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Construction of elastic fields in the problem from the action of body forces of a cyclic nature

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Abstract

The paper presents a method for determining the stress-strain state of transversely isotropic bodies of revolution under the action of non-axisymmetric stationary volumetric forces. This problem involves the use of boundary state method definitions. The basis of the space of internal states is formed using fundamental polynomials. The polynomial is placed in any position of the displacement vector of the plane auxiliary state, and the spatial state is determined by the transition formulaes. The set of such states forms a finite-dimensional basis according to which, after orthogonalization, the desired state is expanded into Fourier series with the same coefficients. Series coefficients are scalar products of vectors of given and basic volumetric forces. Finally, the search for an elastic state is reduced to solving quadratures.

The solutions of problems of the theory of elasticity for a transversely isotropic circular cylinder from the action of volumetric forces given by various cyclic laws (sine and cosine) are analyzed. Recommendations are given for constructing the basis of internal states depending on the form of the function of given volumetric forces. The analysis of the series convergence and the estimation of the solution accuracy in graphical form are given.

Keywords: boundary state method, transversely isotropic materials, body forces, state space, non-axisymmetric deformation.

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1. Introduction. The development of existing and the creation of new methods for calculating the stress-strain state of bodies made of materials with complex structure and rheology, for the most part, relies on a general or fundamental solution of a particular problem of elasticity theory. S.G. Lekhnitsky, A.Ya. Alexandrov, Yu.I. Soloviev, A.S. Kosmodamiansky made a fundamental contribution to the creation of general solutions for an anisotropic medium, etc. However, these solutions were developed in the last century. Naturally, modern scientists have obtained solutions to particular problems that can be used to build mathematical models based on various methods of mechanics. This is especially true of analytical or numerical-analytical methods, which allow obtaining a solution as a function of several variables (coordinates, time, temperatures, etc.). The development of analytical methods has recently prevailed over numerical methods, where the result of the solution is a table of values of a particular quantity in the entire (and sometimes not in the entire) area of the body.

In the field of implementation of various methods for analyzing the stress-strain state of elastostatic solids, taking into account the influence of body forces, the following works can be distinguished. In [1], an isotropic elastic body bounded by concentric spheres and subjected to axisymmetric unsteady body forces was studied. In [2, 3], using expansions of the displacement vector components into series in terms of the circumferential and radial coordinates, analytical solutions were obtained for the equilibrium problems of thick-walled transversally isotropic composite spheres and those under the action of internal pressure and body forces. In [4], forced deformations arising from the effects of surface and bulk forces were studied. In [5], in addition to the two complex Kolosov–Muskhelishvili potentials, a third potential was proposed that takes into account the influence of body forces. Analytical solutions of some problems of plane deformation are given. The work [6] is devoted to the development of the orthogonal projection method. Problems of the theory of elasticity with the participation of body and surface forces in the functional energy spaces of stress and strain tensors were studied.

In [7,8], the method for determining the stress-strain state of isotropic elastic bodies from the action of body forces of a non-potential nature is reduced.

For transversally isotropic bodies bounded by coaxial surfaces of revolution by means of the method of boundary states, the first main [9] and the second main [10] problems of the theory of elasticity are solved with simultaneous action of body forces on the body. By an identical method, the contact problem was solved [11].

Works [12,13] are devoted to the determination of elastic fields from the action of axisymmetric body forces on a transtropic bounded body of revolution, together with the action of surface forces and a steady temperature field.

The purpose of this work is to develop the analytical method for determining the stress-strain state, proposed in [7], for the class of transversally isotropic bodies of revolution and under the action of body forces specified by the cyclic law. Body forces are non-axisymmetric in nature and depend on three cylindrical coordinates.

2. Problem Statement. We consider the elastic equilibrium of a transversally isotropic body bounded by one or more coaxial surfaces of revolution (Fig. 1) under the action of non-axisymmetric body forces $\boldsymbol{X} = \{R, Q, Z\}$ given by the cyclic law. The axis of anisotropy of a transtropic body coincides with the geometric axis of rotation z.

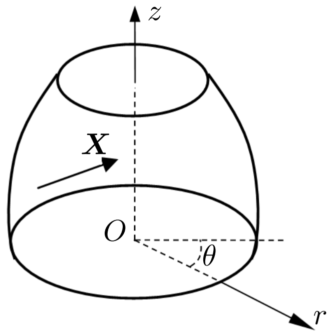

Figure 1. The transversely isotropic body of revolution

The task is to determine the stress-strain state that occurs in the body under the action of body forces.

3. Constitutive Relations of the Theory of Elasticity. In the general case of deformation of a transversally isotropic body in a cylindrical coordinate system, the following relations take place.

Differential equilibrium equation [14]:

$$\begin{aligned}
&\frac{\partial \tau_{zr}}{\partial z}+\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\sigma_r-\sigma_\theta}{r}+R=0,\\
&\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta}+\frac{\tau_{zr}}{r}+Z=0,\\
&\frac{\partial \tau_{z\theta}}{\partial z}+\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+2\frac{\tau_{r\theta}}{r}+Q=0,
\end{aligned} \tag{1}$$

where R, Z, Q — mass forces.

The Cauchy relations [14]:

$$\begin{aligned}
&\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_z=\frac{\partial w}{\partial z},\quad \varepsilon_\theta=\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r},\\
&\gamma_{zr}=\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z},\quad \gamma_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r},\quad \gamma_{z\theta}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}.
\end{aligned} \tag{2}$$

Deformation compatibility equations [15]:

$$\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}-\frac{1}{r}\frac{\partial \varepsilon_r}{\partial r}-\frac{2}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_{r\theta}}{\partial \theta}\right)=0,\\
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_{z\theta}}{\partial \theta}\right)+\frac{1}{r}\frac{\partial^2 \varepsilon_{r\theta}}{\partial \theta \partial z}-\frac{1}{r^2}\frac{\partial^2 \varepsilon_{zr}}{\partial \theta^2}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varepsilon_{\theta\theta}}{\partial z}\right)+\frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial z}=0,\\
&-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r\varepsilon_{z\theta})}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_{r\theta}}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varepsilon_{zr}}{\partial \theta}\right)-\frac{1}{r}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta \partial z}=0,\\
&\frac{1}{r^2}\frac{\partial^2 \varepsilon_{zz}}{\partial \theta^2}+\frac{1}{r}\frac{\partial \varepsilon_{zz}}{\partial r}+\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial z^2}-\frac{2}{r}\frac{\partial^2 \varepsilon_{z\theta}}{\partial \theta \partial z}-\frac{2}{r}\frac{\partial \varepsilon_{zr}}{\partial z}=0,\\
&\frac{1}{r}\frac{\partial^2 \varepsilon_{zr}}{\partial \theta \partial z}+r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varepsilon_{z\theta}}{\partial z}\right)-\frac{\partial^2 \varepsilon_{r\theta}}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varepsilon_{zz}}{\partial \theta}\right)=0,
\end{aligned} \tag{3}$$

$$\frac{\partial^2 \varepsilon_{rr}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial r^2} - 2\frac{\partial^2 \varepsilon_{zr}}{\partial r \partial z} = 0.$$

The generalized Hooke's law [14]:

$$\begin{gathered}
\varepsilon_z = \frac{1}{E_z}\big[\sigma_z - \nu_z(\sigma_r + \sigma_\theta)\big], \quad \varepsilon_r = \frac{1}{E_r}(\sigma_r - \nu_r\sigma_\theta) - \frac{\nu_z}{E_z}\sigma_z, \\
\varepsilon_\theta = \frac{1}{E_r}(\sigma_\theta - \nu_r\sigma_r) - \frac{\nu_z}{E_z}\sigma_z, \\
\gamma_{zr} = \frac{1}{G_z}\tau_{zr}, \quad \gamma_{z\theta} = \frac{1}{G_z}\tau_{z\theta}, \quad \gamma_{r\theta} = \frac{1}{G_r}\tau_{r\theta} = \frac{2(1+\nu_r)}{E_r}\tau_{r\theta}.
\end{gathered} \tag{4}$$

Here u, v, w are the displacement vector components along the x, y, z axes, respectively; ε_r, ε_θ, ε_z, $\gamma_{r\theta}$, γ_{zr}, $\gamma_{z\theta}$ are strain tensor components; σ_r, σ_θ, σ_z, $\tau_{r\theta}$, τ_{zr}, $\tau_{z\theta}$ are stress tensor components; R, Q, Z are the components of the body force vector $\boldsymbol{X}$ along the corresponding axes; E_z and E_r are the elastic modules in the z-axis direction and in the isotropy plane, respectively; ν_z — Poisson's ratio, which characterizes compression along the r axis during tension along the z axis; ν_r — Poisson's ratio characterizing the transverse compression in the plane of isotropy during tension in the same plane; G_r and G_z are the shear modules in the plane of isotropy and perpendicular to it.

4. General Solution of the Elastostatics Problem. In [14], the method of integral overlays established the dependence between the spatial stress-strain state of an elastic transversely isotropic body of revolution and some auxiliary two-dimensional states, the components of which depend on two coordinates z and y (variables). The axis is perpendicular to the zy plane. As plane auxiliary states, we use the plane deformation $\boldsymbol{u}^{pl} = \{u_y^{pl}, u_\eta^{pl}, u_z^{pl}\}$ that occurs in infinite cylinders having at each point a plane of elastic symmetry parallel to the zy plane (direction η).

The transition to the spatial state in cylindrical coordinates is carried out according to the dependencies:

$$\begin{gathered}
u_n = \frac{1}{2\pi}\left(\int_0^\pi (u_y^{pl} + u_\eta^{pl})\cos[(n-1)\beta]d\beta + \int_0^\pi (u_y^{pl} - u_\eta^{pl})\cos[(n+1)\beta]d\beta\right), \\
v_n = \frac{1}{2\pi}\left(\int_0^\pi (u_y^{pl} + u_\eta^{pl})\cos[(n-1)\beta]d\beta - \int_0^\pi (u_y^{pl} - u_\eta^{pl})\cos[(n+1)\beta]d\beta\right), \\
w_n = \frac{1}{\pi}\int_0^\pi u_z^{pl}\cos(n\beta)d\beta, \qquad y = r\cos(\beta);
\end{gathered} \tag{5}$$

$$\begin{gathered}
u = \sum_{n=a}^{b}[u_n\cos(n\theta) + u_n\sin(n\theta)], \\
v = \sum_{n=a}^{b}[-v_n\sin(n\theta) + v_n\cos(n\theta)], \\
w = \sum_{n=a}^{b}[w_n\cos(n\theta) + w_n\sin(n\theta)], \quad a = 0,\ b = \infty.
\end{gathered} \tag{6}$$

Deformations are calculated through the Cauchy relations (2) and are checked for consistency by relations (3). Stresses are determined through Hooke's law (4), and body forces from equilibrium equations (1).

5. Solution Method. The determination of the elastic state of an anisotropic body is carried out by means similar to the means of the boundary state method [16]. The following sets are accepted as a basis in the space of internal states Ξ:

$$\Xi = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots\}, \quad \xi_k = \{u_i^{(k)}, \varepsilon_{ij}^{(k)}, \sigma_{ij}^{(k)}, X_i^{(k)}\}.$$

The papers [12, 13] are devoted to a method for determining the stress-strain state of isotropic bodies from the action of non-conservative continuous body forces. Here we use the same approach.

To construct the displacement field for the body from the action of body forces for planar auxiliary states, the fundamental system of polynomials $y^\alpha z^\beta$ is used, which can be placed in any position of the displacement vector $\mathbf{u}^{pl}(y, z)$, forming some admissible elastic state:

$$\boldsymbol{u}^{pl} = \begin{Bmatrix} u_y^{pl} \\ u_\eta^{pl} \\ u_z^{pl} \end{Bmatrix} \in \left\{ \begin{pmatrix} y^\alpha z^\beta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^\alpha z^\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y^\alpha z^\beta \end{pmatrix} \right\}.$$

Further, according to (5) and (6), the components of the displacement vector $\boldsymbol{u}(r, \theta, z)$ of the spatial state are determined, and the corresponding tensors of strains, stresses, and body forces are determined along the chain (2), (4), (1).

By enumeration of all possible options within $\alpha + \beta \leqslant n$, $(n = 1, 2, 3, \ldots)$, one can obtain a set of states and form a finite-dimensional basis that allows one to expand an arbitrary vector of continuous body forces in a Fourier series in its elements as the number n increases to infinity.

After constructing the basis of states, its orthonormalization is carried out using the recursive-matrix orthogonalization algorithm [17]. The algorithm uses the Gram–Schmidt orthogonalization process in which cross dot products are calculated by the formula (for example, for the 1st and 2nd states):

$$\left(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}\right) = \int_V \boldsymbol{X}^{(1)} \cdot \boldsymbol{X}^{(2)} dV;$$

$$\boldsymbol{X}^{(k)} = X_i^{(k)} = \left\{R^{(k)}(r, \theta, z), Q^{(k)}(r, \theta, z), Z^{(k)}(r, \theta, z)\right\}.$$

Any continuous vector of body forces can be represented as a Fourier series of an orthonormal basis decomposed into elements:

$$\boldsymbol{X} = \sum_{k=1}^{\infty} c_k \boldsymbol{X}^{(k)}, \quad c_k = \left(\boldsymbol{X}, \boldsymbol{X}^{(k)}\right), \tag{7}$$

where $\boldsymbol{X} = \{R, Q, Z\}$ are given body forces.

Each basis vector $\boldsymbol{X}^{(k)}$ corresponds to the displacement vector and strain and stress tensors, which together form an internal state from the action of body forces

$$\xi_0 = \sum_{k=1}^{\infty} c_k \xi_k$$

or in expanded form:

$$u_i = \sum_{k=1}^{\infty} c_k u_i^{(k)}, \quad \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^{(k)}, \quad \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^{(k)}, \quad X_i = \sum_{k=1}^{\infty} c_k X_i^{(k)}. \qquad (8)$$

6. Solving Problem. Let us study the elastic equilibrium of a transversally isotropic circular cylinder made of large dark gray siltstone rock [18]. After the procedure of non-dimensionalization of the parameters of the problem, the analogy of which is presented in [19], the elastic characteristics of the material were $E_z = 6.21$, $E_r = 5.68$, $G_r = 2.29$, $G_z = 2.55$, $\nu_z = 0.22$, $\nu_r = 0.24$ and the cylinder occupies area $V = \{(z, r) \mid 0 \leqslant r \leqslant 1,\ -1 \leqslant z \leqslant 1\}$.

To solve the problem, when all three components of a given vector of body forces are not equal to zero, a rather large “segment” of the basis of internal states is required. In this case, it is advisable to use the principle of independence of the action of forces and solve three separate problems, each of which is given $\boldsymbol{X} = \{R, 0, 0\}$, $\boldsymbol{X} = \{0, Q, 0\}$, $\boldsymbol{X} = \{0, 0, Z\}$, and add the resulting elastic fields.

In the practical implementation of the technique for solving problems and testing it for various types of functions of given body forces, it turned out that not for any type of functions of body forces there is a solution. The possibility of obtaining a rigorous or approximate solution depends on the method of forming the basis.

When constructing the basis of internal states, it is necessary to strive for the greatest simplicity of the form of functions that describe the components of the elastic field. Therefore, let us first consider the basis formed from the left parts of expressions (6) and summation thresholds $a = 0$, $b = 1$:

$$u = \sum_{n=a}^{b} [u_n \cos(n\theta)], \quad v = \sum_{n=a}^{b} [-v_n \sin(n\theta)], \quad w = \sum_{n=a}^{b} [w_n \cos(n\theta)]. \qquad (9)$$

In this case, the problem will be solved if the given body forces R, Q, Z contain trigonometric functions $\cos\theta$, $\sin\theta$, $\cos\theta$, respectively, for example:

$$R = r^m z^k (1 - p\cos\theta), \quad m, k \in \mathbb{N};\ p \in \mathbb{Z}. \qquad (10)$$

Otherwise, the scalar products and Fourier coefficients (7) will be equal to zero.

If we form a basis from the right parts of expressions (6) and summation thresholds $a = 0$, $b = 1$:

$$u = \sum_{n=a}^{b} [u_n \sin(n\theta)], \quad v = \sum_{n=a}^{b} [v_n \cos(n\theta)], \quad w = \sum_{n=a}^{b} [w_n \sin(n\theta)], \qquad (11)$$

then an approximate solution can be obtained if the body forces R, Q, Z contain the trigonometric functions $\sin\theta$, $\cos\theta$, $\sin\theta$, respectively.

If the summation thresholds $a = 1$ and $b = 1$ are used in expressions (9) and (10), then the body forces of the form (10) cannot be restored; in this case, an approximate solution of the problem is sought for a function of the form $r^m z^k p\cos\theta$ or $r^m z^k p\sin\theta$.

In the case when body forces have the form $r^m z^k(\cos\theta + \sin\theta)$, it is already necessary to use expressions (6) in full with summation thresholds $a = 0$, $b = 1$. In this case, it is possible to obtain not only approximate, but also rigorous solutions.

For the latter case, we will give an example of solving the problem when the model volumetric forces are generated, for example, by the magnetic induction of the stator winding of an asynchronous machine [20]:

$$\boldsymbol{X} = \left\{r^3 z^2(\sin\theta + \cos\theta), 0, 0\right\}. \tag{12}$$

After constructing a basis according to relations (6), excluding the basis elements for which $\boldsymbol{X} = 0$, as well as linearly dependent elements in the process of orthogonalization, the basis components of body forces are presented in Table 1 (showing 7 items).

Table 1

Components of the body force of an orthonormal basis

n	R	Q	Z
ξ_1	$-0.2(\cos\theta + \sin\theta)$	$-0.2(\cos\theta - \sin\theta)$	0
ξ_2	0	0 -0.282	
ξ_3	$-0.172z(\cos\theta + \sin\theta)$	$-0.172z(\cos\theta - \sin\theta)$	0
ξ_4	0	0	$-0.244z$
ξ_5	$-0.399r$	0	0
ξ_6	0	$-0.399r$	0
ξ_7	0	0	$-0.399r(\cos\theta + \sin\theta)$

We use a basis of internal states of 50 elements. Non-zero Fourier coefficients: $c_1 = -1.3368$, $c_8 = -1.1957$, $c_{13} = -1.8712$, $c_{14} = 0.4678$, $c_{32} = -1.6736$, $c_{33} = 0.4184$, $c_{38} = -0.2684$, $c_{39} = 0.0671$. As a result of the solution for R and Q, approximate solutions are obtained, for Z — strict ($Z_0 = 0$).

The solution is formed by relations (8). The accuracy is estimated by comparing the given body forces (dashed line) with those restored as a result of the solution (solid line) (Fig. 2).

According to the first graph of Fig. 2, the maximum error is at points $\pi/4$ and $5\pi/4$, therefore, to assess the accuracy of the restored force R depending on r and z, it is advisable to carry out for a section with an angular coordinate of $\pi/4$ (plots *3*, *5* in Fig. 2). In the second graph of Fig. 2, the maximum error is at point $3\pi/4$, so the verification of the force Q depending on r and z is considered in a section with an angular coordinate of $3\pi/4$ (plots *4*, *6* in Fig. 2).

The maximum error of the problem was 25 % and was determined at point $(1, \pi/4, 0)$ (plot *5* in Fig. 2). The error is overcome by increasing the number of basis elements used. When using a basis of 70 elements, two non-zero Fourier coefficients are added: $c_{69} = -0.24$, $c_{70} = 0.06$ and the accuracy of the solution is greatly improved. On Fig. 3 shows plots *5* and *6* of Fig. 2 with 70 elements of the basis retained.

The final internal state ξ_0 is built on 70 basic elements and looks like:

$$u_0 = \left(31.123z^4 - 1369.44r^2z^4 - 1971.16r^4z^4 + 165.557z^6 + \right.$$
$$\left. + 1390.77r^2z^6 - 74.417z^8\right)(\cos\theta + \sin\theta) \cdot 10^{-5},$$

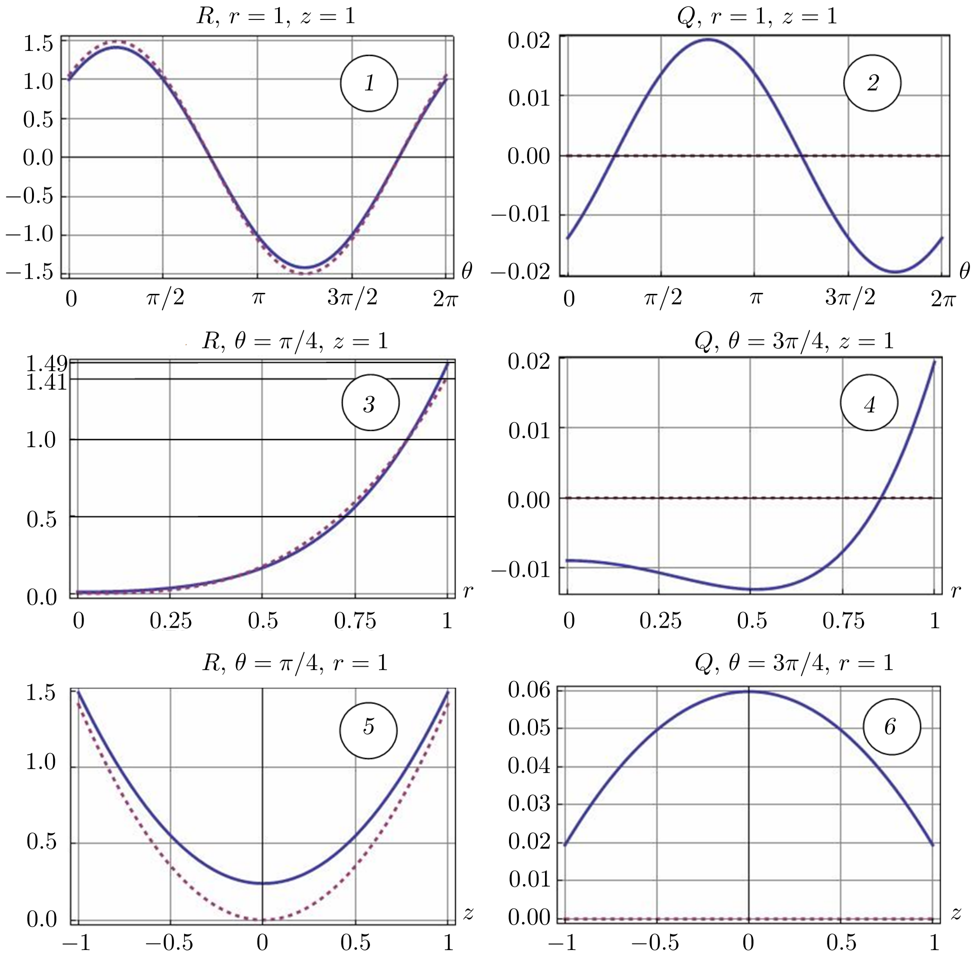

Figure 2. Verification of volumetric forces with 50 retention elements of the basis

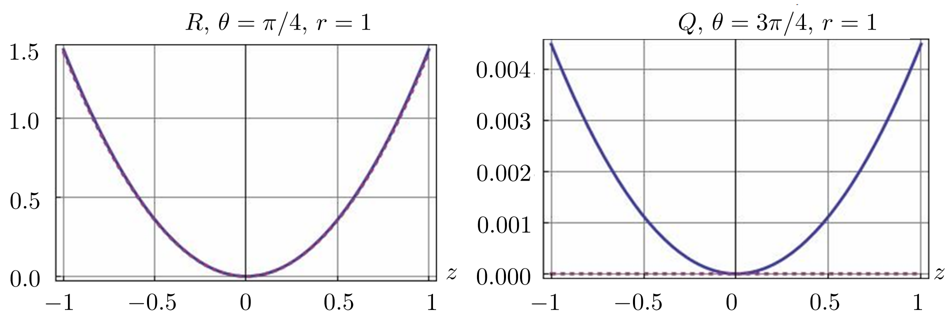

Figure 3. Verification of volumetric forces with 70 retention elements of the basis

$$v_0 = \big(31.123z^4 - 124.494r^2z^4 + 103.745r^4z^4 + 165.557z^6 + \\ + 264.805r^2z^6 - 74.417z^8\big)(\cos\theta - \sin\theta)\cdot 10^{-5},$$

$$w_0 = \big(500.016rz^5 + 1250.04r^3z^5 - 436.69rz^7\big)(\cos\theta + \sin\theta)\cdot 10^{-5};$$

$$R_0 = \big(-952.381z^2 + 4190.8r^2z^2 + 60317.5r^4z^2\big)(\cos\theta + \sin\theta)\cdot 10^{-5},$$

$$Q_0 = \big(-952.381z^2 + 3809.5r^2z^2 - 3174.6r^4z^2\big)(\cos\theta - \sin\theta)\cdot 10^{-5}.$$

The isolines of the obtained characteristics of the elastic field are shown in Fig. 4. The isolines on the plots are shown to scale. The true value of the displayed value is equal to the value on the graph, multiplied by the coefficient κ.

An approximate solution can also be obtained for a body force of the form $r^m(z+p)^k\cos\theta$ or $r^m(z+p)^k\sin\theta$, and for $m=k=2$ is a strict solution.

If for sin and cos in expression (12) there are different coefficients, for example $r^mz^k(p\sin\theta + l\cos\theta)$, then the solution cannot be obtained. This is due to the same coefficients (one) for the corresponding functions in the basic expressions (6).

In the case when body forces depend on $\sin(n\theta)$ or $\cos(n\theta)$, $n=2,3,\ldots$, in expressions (6), (9), (11) it is necessary to use summation thresholds $a=n$, $b=n$.

Consider a function that describes, for example, the body force R of the following form $R = r^mz^kp\cos(n\theta)$. The peculiarity of the solution at $n>1$ is that the restored body forces differ in amplitude from those given by a certain constant — a correction factor κ, which is calculated through the given R and restored R_0 component of the body forces for fixed coordinates r and z: $\kappa = \frac{R}{R_0}\big|_{r,z}$.

Then all other characteristics of the resulting elastic field are multiplied by a factor of κ.

Let the body force $\boldsymbol{X} = \{0, r^2z\cos(3\theta), 0\}$ be given. The basis is formed using expressions (11) and 76 elements of the basis are used to solve this problem (we will not give Fourier coefficients). The result is presented graphically in Fig. 5 (values of body forces R, Q are shown on the surface $r=1$, $z=1$).

Restored expressions for body forces:

$$R_0 = \big(0.25r^2z - 2r^4z + 5.4r^6z - 6r^8z + 2.357r^{10}z\big)\sin(3\theta);$$

$$Q_0 = \big(0.25r^2z + 2r^4z - 5.4r^6z + 6r^8z - 2.357r^{10}z\big)\cos(3\theta); \quad Z_0 = 0.$$

Correction factor $\kappa = r^2/(0.25r^2 + 2r^4 - 5.4r^6 + 6r^8 - 2.357r^{10})$. In this problem κ depends only on r.

Finally, the solution looks like $\xi = \kappa\xi_0$. At $r=1$, $z=1$, the coefficient $\kappa = 2.029$ and the error for R increased, but the result is still satisfactory (the maximum error was 1.5 %).

7. Conclusion. In this paper, the solution of the problem of the theory of elasticity from the action of body forces is constructed as follows. The dependence of the displacement vector of the planar auxiliary state on coordinates $y^\alpha z^\beta$ is specified, and on its basis the displacement vector of the spatial state, which depends on coordinates r, θ, z, is determined. For such a vector, the strain tensor is determined by the Cauchy relation, the stress tensor is determined from Hooke's law, and the body forces are determined from the equilibrium equation. This constructs a strict particular solution of the problem corresponding to the

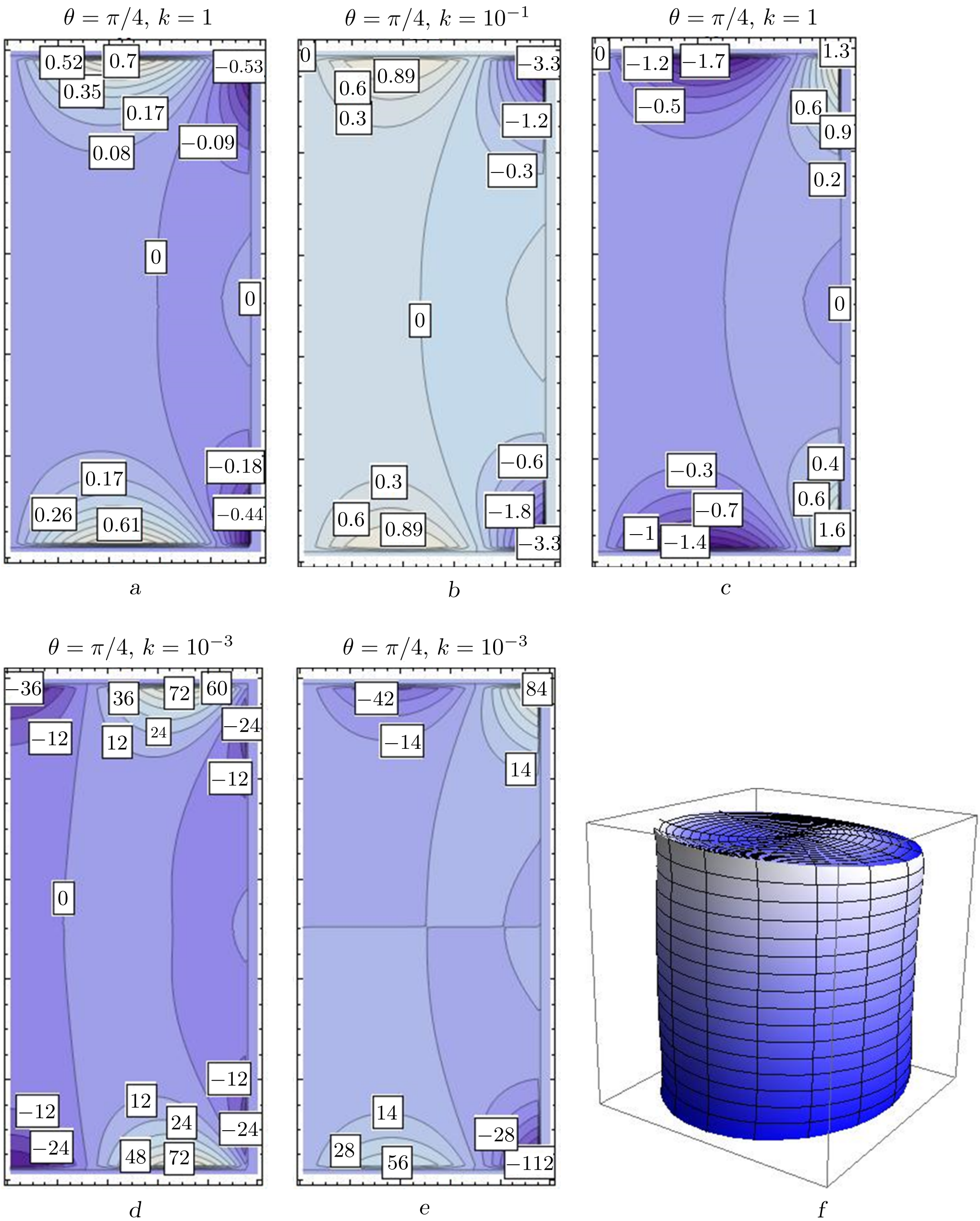

Figure 4. Characteristics of the elastic field: a — component of the stress tensor $\sigma_{\theta\theta}$, b — component of the stress tensor σ_{rr}, c — component of the stress tensor σ_{zz}, d — component of the displacement vector u, e — component of the displacement vector w, f — deformed state contour

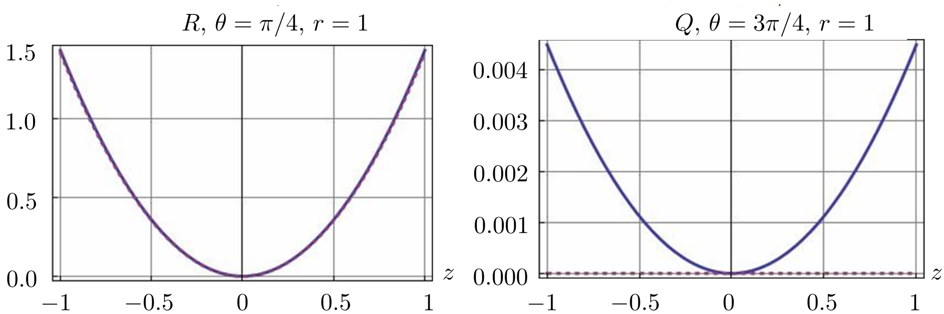

Figure 5. Verification of bulk forces in the problem with the coefficient

displacement function given at each point of the body. Going through $\alpha + \beta \leqslant n$ ($n = 1, 2, 3, \ldots$), a set of strict particular solutions of the problem of linear elasticity theory is constructed: displacement vectors $\boldsymbol{u}_k$, strain tensors $\boldsymbol{\varepsilon}_k$, stress tensors $\boldsymbol{\sigma}_k$, body force vectors $\boldsymbol{X}_k$. Leaving among these solutions only linearly independent ones and implementing them orthogonalization in accordance with relation (7), we obtain a basis according to which the corresponding vectors or tensors are expanded into series with the same coefficients (7). Therefore, the presented approach allows us to immediately construct a solution the problem with given body forces.

Competing interests. We declare that we have no conflict of interest regarding the authorship and publication of this article.

Authors' contributions and responsibilities. Each author has participated in the article concept development and in the manuscript writing. The authors are absolutely responsible for submitting the final manuscript in print. Each author has approved the final version of manuscript.

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Построение упругих полей в задаче от действия объемных сил циклического характера

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Аннотация

Представлен метод определения напряженно-деформированного состояния трансверсально-изотропных тел вращения, возникающего под действием неосесимметричных стационарных объемных сил. Поставленная задача предполагает использование понятий метода граничных состояний. Базис пространства внутренних состояний формируется с помощью фундаментальных полиномов. Многочлен ставится в любое положение вектора смещения плоского вспомогательного состояния и по формулам перехода формируется пространственное состояние. Множество таких состояний образует конечномерный базис, по которому после ортогонализации искомое состояние разлагается в ряды Фурье с теми же коэффициентами. Коэффициенты рядов представляют собой скалярные произведения векторов заданной и базисной объемных сил. Наконец, поиск упругого состояния сводится к решению квадратур.

Анализируются решения задач теории упругости для трансверсально-изотропного кругового цилиндра от действия объемных сил, заданных различными циклическими законами (синуса и косинуса). Даны рекомендации по построению базиса внутренних состояний в зависимости от вида функции заданных объемных сил. Даны анализ сходимости рядов и оценка точности решения в графическом виде.

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