



# Short Communications

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## On the Neuber theory of micropolar elasticity. A pseudotensor formulation

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### Abstract

The present paper deals with a pseudotensor formulation of the Neuber theory of micropolar elasticity. The dynamic equations of the micropolar continuum in terms of relative tensors (pseudotensors) are presented and discussed. The constitutive equations for a linear isotropic micropolar solid is given in the pseudotensor form. The final forms of the dynamic equations for the isotropic micropolar continuum in terms of displacements and microrotations are obtained in terms of relative tensors. The refinements of Neuber's dynamic equations are discussed. Those are also considered in the cylindrical coordinate net.


**Keywords:** micropolarity, elasticity, continuum, microrotation, pseudoscalar, relative tensor, weight, constitutive equation.

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## 1. Preliminary remarks

The classical theories of continuum mechanics often is not acceptable for mathematical modeling of the modern materials behavior (for example, elastic meta-materials [1, 2] or biomaterials [3]: among them sands, soils and other granular elastic media, even perfectly plastic media exhibiting irreversible volume transformation (for instance, the Coulomb–Mohr media), fibrous media, honeycomb structures, reinforced composite materials, bones, vessels, muscles, and other tissues). In those materials the waves of microtations coupled to the displacements waves are observed due to the microstructure effects. Moreover, mirror modes of propagating waves in hemitropic media are caused by a physical mechanism manifested in the hemitropic elastic equations as their sensitivity to mirror reflections and 3D-space inversions.

A first variant of asymmetric elasticity theory was developed by the Cosserats brothers (1909) in the pioneering work [4]. Further consideration of the micropolar theory for finite deformation have been carried out by Truesdell and Toupin [5]. The Aero and Kuvshinskii derived linear constitutive equation of micropolar continuum in [6]. The material anisotropy of micropolar media has been considered and discussed in the Mindlin studies (see for example [7, 8]). The problem of stress concentration is the subject of the Neuber papers [9–11]. An extension of micropolar theory to the hemitropic case can be found in [12, 13].

In the general case of micropolar anisotropy the elastic material is specified by the 171 constitutive constant, which extremely complicates the equations analysis while solving applied problems. A semi-isotropic (hemitropic) solid is determining by nine constitutive constants of which only three new dimensionless ones if compared to the isotropic case. Literary search shows that papers devoted to micropolar theory often contain errors and misprints in the fundamental equations (see for example [10, 14]) making them difficult to understand.

Another important issue in mathematical modelling of micropolar material behaviour is a deficiency of relative tensors technique [5, 15–25], since micropolar characteristics actually are relative tensors. The relative tensors notation provides a deep insight to the physical and geometric nature of considered physical fields. Nonetheless, relative tensors notation in the continuum mechanics is not widespread. The most recent relative tensor formulation of hemitropic micropolar continuum in application to growing solid mechanics is discussed in [26].

The present paper is arranged as follows. The second section is devoted to a number of fundamental definitions from relative tensors algebra. The covariant derivative of an arbitrary relative tensor is considered.

In Sec. 3 Neuber's dynamic equations are derived in terms of relative tensors. The constitutive equations for linear isotropic micropolar continuum are furnished by pseudotensors notation. The weights of relative tensors of linear micropolar elastic medium and constitutive scalars are verified and presented by tables 1 and 2. The final form of Neuber's dynamic equations in an arbitrary curvilinear coordinate system is obtained. The misprints in Neuber's dynamic equations known from [10] are corrected.

The Sec. deals with a formulation of dynamic equations in cylindrical coordinate net. The obtained equations are of crucial importance for investigating wave propagation in long cylindrical waveguides.

The final section contains concluding remarks.

## 2. Relative tensors algebra and covariant differentiation

The permutation symbols and the fundamental orienting pseudoscalar are fundamental objects of relative tensor theory. This theory is a subject of many discussions found in multidimensional geometry tutorials and tensor analysis books [5, 15–25, 27]. A re-orientation of a coordinate frame (left-handed into right-handed or vice versa) can be afforded by re-enumeration of coordinate axes, thus allowing to introduce the fundamental object of relative tensor algebra and multidimensional geometry — the Levi–Civita permutation symbols [18]. It is well known, that the permutation symbols determined according to

$$\epsilon^{[-1]}_{ijk} = \epsilon^{[+1]}_{ijk} = \begin{cases} +1 & \text{for triplets } (1, 2, 3), (2, 3, 1), (3, 1, 2); \\ -1 & \text{for triplets } (3, 2, 1), (1, 3, 2), (2, 1, 3); \\ 0 & \text{in all other cases;} \end{cases}$$

are not absolute tensors. In fact, permutation symbols  $\epsilon^{[-1]}_{ijk}$  and  $\epsilon^{[+1]}_{ijk}$  are the relative tensors (pseudotensors) of the weight  $-1$  (w.g.t. =  $-1$ ) and at the same time — relative contravariant tensors of the weight  $+1$  (w.g.t. =  $+1$ ). Hereinafter, position above a root symbol is reserved for weight of a relative tensor which is additionally embraced by square brackets.

We proceed to discussion of an orienting pseudoscalar (relative scalar of weight  $+1$  (w.g.t. =  $+1$ )), defined by the sequential application of inner and cross products to the covariant basis vectors:

$$e^{[+1]} = \mathbf{i}_1 \cdot (\mathbf{i}_2 \times \mathbf{i}_3) \tag{1}$$

and the relative scalar of the negative weight  $-1$  (w.g.t. =  $-1$ ):

$$e^{[-1]}_{-1} = \mathbf{i} \cdot (\mathbf{i} \times \mathbf{i}).$$

Note that the pseudoscalar (1) is related to the parallelepiped volume built on the vectors  $\mathbf{i}_s$ . In further considerations, we will omit the weight indication for fundamental symbols such as  $e$ ,  $\epsilon_{ijk}$ ,  $\epsilon^{ijk}$  and also true for zero weight relative tensors. Here once again, we emphasize that  $e > 0$  for a right-handed coordinate system,  $e < 0$  for a left-handed coordinate system.

In general, the transformation formula for a relative tensor of weight  $W$  reads by [15–17]

$$\overline{T}^{lm\dots n}_{ij\dots k} = \Delta^W (\partial_p \overline{x}^l) (\partial_q \overline{x}^m) \dots (\partial_s \overline{x}^n) (\overline{\partial}_i x^a) (\overline{\partial}_j x^b) \dots (\overline{\partial}_k x^c) T^{pq\dots s}_{ab\dots c},$$

where

$$\Delta = \det(\overline{\partial}_j x^i), \quad \partial_p = \frac{\partial}{\partial x^p}, \quad \overline{\partial}_p = \frac{\partial}{\partial \overline{x}^p}.$$

Here, an overlined symbol should be considered as related to new coordinates  $\overline{x}^k$  ( $k = 1, 2, 3$ ),  $\Delta$  denotes the transformation Jacobian.

Covariant derivative of the relative tensor  $T^{lm\dots n}_{ij\dots k}$  of a given weight  $W$  is similarly defined by the corresponding derivative for an absolute tensor [15, 17, 21]:

$$\begin{aligned} \nabla_p [W] T_{ij\dots k}^{lm\dots n} &= \partial_p [W] T_{ij\dots k}^{lm\dots n} + T_{ij\dots k}^{sm\dots n} \Gamma_{sp}^l + \dots + T_{ij\dots k}^{lm\dots s} \Gamma_{ip}^s - \\ &\quad - \Gamma_{sp}^l [W] T_{sj\dots k}^{lm\dots n} - \dots - \Gamma_{sp}^l [W] T_{ij\dots s}^{lm\dots n} - W [W] T_{ij\dots k}^{lm\dots n} \Gamma_{sp}^s. \end{aligned} \quad (2)$$

### 3. Reminder and refinement of Neuber's micropolar elasticity theory

The applying relative tensors formalism to the Neuber's theory allows to clarify its physical sense. The dynamic equations in terms of relative tensors can be presented in contrary to [9]

$$\begin{aligned} \nabla_\lambda t^{\lambda\mu} &= \rho \partial.\partial.V^\mu, \\ \nabla_\lambda [m]^{[-1]\lambda\mu} + \epsilon^{\mu\lambda\eta} t_{\lambda\eta} &= -\theta \partial.\partial.\omega^{[+1]\mu}. \end{aligned} \quad (3)$$

In the latter equation contrary to the Neuber theory we use  $\epsilon^{\mu\lambda\eta}$  despite of weights unbalance.

The equations (3) in a curvilinear coordinate net can be rearranged due to (2) as follows

$$\begin{aligned} \partial_\lambda t^{\lambda\mu} + t^{\eta\mu} \Gamma_{\eta\lambda}^\lambda + t^{\lambda\eta} \Gamma_{\eta\lambda}^\mu &= \rho \partial.\partial.V^\mu, \\ \partial_\lambda [m]^{[-1]\lambda\mu} + [m]^{[-1]\mu\eta} \Gamma_{\eta\lambda}^\lambda + [m]^{[-1]\lambda\mu} \Gamma_{\eta\lambda}^\mu + [m]^{[-1]\lambda\mu} \Gamma_{\eta\lambda}^\eta &+ \epsilon^{\mu\lambda\eta} t_{\lambda\eta} = -\theta \partial.\partial.\omega^{[+1]\mu}. \end{aligned} \quad (4)$$

Linear isotropic micropolar elastic constitutive equations [9] in terms of rela-

Table 1

Relative tensors of the isotropic micropolar elasticity

Standard terminology	Notation adopted in [26]	Neuber's notation	Weight	Transformation to absolute tensor
displacements vector	$u^k$	$V^\mu$	0	
asymmetric strain tensor	$\epsilon_{ij}$	$d^{\lambda\mu}$	0	
force stress tensor	$\sigma^{ik}$	$t^{\lambda\mu}$	0	
mass density	$\rho$	$\rho$	0	
couple stress tensor	$\mu_{\cdot k}^i$	$m^{\lambda\mu}$	-1	$m^{\lambda\mu} = e [m]^{[-1]\lambda\mu}$
microinertia	$\mathfrak{I}$	$\theta$	-2	$\theta = e^2 \theta^{[-2]}$
microrotation vector	$\phi^i$	$\omega^\mu$	+1	$\omega^\mu = \frac{1}{e} \omega^{[+1]\mu}$
wryness tensor	$\kappa_{\cdot i}^s$	$k_{\lambda\mu}$	+1	$k_{\lambda\mu} = \frac{1}{e} k_{\lambda\mu}^{[+1]}$

tive tensors are furnished by

$$t^{\lambda\mu} = G[(1 + e^{2\frac{[-2]}{a}})\nabla^\lambda V^\mu + (1 - e^{2\frac{[-2]}{a}})\nabla^\mu V^\lambda + 2\frac{[-2]}{a}\epsilon^{\mu\lambda\eta}\omega_\eta^{[+1]} + 2\nu(1 - 2\nu)^{-1}g^{\lambda\mu}\nabla_\eta V^\eta], \tag{5}$$

$$\frac{[-1]}{m}\lambda\mu = 4G\frac{[-1]}{l}\frac{[-1]}{l}[\nabla^\lambda\omega^{[+1]\mu} + b\nabla^\mu\omega^{[+1]\lambda} + cg^{\lambda\mu}\nabla_\eta\omega^{[+1]\eta}].$$

In the above formulae the constitutive scalars and pseudoscalars are denoted by:

- $G$  is the shear modulus of elasticity;
- $\nu$  is the Poisson ratio;
- $\frac{[-1]}{l}$  is the micropolar characteristic length;
- $\frac{[-2]}{a}$ ,  $b$ ,  $c$  are dimensionless constitutive scalars.

Upon substituting constitutive equations (5) in equations (3) the Neuber dynamic equations in terms of relative tensors read by

$$\begin{aligned} (1 + e^{2\frac{[-2]}{a}})\Delta V^\mu + ((1 - 2\nu)^{-1} - e^{2\frac{[-2]}{a}})\nabla^\mu\nabla_\lambda V^\lambda + \\ + 2\frac{[-2]}{a}\epsilon^{\mu\lambda\eta}\nabla_\lambda\omega_\eta^{[+1]} = \rho G^{-1}\partial.\partial.V^\mu, \\ (\frac{[-2]}{a} - \frac{[-1]}{l}\frac{[-1]}{l}\Delta)\omega^{[+1]\mu} - \frac{[-1]}{l}\frac{[-1]}{l}(b + c)\nabla^\mu\nabla_\lambda\omega^{[+1]\lambda} - \\ - \frac{[-2]}{2}\epsilon^{\mu\eta\sigma}\nabla_\eta V_\sigma = -\frac{[-2]}{\theta}(4G)^{-1}\partial.\partial.\omega^{[+1]\mu}. \end{aligned} \tag{6}$$

Note that in the original paper [9] the multiplier  $l^2$  is omitted in the second term of the second equation of the system (6).

Dynamic equations of linear micropolar elasticity by notation introduced in [13, 26] in relative tensors are represented by

$$\begin{aligned} (1 + e^{2\frac{[-2]}{c_1}})\nabla^s\nabla_s u^i + (1 - e^{2\frac{[-2]}{c_1}} + 2\nu(1 - 2\nu)^{-1})\nabla^i\nabla_k u^k + \\ + 2\frac{[-2]}{c_1}\epsilon^{ikl}\nabla_k\phi_l = \rho G^{-1}\partial.\partial.u^i, \\ \frac{[-1]}{L}\frac{[-1]}{L}[(1 + e^{-2\frac{[+2]}{c_2}})\nabla^s\nabla_s\phi_i + (1 - e^{-2\frac{[+2]}{c_2}} + 2c_3)\nabla_i\nabla_k\phi^k] - \\ - 2\frac{[-2]}{c_1}(2\phi_i - \epsilon_{ikl}g^{ks}u^l) = \mathfrak{J}G^{-1}\partial.\partial.\phi_i. \end{aligned} \tag{7}$$

Comparison of equations (6) and (7) leads to the relations between the micropolar constitutive constants in the form

$$\frac{[-1]}{L}\frac{[-1]}{L} = 2\frac{[-1]}{l}\frac{[-1]}{l}(1 + b), \quad \frac{[-2]}{c_1} = \frac{[-2]}{a}, \quad \frac{[+2]}{c_2} = \frac{1 - b}{1 + b}, \quad c_3 = \frac{c}{1 + b}.$$

The weights of the Neuber constitutive scalars and pseudoscalars  $l, a, b, c$  are given by table 2. The weight of constitutive scalar  $b$  in table 2 is verified by formula

$$b = \frac{e^2 - \frac{[+2]}{c_2}}{e^2 + \frac{[+2]}{c_2}}.$$

Table 2

**Micropolar constitutive scalars of Neuber’s theory**

Standard terminology	Root notation	Weight	Transformation to absolute tensor
shear modulus of elasticity	$G$	0	
the Poisson ratio	$\nu$	0	
micropolar characteristic length dimensionless	$l$	-1	$l = e^{[-1]} l$
micropolar modulus i dimensionless	$a$	-2	$a = e^{2[-2]} a_1$
micropolar modulus ii dimensionless	$b$	0	
micropolar modulus iii dimensionless	$c$	0	

**4. Neuber’s dynamic equations in cylindrical coordinates**

We proceed the paper to consideration of Neuber’s dynamic equations in cylindrical coordinate net  $(r, \varphi, z)$ . It is convenient to assume that the reference plane of the former is the Cartesian  $xy$ -plane (with equation  $z = 0$ ), and the cylindrical axis is the Cartesian  $z$ -axis. Then the  $z$ -coordinate is the same in both systems. The transformation formulae between cylindrical  $(r, \varphi, z)$  and Cartesian coordinates  $(x, y, z)$  can be furnished by

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z.$$

The nonzero components of metric tensor and Christoffel symbols are determined by

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad \Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = r^{-1}.$$

Thus, the dynamic equations (4) can be presented in following form

$$\begin{aligned} &\partial_r(rt_{rr}) + \partial_\varphi t_{\varphi r} + r\partial_z t_{zr} - t_{\varphi\varphi} = r\rho\partial.\partial.V^r, \\ &\partial_\varphi(t_{\varphi\varphi}) + \partial_\varphi(rt_{r\varphi}) + r\partial_z t_{z\varphi} + t_{\varphi r} = r\rho\partial.\partial.V^\varphi, \\ &r\partial_z(t_{zz}) + \partial_\varphi t_{\varphi z} + \partial_r(rt_{zr}) = r\rho\partial.\partial.V^z, \\ &\partial_r(r^{[-1]}m_{rr}) - m_{\varphi\varphi}^{[-1]} + \partial_\varphi m_{\varphi r}^{[-1]} + r\partial_z m_{zr}^{[-1]} + m_{rr}^{[-1]} + r(t_{\varphi z} - t_{z\varphi}) = r\theta\partial.\partial.\omega^{[-2] [+1]}_r, \quad (8) \\ &\partial_\varphi m_{\varphi\varphi}^{[-1]} + \partial_r(r^{[-1]}m_{r\varphi}) + m_{\varphi r}^{[-1]} + r\partial_z m_{z\varphi}^{[-1]} + r^{-1}m_{r\varphi}^{[-1]} + r(t_{zr} - t_{rz}) = r\theta\partial.\partial.\omega^{[-2] [+1]}_\varphi, \\ &\partial_r(r^{[-1]}m_{rz}) + \partial_\varphi m_{\varphi z}^{[-1]} + r\partial_z m_{zz}^{[-1]} + m_{rz}^{[-1]} + r(t_{r\varphi} - t_{\varphi r}) = r\theta\partial.\partial.\omega^{[-2] [+1]}_z. \end{aligned}$$

The obtained equations (8) are of crucial importance for investigating waves propagation in long cylindrical waveguides.

## 5. Conclusions

- (i) The Neuber dynamic equations of the linear micropolar continuum in terms of relative tensors (pseudotensors) in an arbitrary curvilinear coordinate system are presented and discussed. The misprints in original Neuber's dynamic equations are eliminated.
- (ii) The constitutive equations for linear isotropic micropolar continuum are furnished by pseudotensors notation.
- (iii) The weights of relative tensors of linear micropolar elastic medium and the Neuber constitutive scalars are verified and given by tables 1 and 2.
- (iv) The final forms of the dynamic equations for the isotropic micropolar continuum in terms of displacements and microrotations are obtained in terms of relative tensors.
- (v) The refinements of the final form of Neuber's dynamic equations are discussed.

**Competing interests.** We declare that we have no competing interests.

**Author's Responsibilities.** We take full responsibility for submitting the final manuscript in print. We approved the final version of the manuscript.

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## К теории микрополярной упругости Нейбера. Псевдотензорная формулировка

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### Аннотация

Рассматривается псевдотензорная формулировка теории микрополярной упругости Нейбера. Приведены и обсуждаются динамические уравнения микрополярного континуума в терминах относительных тензоров (псевдотензоров). Даны определяющие уравнения для линейного изотропного микрополярного твердого тела. Окончательные формы динамических уравнений для изотропного микрополярного континуума в терминах смещений и микровращений получены в терминах относительных тензоров. Устранены недочеты в окончательной форме динамических уравнений Нейбера. Получены динамические уравнения Нейбера в цилиндрической системе координат.


**Ключевые слова:** микрополярность, упругость, континуум, микровращение, псевдоскаляр, относительный тензор, вес, определяющее уравнение.

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