad \xi_6(\theta_1) = \mathrm{e}^{\theta_1}, \quad \xi_7(\theta_1) = \mathrm{e}^{\theta_1}.$$

In this way, we obtained the following 7th step approximate solution of Eq. (10) in Laplace transforms space:

$$\Omega_7(\theta_1,\nu) = \mathrm{e}^{\theta_1}\Big(\frac{1}{\nu} + \frac{1}{\nu^{\beta+1}} + \frac{1}{\nu^{2\beta+1}} + \frac{1}{\nu^{3\beta+1}} + \\ + \frac{1}{\nu^{4\beta+1}} + \frac{1}{\nu^{5\beta+1}} + \frac{1}{\nu^{6\beta+1}} + \frac{1}{\nu^{7\beta+1}}\Big). \quad (13)$$

By applying $\mathcal{L}^{-1}$ to Eq. (13), we obtain the approximate 7th step solution in the original space, which has the following form:

$$\aleph_7(\theta_1,\omega) = \mathrm{e}^{\theta_1}\Big(1 + \frac{\omega^{\beta}}{\Gamma[\beta+1]} + \frac{\omega^{2\beta}}{\Gamma[2\beta+1]} + \frac{\omega^{3\beta}}{\Gamma[3\beta+1]} + \\ + \frac{\omega^{4\beta}}{\Gamma[4\beta+1]} + \frac{\omega^{5\beta}}{\Gamma[5\beta+1]} + \frac{\omega^{6\beta}}{\Gamma[6\beta+1]} + \frac{\omega^{7\beta}}{\Gamma[7\beta+1]}\Big).$$

When $\beta = 1.0$, the 7th step approximate solution is

$$\aleph_7(\theta_1,\omega) = \mathrm{e}^{\theta_1}\Big(1 + \frac{\omega}{\Gamma[2]} + \frac{\omega^2}{\Gamma[3]} + \frac{\omega^3}{\Gamma[4]} + \frac{\omega^4}{\Gamma[5]} + \frac{\omega^5}{\Gamma[6]} + \frac{\omega^6}{\Gamma[7]} + \frac{\omega^7}{\Gamma[8]}\Big). \quad (14)$$

The first eight terms of the series of the exact solution to $\aleph(\theta,\omega) = \mathrm{e}^{\theta_1+\omega}$ are represented by Eq. (14).

PROBLEM 2. In our second illustration, we take the linear TFPDE provided below [32]:

$$D_{\omega}^{2\beta}\aleph(\theta_1,\theta_2,\theta_3,\omega) = \frac{1}{2}\theta_1^2\aleph_{\theta_1\theta_1}(\theta_1,\theta_2,\theta_3,\omega) + \frac{1}{2}\theta_2^2\aleph_{\theta_2\theta_2}(\theta_1,\theta_2,\theta_3,\omega) + \\ + \frac{1}{2}\theta_3^2\aleph_{\theta_3\theta_3}(\theta_1,\theta_2,\theta_3,\omega) + \theta_1^2 + \theta_2^2 + \theta_3^2, \quad (15)$$

where $0 < \beta \leqslant 1$, $(\theta_1,\theta_2,\theta_3,\omega) \in \left(\mathbb{R}^+\right)^4$, and with the following initial conditions:

$$\aleph(\theta_1,\theta_2,\theta_3,0) = 0, \quad D_{\omega}^{\beta}\aleph(\theta_1,\theta_2,\theta_3,0) = \theta_1^2 + \theta_2^2 - \theta_3^2.$$

By applying $\mathcal{L}$ to Eq. (15),

$$\mathcal{L}[D_{\omega}^{2\beta}\aleph(\theta_1,\theta_2,\theta_3,\omega)] = \mathcal{L}\Big[\frac{1}{2}\theta_1^2\aleph_{\theta_1\theta_1}(\theta_1,\theta_2,\theta_3,\omega) + \frac{1}{2}\theta_2^2\aleph_{\theta_2\theta_2}(\theta_1,\theta_2,\theta_3,\omega) + \\ + \frac{1}{2}\theta_3^2\aleph_{\theta_3\theta_3}(\theta_1,\theta_2,\theta_3,\omega) + \theta_1^2 + \theta_2^2 + \theta_3^2\Big], \quad (16)$$

By following the steps that are established in Section 3, we get the following result from Eq. (16):

$$\Omega(\theta_1,\theta_2,\theta_3,\nu)=\frac{1}{\nu^{\beta+1}}(\theta_1^2+\theta_2^2-\theta_3^2)+\frac{\theta_1^2}{2\nu^{2\beta}}D_{\theta_1\theta_1}\Omega(\theta_1,\theta_2,\theta_3,\nu)+ \\ +\frac{\theta_2^2}{2\nu^{2\beta}}D_{\theta_2\theta_2}\Omega(\theta_1,\theta_2,\theta_3,\omega)+\frac{\theta_3^2}{2\nu^{2\beta}}D_{\theta_3\theta_3}\Omega(\theta_1,\theta_2,\theta_3,\omega)+ \\ +\frac{1}{\nu^{2\beta+1}}(\theta_1^2+\theta_2^2+\theta_3^2). \quad (17)$$

Consider the series solutions of Eq. (17), which have the following form:

$$\Omega(\theta_1,\theta_2,\theta_3,\nu)=\sum_{r=0}^{\infty}\frac{\xi_r(\theta_1,\theta_2,\theta_3)}{\nu^{r\beta+1}},\quad \nu>0.$$

The jth truncated series is as

$$\Omega_j(\theta_1,\theta_2,\theta_3,\nu)=\sum_{r=0}^{j}\frac{\xi_r(\theta_1,\theta_2,\theta_3)}{\nu^{r\beta+1}},\ \nu>0. \quad (18)$$

As a result of applying the Lemma 1(iii) and Theorem1, we obtained the following results:

$$\lim_{\nu\to\infty}\big(\nu\Omega(\theta_1,\theta_2,\theta_3,\nu)\big)=\xi_0(\theta_1,\theta_2,\theta_3)=\aleph(\theta_1,\theta_2,\theta_3,0)=0,$$
$$\xi_1(\theta_1,\theta_2,\theta_3)=D_\omega^\beta\aleph(\theta_1,\theta_2,\theta_3,0)=\theta_1^2+\theta_2^2-\theta_3^2.$$

So, Eq. (18) becomes as follows:

$$\Omega_j(\theta_1,\theta_2,\theta_3,\nu)=\frac{(\theta_1^2+\theta_2^2-\theta_3^2)}{\nu^{\beta+1}}+\sum_{r=2}^{\kappa}\frac{\xi_r(\theta_1,\theta_2,\theta_3)}{\nu^{r\beta+1}},\quad \nu>0. \quad (19)$$

The following is the LRF for Eq. (17):

$$L\,\mathrm{Res}(\theta_1,\theta_2,\theta_3,\nu)=\Omega(\theta_1,\theta_2,\theta_3,\nu)-\frac{1}{\nu^{\beta+1}}(\theta_1^2+\theta_2^2-\theta_3^2)- \\ -\frac{\theta_1^2}{2\nu^{2\beta}}D_{\theta_1\theta_1}\Omega(\theta_1,\theta_2,\theta_3,\nu)-\frac{\theta_2^2}{2\nu^{2\beta}}D_{\theta_2\theta_2}\Omega(\theta_1,\theta_2,\theta_3,\nu)- \\ -\frac{\theta_3^2}{2\nu^{2\beta}}D_{\theta_3\theta_3}\Omega(\theta_1,\theta_2,\theta_3,\nu)-\frac{1}{\nu^{2\beta+1}}(\theta_1^2+\theta_2^2+\theta_3^2).$$

The following is the jth truncated LRF for Eq. (17):

$$\mathcal{L}\,\mathrm{Res}_j(\theta_1,\theta_2,\theta_3,\nu)=\Omega_j(\theta_1,\theta_2,\theta_3,\nu)-\frac{1}{\nu^{\beta+1}}(\theta_1^2+\theta_2^2-\theta_3^2)- \\ -\frac{\theta_1^2}{2\nu^{2\beta}}D_{\theta_1\theta_2}\Omega_j(\theta_1,\theta_2,\theta_3,\nu)-\frac{\theta_2^2}{2\nu^{2\beta}}D_{\theta_2\theta_2}\Omega_j(\theta_1,\theta_2,\theta_3,\nu)-$$

$$- \frac{\theta_3^2}{2\nu^{2\beta}} D_{\theta_3\theta_3}\Omega_j(\theta_1,\theta_2,\theta_3,\nu) - \frac{1}{\nu^{2\beta+1}}(\theta_1^2+\theta_2^2+\theta_3^2). \quad (20)$$

In Eq. (19) and Eq. (20), use $j = \overline{1,7}$ to find the undetermined coefficients $\xi_j(\theta_1,\theta_2)$, and then solve the following:

$$\lim_{\nu\to\infty}\left(\nu^{j\beta+1}\mathcal{L}\,\mathrm{Res}_j(\theta_1,\theta_2,\theta_3,\nu)\right) = 0.$$

The following outcomes obtained:

$$\begin{aligned}
&\xi_2(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2+\theta_3^2, && \xi_3(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2-\theta_3^2,\\
&\xi_4(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2+\theta_3^2, && \xi_5(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2-\theta_3^2,\\
&\xi_6(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2+\theta_3^2, && \xi_7(\theta_1,\theta_2,\theta_3) = \theta_1^2+\theta_2^2-\theta_3^2.
\end{aligned}$$

In this way, we obtained the following 7th step approximate solution of Eq. (17) in Laplace transforms space:

$$\Omega_7(\theta_1,\theta_2,\theta_3,\nu) = \frac{\theta_1^2+\theta_2^2-\theta_3^2}{\nu^{\beta+1}} + \frac{\theta_1^2+\theta_2^2+\theta_3^2}{\nu^{2\beta+1}} + \frac{\theta_1^2+\theta_2^2-\theta_3^2}{\nu^{3\beta+1}} + \\ + \frac{\theta_1^2+\theta_2^2+\theta_3^2}{\nu^{4\beta+1}} + \frac{\theta_1^2+\theta_2^2-\theta_3^2}{\nu^{5\beta+1}} + \frac{\theta_1^2+\theta_2^2+\theta_3^2}{\nu^{6\beta+1}} + \frac{\theta_1^2+\theta_2^2-\theta_3^2}{\nu^{7\beta+1}}. \quad (21)$$

By applying $\mathcal{L}^{-1}$ to Eq. (21), we obtain the approximate 7th step solution in the original space, which has the following form:

$$\aleph_7(\theta_1,\theta_2,\theta_3,\omega) = \\ = (\theta_1^2+\theta_2^2-\theta_3^2)\left(\frac{\omega^{\beta}}{\Gamma[\beta+1]} + \frac{\omega^{3\beta}}{\Gamma[3\beta+1]} + \frac{\omega^{5\beta}}{\Gamma[5\beta+1]} + \frac{\omega^{7\beta}}{\Gamma[7\beta+1]}\right) + \\ + (\theta_1^2+\theta_2^2+\theta_3^2)\left(\frac{\omega^{2\beta}}{\Gamma[2\beta+1]} + \frac{\omega^{4\beta}}{\Gamma[4\beta+1]} + \frac{\omega^{6\beta}}{\Gamma[6\beta+1]}\right).$$

When $\beta = 1.0$, the 7th step approximate solution is

$$\aleph_7(\theta_1,\theta_2,\theta_3,\omega) = (\theta_1^2+\theta_2^2-\theta_3^2)\left(\frac{\omega}{\Gamma[2]} + \frac{\omega^3}{\Gamma[4]} + \frac{\omega^5}{\Gamma[6]} + \frac{\omega^7}{\Gamma[8]}\right) + \\ + (\theta_1^2+\theta_2^2+\theta_3^2)\left(\frac{\omega^2}{\Gamma[3]} + \frac{\omega^4}{\Gamma[5]} + \frac{\omega^6}{\Gamma[7]}\right). \quad (22)$$

The first seven terms of the series of the exact solution to

$$\aleph(\theta_1,\theta_2,\theta_3,\omega) = (\theta_1^2+\theta_2^2-\theta_3^2)\sinh\omega + (\theta_1^2+\theta_2^2+\theta_3^2)(\cosh\omega - 1)$$

are represented by Eq. (22).

Problem 3. In our third illustration, we take the nonlinear TFPDE provided below [33]:

$$D_\omega^{2\beta}\aleph(\theta_1,\theta_2,\omega) = \frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\aleph_{\theta_1\theta_1}(\theta_1,\theta_2,\omega)\aleph_{\theta_2\theta_2}(\theta_1,\theta_2,\omega)\big) - \\ - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\theta_1\theta_2\aleph_{\theta_1}(\theta_1,\theta_2,\omega)\aleph_{\theta_2}(\theta_1,\theta_2,\omega)\big) - \aleph(\theta_1,\theta_2,\omega), \quad (23)$$

where $0<\beta\leqslant 1$, $(\theta_1,\theta_2,\omega)\in(\mathbb{R}^+)^3$, and with the following initial conditions:

$$\aleph(\theta_1,\theta_2,0) = \mathrm{e}^{\theta_1\theta_2}, \quad D_\omega^\beta\aleph(\theta_1,\theta_2,0) = \mathrm{e}^{\theta_1\theta_2}.$$

By applying $\mathcal{L}$ to Eq. (23),

$$\mathcal{L}[D_\omega^{2\beta}\aleph(\theta_1,\theta_2,\omega) = \mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\aleph_{\theta_1\theta_1}(\theta_1,\theta_2,\omega)\aleph_{\theta_2\theta_2}(\theta_1,\theta_2,\omega)\big) - \\ - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\theta_1\theta_2\aleph_{\theta_1}(\theta_1,\theta_2,\omega)\aleph_{\theta_2}(\theta_1,\theta_2,\omega)\big) - \aleph(\theta_1,\theta_2,\omega)\Big]. \quad (24)$$

By following the steps that are established in Section 3, we get the following result from Eq. (24):

$$\Omega(\theta_1,\theta_2,\nu) = \frac{\mathrm{e}^{\theta_1\theta_2}}{\nu} + \frac{\mathrm{e}^{\theta_1\theta_2}}{\nu^{\beta+1}} + \\ + \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(D_{\theta_1\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]D_{\theta_2\theta_2}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]\big)\Big] - \\ - \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\theta_1\theta_2D_{\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]D_{\theta_2}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]\big)\Big] - \\ - \frac{1}{\nu^{2\beta}}\Omega(\theta_1,\theta_2,\nu). \quad (25)$$

Consider the series solutions of Eq. (25), which have the following form:

$$\Omega(\theta_1,\theta_2,\nu) = \sum_{r=0}^{\infty}\frac{\xi_r(\theta_1,\theta_2)}{\nu^{r\beta+1}}, \quad \nu>0.$$

The jth truncated expansion is as

$$\Omega_j(\theta_1,\theta_2,\nu) = \sum_{r=0}^{j}\frac{\xi_r(\theta_1,\theta_2)}{\nu^{r\beta+1}}, \quad \nu>0.$$

As a result of applying the Lemma 1(iii) and Theorem 1, we obtained the following results:

$$\lim_{\nu\to\infty}\big(\nu\Omega(\theta_1,\theta_2,\nu)\big) = \aleph(\theta_1,\theta_2,0) = \xi_0(\theta_1) = \mathrm{e}^{\theta_1\theta_2}, \xi_1(\theta_1) = D_\omega^\beta\aleph(\theta_1,\theta_2,0) = \mathrm{e}^{\theta_1\theta_2}.$$

So, Eq. (26) becomes

$$\Omega_j(\theta_1,\theta_2,\nu) = \frac{e^{\theta_1\theta_2}}{\nu} + \frac{e^{\theta_1\theta_2}}{\nu^{\beta+1}} + \sum_{r=2}^{j} \frac{\xi_r(\theta_1,\theta_2)}{\nu^{r\beta+1}}, \quad \nu > 0. \tag{26}$$

The following is the LRF for Eq. (25):

$$\begin{aligned}\mathcal{L}\,\mathrm{Res}(\theta_1,\theta_2,\nu) &= \Omega(\theta_1,\theta_2,\nu) - \frac{e^{\theta_1\theta_2}}{\nu} - \frac{e^{\theta_1\theta_2}}{\nu^{\beta+1}} - \\ &- \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(D_{\theta_1\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]D_{\theta_2\theta_2}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]\big)\Big] + \\ &+ \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\theta_1\theta_2 D_{\theta_1}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]D_{\theta_2}\mathcal{L}^{-1}[\Omega(\theta_1,\theta_2,\nu)]\big)\Big] + \\ &+ \frac{1}{\nu^{2\beta}}\Omega(\theta_1,\theta_2,\nu).\end{aligned}$$

The following is the jth truncated LRF for Eq. (25):

$$\begin{aligned}\mathcal{L}\,\mathrm{Res}_j(\theta_1,\theta_2,\nu) &= \Omega_j(\theta_1,\theta_2,\nu) - \frac{e^{\theta_1\theta_2}}{\nu} - \frac{e^{\theta_1\theta_2}}{\nu^{\beta+1}} - \\ &- \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(D_{\theta_1\theta_1}\mathcal{L}^{-1}[\Omega_j(\theta_1,\theta_2,\nu)]D_{\theta_2\theta_2}\mathcal{L}^{-1}[\Omega_j(\theta_1,\theta_2,\nu)]\big)\Big] + \\ &+ \frac{1}{\nu^{2\beta}}\mathcal{L}\Big[\frac{\partial^2}{\partial\theta_1\partial\theta_2}\big(\theta_1\theta_2 D_{\theta_1}\mathcal{L}^{-1}[\Omega_j(\theta_1,\theta_2,\nu)]D_{\theta_2}\mathcal{L}^{-1}[\Omega_j(\theta_1,\theta_2,\nu)]\big)\Big] + \\ &+ \frac{1}{\nu^{2\beta}}\Omega_j(\theta_1,\theta_2,\nu).\end{aligned} \tag{27}$$

In Eq. (26) and Eq. (27), use $j = \overline{1,7}$ to find the undetermined coefficients $\xi_j(\theta_1,\theta_2)$, and then solve the following:

$$\lim_{\nu\to\infty}\big(\nu^{j\beta+1}\mathcal{L}\,\mathrm{Res}_j(\theta_1,\theta_2,\nu)\big) = 0.$$

The following outcomes obtained:

$$\begin{aligned}&\xi_2(\theta_1,\theta_2) = -e^{\theta_1\theta_2}, \quad \xi_3(\theta_1,\theta_2) = -e^{\theta_1\theta_2}, \quad \xi_4(\theta_1,\theta_2) = e^{\theta_1\theta_2},\\ &\xi_5(\theta_1,\theta_2) = e^{\theta_1\theta_2}, \quad \xi_6(\theta_1,\theta_2) = -e^{\theta_1\theta_2}, \quad \xi_7(\theta_1,\theta_2) = -e^{\theta_1\theta_2}.\end{aligned}$$

In this way, we obtained the following 7th step approximate solution of Eq. (25) in Laplace transforms space:

$$\begin{aligned}\Omega_7(\theta_1,\theta_2,\nu) = e^{\theta_1\theta_2}\Big(&\frac{1}{\nu} + \frac{1}{\nu^{\beta+1}} - \frac{1}{\nu^{2\beta+1}} - \frac{1}{\nu^{3\beta+1}} + \\ &+ \frac{1}{\nu^{4\beta+1}} + \frac{1}{\nu^{5\beta+1}} - \frac{1}{\nu^{6\beta+1}} - \frac{1}{\nu^{7\beta+1}}\Big).\end{aligned} \tag{28}$$

By applying $\mathcal{L}^{-1}$ to Eq. (28), we obtain the approximate 7th step solution in the original space, which has the following form:

$$\aleph_7(\theta_1,\theta_2,\omega) = \mathrm{e}^{\theta_1\theta_2}\Big(1 + \frac{\omega^{\beta}}{\Gamma[\beta+1]} - \frac{\omega^{2\beta}}{\Gamma[2\beta+1]} - \frac{\omega^{3\beta}}{\Gamma[3\beta+1]} + \frac{\omega^{4\beta}}{\Gamma[4\beta+1]} + + \frac{\omega^{5\beta}}{\Gamma[5\beta+1]} - \frac{\omega^{6\beta}}{\Gamma[6\beta+1]} - \frac{\omega^{7\beta}}{\Gamma[7\beta+1]}\Big).$$

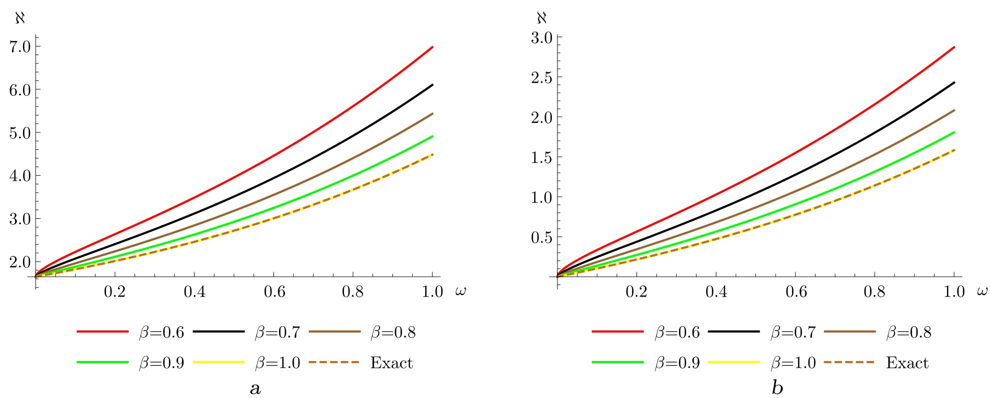
When $\beta = 1.0$, the 7th step approximate solution is

$$\aleph_7(\theta_1,\theta_2,\omega) = \mathrm{e}^{\theta_1\theta_2}\Big(1 + \frac{\omega}{\Gamma[2]} - \frac{\omega^2}{\Gamma[3]} - \frac{\omega^3}{\Gamma[4]} + \frac{\omega^4}{\Gamma[5]} + \frac{\omega^5}{\Gamma[6]} - \frac{\omega^6}{\Gamma[7]} - \frac{\omega^7}{\Gamma[8]}\Big). \quad (29)$$

The first eight terms of the series of the exact solution to $\mathrm{e}^{\theta_1\theta_2}(\cos\omega + \sin\omega)$ are represented by Eq. (29).

5. Numerical Simulation and Discussion. In this section, the results of the approximate and exact solutions to the problems are examined graphically and numerically. Error functions can be used to evaluate the accuracy of the approximate analytical approach, so it is necessary to specify the errors in the approximate solutions provided by the LRPSM. We used the Abs-E, Rel-E, and Rec-E functions to demonstrate the accuracy and efficiency of LRPSM.

Fig. 1 depicts the 2D graphs of the comparative study of the exact and approximate solutions obtained by the proposed method in Problems 1–3. These figures show the 2D plots of the exact and 7th step approximate solutions attained by LRPSM for Problems 1–3, when β = 0.6, 0.7, 0.8, 0.9, and 1.0 in the range



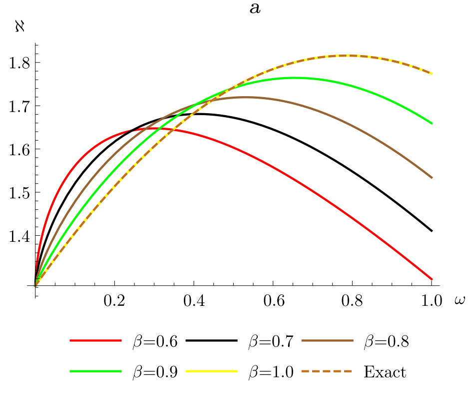


Figure 1. The approximate and exact results of $\aleph(\theta_1,\omega)$ (a), $\aleph(\theta_1,\theta_2,\theta_3,\omega)$ (b), and $\aleph(\theta_1,\theta_2,\omega)$ (c) for various amounts of β in the range $\omega \in [0, 1.0]$, when $\theta_1 = 0.5$ (b), $\theta_1 = \theta_2 = \theta_3 = 0.5$ (b), and $\theta_1 = \theta_2 = 0.5$ (c)

$\omega \in [0, 1.0]$. These graphs indicate that when $\beta \to 1.0$ is applied, the approximative solution converges to the exact solution. The exact and approximation solutions overlap at $\beta = 1.0$, demonstrating the accuracy and reliability of the proposed method.

Fig. 2 shows the Abs-E of the proposed method's 7th step approximate and exact solutions to Problems 1–3 in the range $\omega \in [0, 0.5]$ for $\beta = 1.0$.

Fig. 3 depicts a comparison of the Rel-E of the exact and 7th step approximate solutions to problems $1 - 3$ with $\beta = 1.0$ in the range $\omega \in [0, 1.0]$. The graphical analysis of the approximate and exact findings in the form of Abs-E and Rel-E demonstrates the reliability and precision of LRPSM.

Fig. 4 depicts the comparison study using the 3D plots in terms of the Abs-E of the approximate finding from the seven iterations and the exact result found using the suggested method to Problems 1–3, respectively, at $\beta = 1.0$ in the ranges $\omega \in [0, 0.5]$ and $\theta \in [0, 0.5]$.

Fig. 5 depicts the comparison study using the 3D curve in terms of the Rel-E of the approximate finding from the seven iterations and the exact result found using the suggested method to Problems 1–3, respectively, at $\beta = 1.0$ in the range $\omega \in [0, 0.5]$ and $\theta \in [0, 0.5]$.

The study has revealed that the 7th step approximate solutions of the proposed method are very similar to the exact solution. The reliability and accuracy of LRPSM is demonstrated by graphical analysis of approximate and exact results in the form of Abs-E and Rel-E.

Tables 1–3 show Abs-E and Rel-E in the range $\omega \in [0, 1.0]$ between the approximate solution obtained from the seven iterations and the exact solution obtained by LRPSM at $\beta = 1.0$ for appropriately chosen values. The amplitudes of Abs-E and Rel-E are shown in Table 1 which range from $2.29781 \cdot 10^{-11}$ to $3.04568 \cdot 10^{-3}$ and from $2.07914 \cdot 10^{-11}$ to $4.55539 \cdot 10^{-4}$, respectively, for Problem 1. The inter-

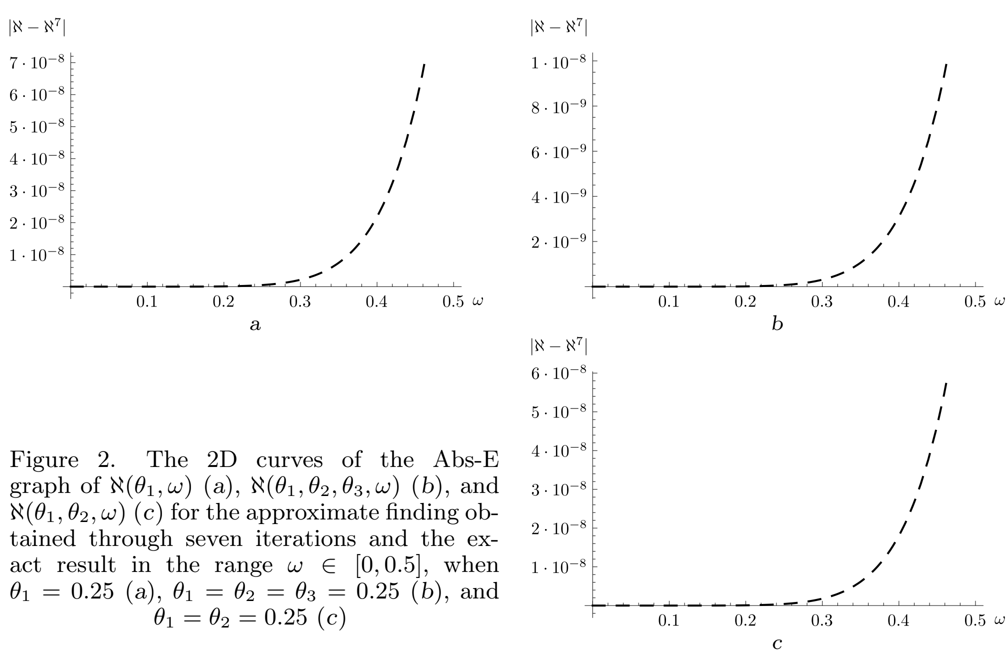
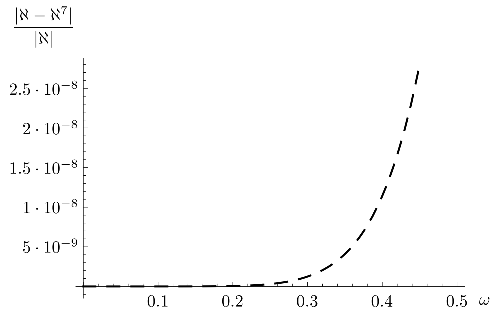
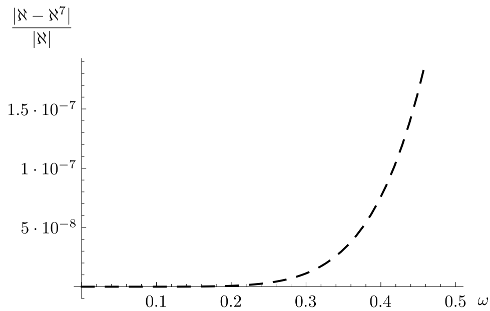


Figure 2. The 2D curves of the Abs-E graph of $\aleph(\theta_1, \omega)$ (a), $\aleph(\theta_1, \theta_2, \theta_3, \omega)$ (b), and $\aleph(\theta_1, \theta_2, \omega)$ (c) for the approximate finding obtained through seven iterations and the exact result in the range $\omega \in [0, 0.5]$, when $\theta_1 = 0.25$ (a), $\theta_1 = \theta_2 = \theta_3 = 0.25$ (b), and $\theta_1 = \theta_2 = 0.25$ (c)



a



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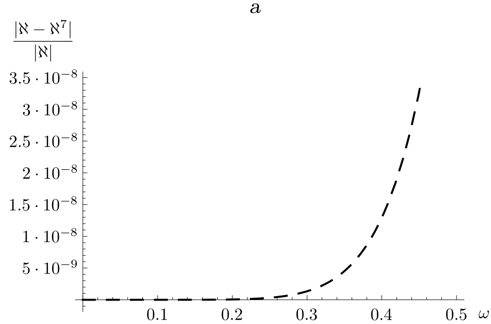
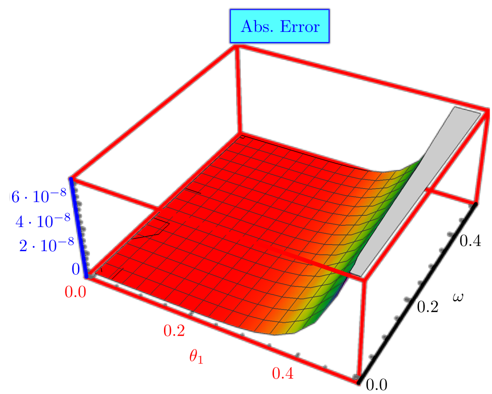


c

Figure 3. The 2D curves of the Rel-E graph of $\aleph(\theta_1, \omega)$ (a), $\aleph(\theta_1, \theta_2, \theta_3, \omega)$ (b), and $\aleph(\theta_1, \theta_2, \omega)$ (c) for the approximate finding obtained through seven iterations and the exact result in the range $\omega \in [0, 0.5]$, when $\theta_1 = 0.25$ (a), $\theta_1 = \theta_2 = \theta_3 = 0.25$ (b), and $\theta_1 = \theta_2 = 0.25$ (c)



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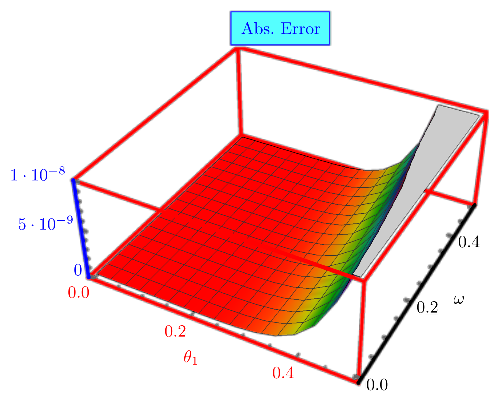

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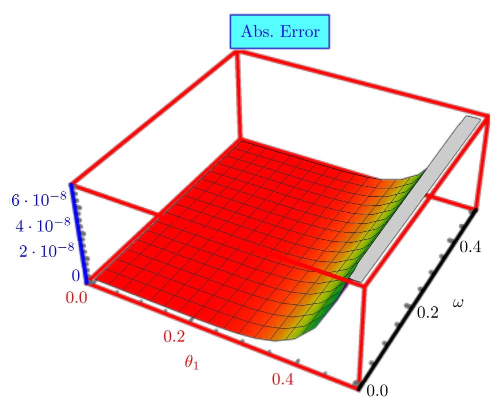
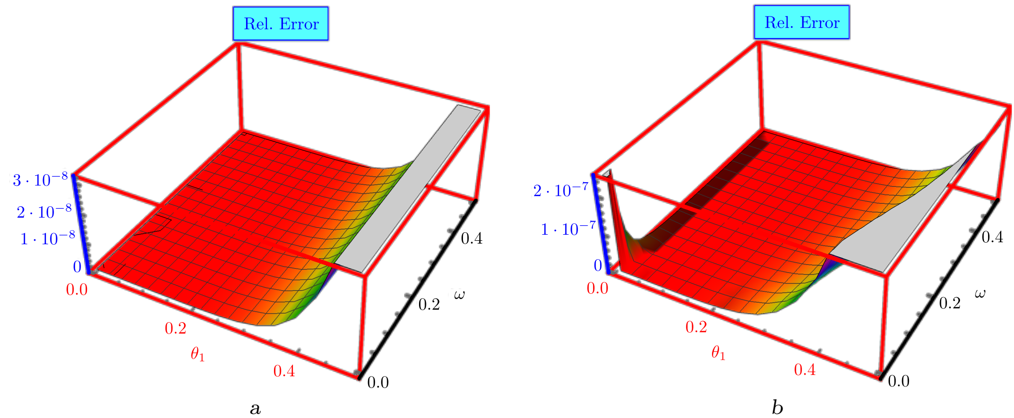


c

Figure 4. The 3D plots of the Abs-E of $\aleph(\theta_1, \omega)$ (a), $\aleph(\theta_1, \theta_2, \theta_3, \omega)$ (b), and $\aleph(\theta_1, \theta_2, \omega)$ (c) for the approximate finding obtained through seven iterations and exact result in the ranges $\omega \in [0, 0.5]$ and $\theta_1 \in [0, 0.5]$, when $\theta_2 = \theta_3 = 0.25$ (b), and $\theta_2 = 0.25$ (c)



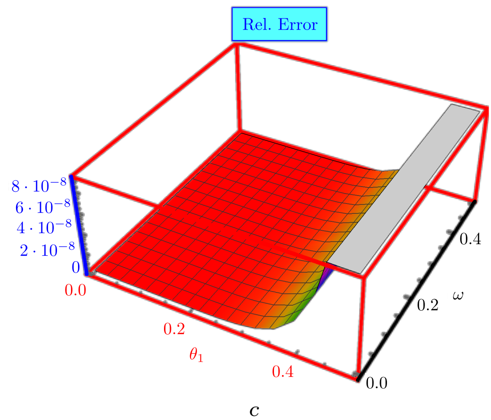


Figure 5. The 3D plots of the Rel-E of $\aleph(\theta_1,\omega)$ (a), $\aleph(\theta_1,\theta_2,\theta_3,\omega)$ (b), and $\aleph(\theta_1,\theta_2,\omega)$ (c) for the approximate finding obtained through seven iterations and exact result in the ranges $\omega\in[0,0.5]$ and $\theta_1\in[0,0.5]$, when $\theta_2=\theta_3=0.25$ (b), and $\theta_2=0.25$ (c)

vals of Abs-E and Rel-E are shown in Table 2 which are from $1.63156\cdot10^{-13}$ to $2.93588\cdot10^{-3}$ and from $1.21370\cdot10^{-9}$ for $1.27164\cdot10^{-3}$, respectively, to Problem 2. Table 3 shows the intervals of Abs-E and Rel-E which are from $2.19100\cdot10^{-11}$ to $2.81468\cdot10^{-3}$ and $2.08398\cdot10^{-11}$ for $8.18227\cdot10^{-4}$, respectively, to Problem 3. Tables 1–3 demonstrate that the 7th step approximate solutions to all numerical equations have remarkably minimal Abs-E and Rel-E. So, the suggested method is very useful for the analysis of different FODEs with a physical interest in the areas of applied mathematics and engineering. As shown in Tables 4–6, Rec-E has been used to numerically demonstrate the process by which the approximate solutions of the Problems 1–3 converge to the exact solution for selected values in the range $\omega\in[0,1]$. As the order of the fractional derivative is increased, the approximate solution of the seven iterations produced using the suggested method quickly converges to the exact solution. The graphical and numerical findings show the accuracy and reliability of the LRPSM.

Table 1 displays the Abs-E and Rel-E at appropriate grid locations in the ranges $\omega\in[0,\ 1.0]$ and $\theta_1\in[0,\ 1.0]$ of the approximate solution attained from the seven iterations and the exact solution to Problem 1 at $\beta=1.0$ using the LRPSM.

Table 2 displays the Abs-E and Rel-E at appropriate grid locations in the ranges $\omega\in[0,1.0]$, $\theta_1\in[0,1.0]$, $\theta_2\in[0,1.0]$ as well as $\theta_3\in[0,1.0]$ of the ap-

Table 1

The Abs-E and Rel-E for Problem 1

(θ_1, ω)	Abs. Errors	Rel. Errors
$(0.05, 0.05)$	$2.29781 \cdot 10^{-11}$	$2.07914 \cdot 10^{-11}$
$(0.15, 0.15)$	$1.87820 \cdot 10^{-8}$	$1.39140 \cdot 10^{-8}$
$(0.25, 0.25)$	$4.51442 \cdot 10^{-7}$	$2.73814 \cdot 10^{-7}$
$(0.35, 0.35)$	$3.81249 \cdot 10^{-6}$	$1.89322 \cdot 10^{-6}$
$(0.45, 0.45)$	$1.93189 \cdot 10^{-5}$	$7.85450 \cdot 10^{-6}$
$(0.55, 0.55)$	$7.22544 \cdot 10^{-5}$	$2.40514 \cdot 10^{-5}$
$(0.65, 0.65)$	$2.20912 \cdot 10^{-4}$	$6.02057 \cdot 10^{-5}$
$(0.75, 0.75)$	$5.85105 \cdot 10^{-4}$	$1.30554 \cdot 10^{-4}$
$(0.85, 0.85)$	$1.39181 \cdot 10^{-3}$	$2.54261 \cdot 10^{-4}$
$(0.95, 0.95)$	$3.04568 \cdot 10^{-3}$	$4.55539 \cdot 10^{-4}$

Table 2

The Abs-E and Rel-E for Problem 2

$(\theta_1, \theta_2, \theta_3, \omega)$	Abs. Errors	Rel. Errors
$(0.05, 0.05, 0.05, 0.05)$	$1.63156 \cdot 10^{-13}$	$1.21370 \cdot 10^{-9}$
$(0.15, 0.15, 0.15, 0.15)$	$1.07593 \cdot 10^{-9}$	$2.59356 \cdot 10^{-7}$
$(0.25, 0.25, 0.25, 0.25)$	$6.44068 \cdot 10^{-8}$	$2.97104 \cdot 10^{-6}$
$(0.35, 0.35, 0.35, 0.35)$	$9.56001 \cdot 10^{-7}$	$1.43769 \cdot 10^{-5}$
$(0.45, 0.45, 0.45, 0.45)$	$7.18225 \cdot 10^{-6}$	$4.58092 \cdot 10^{-5}$
$(0.55, 0.55, 0.55, 0.55)$	$3.59959 \cdot 10^{-5}$	$1.14039 \cdot 10^{-4}$
$(0.65, 0.65, 0.65, 0.65)$	$1.37909 \cdot 10^{-4}$	$2.41227 \cdot 10^{-4}$
$(0.75, 0.75, 0.75, 0.75)$	$4.36368 \cdot 10^{-4}$	$4.54630 \cdot 10^{-4}$
$(0.85, 0.85, 0.85, 0.85)$	$1.19656 \cdot 10^{-3}$	$7.86128 \cdot 10^{-4}$
$(0.95, 0.95, 0.95, 0.95)$	$2.93588 \cdot 10^{-3}$	$1.27164 \cdot 10^{-3}$

Table 3

The Abs-E and Rel-E for Problem 3

$(\theta_1, \theta_2, \omega)$	Abs. Errors	Rel. Errors
$(0.05, 0.05, 0.05)$	$2.19100 \cdot 10^{-11}$	$2.08398 \cdot 10^{-11}$
$(0.15, 0.05, 0.15)$	$1.65204 \cdot 10^{-8}$	$1.41915 \cdot 10^{-8}$
$(0.25, 0.05, 0.25)$	$3.73431 \cdot 10^{-7}$	$2.88416 \cdot 10^{-7}$
$(0.35, 0.05, 0.35)$	$3.02361 \cdot 10^{-6}$	$2.08615 \cdot 10^{-6}$
$(0.45, 0.05, 0.45)$	$1.49760 \cdot 10^{-5}$	$9.15876 \cdot 10^{-6}$
$(0.55, 0.05, 0.55)$	$5.58162 \cdot 10^{-5}$	$2.99928 \cdot 10^{-5}$
$(0.65, 0.05, 0.65)$	$1.73376 \cdot 10^{-4}$	$8.10919 \cdot 10^{-5}$
$(0.75, 0.05, 0.75)$	$4.75627 \cdot 10^{-4}$	$1.91749 \cdot 10^{-4}$
$(0.85, 0.05, 0.85)$	$1.19473 \cdot 10^{-3}$	$4.11041 \cdot 10^{-4}$
$(0.95, 0.05, 0.95)$	$2.81468 \cdot 10^{-3}$	$8.18227 \cdot 10^{-4}$

Table 4

The Rec-E for $\aleph(\theta_1, \omega)$ at different values of β for Problem 1

(θ_1, ω)	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$
$(0.02, 0.02)$	$9.92308 \cdot 10^{-8}$	$6.80134 \cdot 10^{-9}$	$4.41026 \cdot 10^{-10}$	$2.72054 \cdot 10^{-11}$
$(0.12, 0.12)$	$5.80237 \cdot 10^{-5}$	$9.74157 \cdot 10^{-6}$	$1.54730 \cdot 10^{-6}$	$2.33798 \cdot 10^{-7}$
$(0.22, 0.22)$	$5.35032 \cdot 10^{-4}$	$1.21625 \cdot 10^{-4}$	$2.61571 \cdot 10^{-5}$	$5.35152 \cdot 10^{-6}$
$(0.32, 0.32)$	$2.19460 \cdot 10^{-3}$	$6.01676 \cdot 10^{-4}$	$1.56060 \cdot 10^{-4}$	$3.85073 \cdot 10^{-5}$
$(0.42, 0.42)$	$6.28249 \cdot 10^{-3}$	$1.97328 \cdot 10^{-3}$	$5.86366 \cdot 10^{-4}$	$1.65756 \cdot 10^{-4}$
$(0.52, 0.52)$	$1.46622 \cdot 10^{-2}$	$5.12431 \cdot 10^{-3}$	$1.69430 \cdot 10^{-3}$	$7.39038 \cdot 10^{-4}$
$(0.62, 0.62)$	$2.99909 \cdot 10^{-2}$	$1.14451 \cdot 10^{-2}$	$4.13208 \cdot 10^{-3}$	$1.80237 \cdot 10^{-3}$
$(0.72, 0.72)$	$5.59386 \cdot 10^{-2}$	$2.30044 \cdot 10^{-2}$	$8.95017 \cdot 10^{-3}$	$3.90397 \cdot 10^{-3}$
$(0.82, 0.82)$	$9.74598 \cdot 10^{-2}$	$4.27726 \cdot 10^{-2}$	$1.77593 \cdot 10^{-2}$	$7.74645 \cdot 10^{-3}$
$(0.92, 0.92)$	$1.61125 \cdot 10^{-1}$	$7.49016 \cdot 10^{-2}$	$3.29412 \cdot 10^{-2}$	$1.43686 \cdot 10^{-2}$

proximate solution obtained from the seven iterations and the exact solution to Problem 2 at $\beta = 1.0$ using the LRPSM.

Table 3 displays the Abs-E and Rel-E at appropriate grid locations in the ranges $\omega \in [0, 1.0]$, $\theta_1 \in [0, 1.0]$ as well as $\theta_2 \in [0, 1.0]$ of the approximate solution obtained from the seven iterations and the exact solution to Problem 3 at $\beta = 1.0$ using the LRPSM.

Table 4 shows the Rec-E between the approximate solution attained from the seven iterations and exact solutions of Problem 1 acquired by LRPSM at appropriate grid locations in the ranges $\omega \in [0, 1.0]$, and $\theta_1 \in [0, 1.0]$.

Table 5 shows the Rec-E between the approximate solution obtained from the seven iterations and exact solutions of Problem 2 acquired by LRPSM at appropriate grid locations in the ranges $\omega \in [0, 1.0]$, $\theta_1 \in [0, 1.0]$, $\theta_2 \in [0, 1.0]$, and $\theta_3 \in [0, 1.0]$.

Table 6 shows the Rec-E between the approximate solution from the 5th iteration and exact solutions of Problem 3 acquired by LRPSM at appropriate grid locations in the ranges $\omega \in [0, 1.0]$, $\theta_1 \in [0, 1.0]$, and $\theta_2 \in [0, 1.0]$.

6. Conclusion. We used the Laplace transform with the residual power series method to solve time-fractional partial differential equations with variable coefficients in the sense of the Gerasimov–Caputo fractional derivative. Graphics and tables show that the 7th step approximate and exact solutions are in perfect agreement, which demonstrated the efficiency and reliability of the Laplace residual power series method.

Finally, in conclusion, the main features of our recommended method are the following. The residual power series method is useful for obtaining approximate analytical solutions to fractional order problems, but it requires the residual function's $(n-1)\beta$ derivative to determine the coefficients of the series solution, whereas more widely used methods such as the homotopy perturbation, variational iterational, and Adomian decomposition methods require integration. We are all aware of how challenging it is to find the fractional derivatives and the integration, but our method only requires the concept of an infinite limit, which is relatively simple. Therefore, the problem can be solved using our recommended method, which provides a quick and simple way to figure out the coefficients of the series solution. It is not essential to make any significant physical or parametric assumptions related to the problem. Therefore, our method overcomes some of

Table 5

The Rec-E for $\aleph(\theta_1, \theta_2, \omega)$ at different values of β for Problem 2

$(\theta_1, \theta_2, \theta_3, \omega)$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$
$(0.02, 0.02, o.02, 0.02)$	$3.89064 \cdot 10^{-11}$	$2.66667 \cdot 10^{-12}$	$1.72917 \cdot 10^{-13}$	$1.06667 \cdot 10^{-14}$
$(0.12, 0.12, 0.12, 0.12)$	$7.41059 \cdot 10^{-7}$	$1.24416 \cdot 10^{-7}$	$1.97616 \cdot 10^{-8}$	$2.98598 \cdot 10^{-9}$
$(0.22, 0.22, 0.22, 0.22)$	$2.07817 \cdot 10^{-5}$	$4.72416 \cdot 10^{-6}$	$1.01599 \cdot 10^{-6}$	$2.07863 \cdot 10^{-7}$
$(0.32, 0.32, 0.32, 0.32)$	$1.63185 \cdot 10^{-4}$	$4.47392 \cdot 10^{-5}$	$1.16043 \cdot 10^{-5}$	$2.86331 \cdot 10^{-6}$
$(0.42, 0.42, 0.42, 0.42)$	$7.28160 \cdot 10^{-4}$	$2.28710 \cdot 10^{-4}$	$6.79616 \cdot 10^{-5}$	$1.92116 \cdot 10^{-5}$
$(0.52, 0.52, 0.52, 0.52)$	$2.35708 \cdot 10^{-3}$	$8.23775 \cdot 10^{-3}$	$2.72374 \cdot 10^{-4}$	$8.56726 \cdot 10^{-5}$
$(0.62, 0.62, 062, 0.62)$	$6.20169 \cdot 10^{-3}$	$2.36668 \cdot 10^{-3}$	$8.54455 \cdot 10^{-4}$	$2.93468 \cdot 10^{-4}$
$(0.72, 0.72, 0.72, 0.72)$	$1.41151 \cdot 10^{-2}$	$5.80475 \cdot 10^{-3}$	$2.25842 \cdot 10^{-3}$	$8.35884 \cdot 10^{-4}$
$(0.82, 0.82, 0.82, 0.82)$	$2.88624 \cdot 10^{-2}$	$1.26669 \cdot 10^{-2}$	$5.25936 \cdot 10^{-3}$	$2.07738 \cdot 10^{-3}$
$(0.92, 0.92, 0.92, 0.92)$	$5.43487 \cdot 10^{-2}$	$2.52648 \cdot 10^{-2}$	$1.11113 \cdot 10^{-2}$	$4.64872 \cdot 10^{-3}$

Table 6

The Rec-E for $\aleph(\theta_1, \theta_2, \omega)$ at different values of β for Problem 3

$(\theta_1, \theta_2, \omega)$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$
$(0.02, 0.02, 0.02)$	$9.73048 \cdot 10^{-8}$	$6.66933 \cdot 10^{-9}$	$4.32466 \cdot 10^{-10}$	$2.66773 \cdot 10^{-11}$
$(0.12, 0.12, 0.12)$	$5.22088 \cdot 10^{-5}$	$8.76532 \cdot 10^{-6}$	$1.39224 \cdot 10^{-6}$	$2.10368 \cdot 10^{-7}$
$(0.22, 0.22, 0.22)$	$5.06660 \cdot 10^{-4}$	$1.02447 \cdot 10^{-4}$	$2.20326 \cdot 10^{-5}$	$4.50767 \cdot 10^{-6}$
$(0.32, 0.32, 0.32)$	$1.76544 \cdot 10^{-3}$	$4.84017 \cdot 10^{-4}$	$1.25542 \cdot 10^{-4}$	$3.09771 \cdot 10^{-5}$
$(0.42, 0.42, 0.42)$	$4.92422 \cdot 10^{-3}$	$1.54666 \cdot 10^{-3}$	$4.59594 \cdot 10^{-4}$	$1.29920 \cdot 10^{-4}$
$(0.52, 0.52, 0.52)$	$1.14235 \cdot 10^{-2}$	$3.99241 \cdot 10^{-3}$	$1.32005 \cdot 10^{-3}$	$4.15211 \cdot 10^{-4}$
$(0.62, 0.62, 0.62)$	$2.36957 \cdot 10^{-2}$	$9.04270 \cdot 10^{-3}$	$3.26474 \cdot 10^{-3}$	$1.12130 \cdot 10^{-3}$
$(0.72, 0.72, 0.72)$	$4.57254 \cdot 10^{-2}$	$1.88043 \cdot 10^{-2}$	$7.31607 \cdot 10^{-3}$	$2.70782 \cdot 10^{-3}$
$(0.82, 0.82, 0.82)$	$8.40860 \cdot 10^{-2}$	$3.69032 \cdot 10^{-2}$	$1.53223 \cdot 10^{-2}$	$6.05212 \cdot 10^{-3}$
$(0.92, 0.92, 0.92)$	$1.49692 \cdot 10^{-1}$	$6.95868 \cdot 10^{-2}$	$3.06038 \cdot 10^{-2}$	$1.28040 \cdot 10^{-2}$

the inherent limitations of conventional perturbation approaches and works with both weak and strongly nonlinear systems. The He and Adomian polynomials do not need our method, so nonlinear problems can be solved with a relatively small number of calculations. As a result, it performs noticeably better than various series solution methods based on Adomian decomposition and homotopy perturbation methods. In contrast to various analytic approximate methods, the Laplace residual power series method may give analytical expansion solutions for both linear and nonlinear problems without the need for perturbation, linearization, or discretization.

Therefore, our method is simple to apply, accurate and reliable in its results. In the future, as new fractional-order problems emerge in various situations, we plan to solve them using the Laplace residual power series method.

Competing interests. The authors declare that they have no competing interests.

Authors' contributions and responsibilities. The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Availability of Data and Materials. No data were generated or analyzed during the current study.

References

1. Kulish V. V., Lage J. L. Application of fractional calculus to fluid mechanics, *J. Fluids Eng.*, 2002, vol. 124, no. 3, pp. 803–806. DOI: https://doi.org/10.1115/1.1478062.
2. Tarasov V. E. On history of mathematical economics: Application of fractional calculus, *Mathematics*, 2019, vol. 7, no. 6, 509. EDN: QUGYQX. DOI: https://doi.org/10.3390/math7060509.
3. Sun H., Zhang Y., Baleanu D., Chen W., Chen Y. A new collection of real world applications of fractional calculus in science and engineering, *Comm. Nonlinear Sci. Numer. Simul.*, 2018, vol. 64, pp. 213–231. EDN: YGTRMT DOI: https://doi.org/10.1016/j.cnsns.2018.04.019.
4. Dalir M., Bashour M. Applications of fractional calculus, *Appl. Math. Sci.*, 2010, vol. 4, no. 21, pp. 1021–1032.
5. Valério D., Machado J. T., Kiryakova V. Some pioneers of the applications of fractional calculus, *Fract. Calc. Appl. Anal.*, 2014, vol. 17, no. 2, pp. 552–578. EDN: SOSIVZ. DOI: https://doi.org/10.2478/s13540-014-0185-1.
6. Liaqat M. I., Akgül A., Abu-Zinadah H. Analytical investigation of some time-fractional Black–Scholes models by the Aboodh residual power series method, *Mathematics*, 2023, vol. 11, no. 2, 276. DOI: https://doi.org/10.3390/math11020276.
7. Traore A., Sene N. Model of economic growth in the context of fractional derivative, *Alexandria Eng. J.*, 2020, vol. 59, no. 6, pp. 4843–4850. DOI: https://doi.org/10.1016/j.aej.2020.08.047.
8. Hilfer R., Luchko Y., Tomovski Z. Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives, *Fract. Calc. Appl. Anal.*, 2009, vol. 12, no. 3, pp. 299–318.
9. Bas E., Ozarslan R. Real world applications of fractional models by Atangana–Baleanu fractional derivative, *Chaos, Solitons, Fractals*, 2018, vol. 116, pp. 121–125. DOI: https://doi.org/10.1016/j.chaos.2018.09.019.
10. Almeida R. A Caputo fractional derivative of a function with respect to another function, *Comm. Nonlinear Sci. Numer. Simul.*, 2017, vol. 44, pp. 460–481. DOI: https://doi.org/10.1016/j.cnsns.2016.09.006.

11. Khan A., Liaqat M. I., Alqudah M. A., Abdeljawad T. Analysis of the conformable temporal-fractional Swift–Hohenberg equation using a novel computational technique, *Fractals*, 2023, vol. 31, no. 4, 2340050. DOI: https://doi.org/10.1142/S0218348X23400509.
12. Odibat Z., Momani S., Erturk V. S. Generalized differential transform method: Application to differential equations of fractional order, *Appl. Math. Comp.*, 2008, vol. 197, no. 2, pp. 467–477. DOI: https://doi.org/10.1016/j.amc.2007.07.068.
13. Anjum N., He J.-H. Laplace transform: Making the variational iteration method easier, *Appl. Math. Letters*, 2019, vol. 92, pp. 134–138. DOI: https://doi.org/10.1016/j.aml.2019.01.016.
14. Dehestani H., Ordokhani Y., Razzaghi M. Pseudo-operational matrix method for the solution of variable-order fractional partial integro-differential equations, *Engineering with Computers*, 2021, vol. 37, no. 3, pp. 1791–1806. DOI: https://doi.org/10.1007/s00366-019-00912-z.
15. Liaqat M. I., Khan A., Alqudah M. A., Abdeljawad T. Adapted homotopy perturbation method with Shehu transform for solving conformable fractional nonlinear partial differential equations, *Fractals*, 2023, vol. 31, no. 2, 2340027. DOI: https://doi.org/10.1142/S0218348X23400273.
16. Xiang W., Yan S., Wu J., Niu W. Dynamic response and sensitivity analysis for mechanical systems with clearance joints and parameter uncertainties using Chebyshev polynomials method, *Mech. Syst. Signal Proc.*, 2020, vol. 138, 106596. DOI: https://doi.org/10.1016/j.ymssp.2019.106596.
17. Liaqat M. I., Etemad S., Rezapour S., Park C. A novel analytical Aboodh residual power series method for solving linear and nonlinear time-fractional partial differential equations with variable coefficients, *AIMS Math.*, 2022, vol. 7, no. 9, pp. 16917–16948. DOI: https://doi.org/10.3934/math.2022929.
18. Ratas M., Salupere A., Majak J. Solving nonlinear PDEs using the higher order Haar wavelet method on nonuniform and adaptive grids, *Math. Model. Anal.*, 2021, vol. 26, no. 1, pp. 147-169. DOI: https://doi.org/10.3846/mma.2021.12920.
19. Liaqat M. I., Khan A., Akgül A. Adaptation on power series method with conformable operator for solving fractional order systems of nonlinear partial differential equations, *Chaos, Solitons, Fractals*, 2022, vol. 157, 111984. DOI: https://doi.org/10.1016/j.chaos.2022.111984.
20. Zhang Y., Kumar A., Kumar S., et al. Residual power series method for time-fractional Schrödinger equations, *J. Nonlinear Sci. Appl.*, 2016, vol. 9, no. 11, pp. 5821–5829. DOI: https://doi.org/10.22436/jnsa.009.11.10.
21. El-Ajou A., Arqub O. A., Momani S. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: A new iterative algorithm, *J. Comp. Phys.*, 2015, vol. 293, pp. 81–95. DOI: https://doi.org/10.1016/j.jcp.2014.08.004.
22. Saadeh R., Alaroud M., Al-Smadi M., et al. Application of fractional residual power series algorithm to solve Newell–Whitehead–Segel equation of fractional order, *Symmetry*, 2019, vol. 11, no. 12, 1431. DOI: https://doi.org/10.3390/sym11121431.
23. Eriqat T., El-Ajou A., Oqielat M. N., et al. A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations, *Chaos, Solitons, Fractals*, 2020, vol. 138, 109957. DOI: https://doi.org/10.1016/j.chaos.2020.109957.
24. Hesameddini E., Rahimi A. Solving fractional partial differential equations with variable coefficients by the reconstruction of variational iteration method, *Z. Naturforsch., A*, 2015, vol. 70, no. 5, pp. 375–382. DOI: https://doi.org/10.1515/zna-2015-1017.
25. Keskin Y., Karaoglu O., Servi S. The approximate solution of high-order linear fractional differential equations with variable coefficients in terms of generalized Taylor polynomials, *Math. Comput. Appl.*, 2011, vol. 16, no. 3, pp. 617–629. DOI: https://doi.org/10.3390/mca16030617.
26. Sarwar S., Alkhalaf S., Iqbal S., Zahid M. A. A note on optimal homotopy asymptotic method for the solutions of fractional order heat-and wave-like partial differential equations,

Comp. Math. Appl., 2015, vol. 70, no. 5, pp. 942–953. DOI: https://doi.org/10.1016/j.camwa.2015.06.017.

27. Rostamy D., Karimi K. Bernstein polynomials for solving fractional heat- and wave-like equations, *Fract. Calc. Appl. Anal.*, 2012, vol. 15, no. 4, pp. 556–571. DOI: https://doi.org/10.2478/s13540-012-0039-7.
28. Bulut H., Baskonus H. M., Tuluce S. The solutions of partial differential equations with variable coefficient by Sumudu transform method, *AIP Conf. Proc.*, 2012, vol. 1493, pp. 91–95. DOI: https://doi.org/10.1063/1.47654751.
29. Nadeem M., Li F., Ahmad H. Modified Laplace variational iteration method for solving fourth-order parabolic partial differential equation with variable coefficients, *Comp. Math. Appl.*, 2019, vol. 78, no. 6, pp. 2052–2062. DOI: https://doi.org/10.1016/j.camwa.2019.03.053.
30. Dehghan M., Manafian J. The solution of the variable coefficients fourth-order parabolic partial differential equations by the homotopy perturbation method, *Z. Naturforsch., A*, 2009, vol. 64, no. 7–8, pp. 420–430. DOI: https://doi.org/10.1515/zna-2009-7-803.
31. Elzaki T. M., Ezaki S. M. On the Elzaki transform and ordinary differential equation with variable coefficients, *Adv. Theor. Appl. Math.*, 2011, vol. 6, no. 1, pp. 41–46.
32. Khan H., Shah R., Kumam P., Arif M. Analytical solutions of fractional-order heat and wave equations by the natural transform decomposition method, *Entropy*, 2019, vol. 21, no. 6, 597. DOI: https://doi.org/10.3390/e21060597.
33. Khalouta A., Kadem A. A new computational for approximate analytical solutions of nonlinear time-fractional wave-like equations with variable coefficients, *AIMS Math.*, 2020, vol. 5, no. 1, pp. 1–14. DOI: https://doi.org/10.3934/math.2020001.
34. Silva F. S., Moreira D. M., Moret M. A. Conformable Laplace transform of fractional differential equations, *Axioms*, 2018, vol. 7, no. 3, 55. DOI: https://doi.org/10.3390/axioms7030055.
35. Prakasha D. G., Veeresha P., Baskonus H. M. Residual power series method for fractional Swift-Hohenberg equation, *Fractal Fract.*, 2019, vol. 3, no. 1, 9. DOI: https://doi.org/10.3390/fractalfract3010009.

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Эффективный метод аналитического исследования линейных и нелинейных дробно-временных уравнений в частных производных с переменными коэффициентами

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Аннотация

Метод остаточных степенных рядов эффективен для получения приближенных аналитических решений дифференциальных уравнений дробного порядка. Вычисление дробной производной для коэффициентов степенного ряда, аппроксимирующего точное решение дифференциального уравнения, является недостатком этого метода. Другие известные методы приближенного интегрирования, такие как гомотопическое возмущение, разложение Адомиана и методы вариационных итераций, основываются на интегрировании для получения степенного ряда. Известна сложность вычисления дробных производных и интегрирования функций при построении степенного ряда для решения уравнений математической физики дробного порядка, поэтому использование упомянутых выше методов ограничено спецификой решаемой задачи. В настоящей статье получены приближенные и точные аналитические решения уравнений в частных производных переменными коэффициентами при использовании метода рядов остаточных степеней Лапласа в смысле дробной производной Герасимова–Капуто для времени. Этот метод

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помог преодолеть ограничения упомянутых выше способов интегрирования уравнений дробного порядка. Метод остаточных степенных рядов Лапласа лучше использовать при вычислении коэффициентов членов в решении ряда, применяя принцип прямого предела на бесконечности. Он также более эффективен, чем различные методы решения, если не использовать полиномы Адомиана и Не для решения нелинейных задач дробного порядка. В статье исследуются относительные, повторяющиеся и абсолютные ошибки для трех задач математической физики для оценки достоверности предложенного метода. Результаты показывают, что сконструированный метод является альтернативой различным методам для построения решения рядами при решении уравнений в частных производных с дробным временем.

Ключевые слова: преобразование Лапласа, метод остаточных степенных рядов, уравнение в частных производных, производная Герасимова–Капуто.

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