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Khalouta transform via different fractional derivative operators

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Abstract

Recently, the author defined and developed a new integral transform namely the Khalouta transform, which is a generalization of many well-known integral transforms. The aim of this paper is to extend this new integral transform to include different fractional derivative operators. The fractional derivatives are described in the sense of Riemann–Liouville, Liouville–Caputo, Caputo–Fabrizio, Atangana–Baleanu–Riemann–Liouville, and Atangana–Baleanu–Caputo. Theorems dealing with the properties of the Khalouta transform for solving fractional differential equations using the mentioned fractional derivative operators are proven. Several examples are presented to verify the reliability and effectiveness of the proposed technique. The results show that the Khalouta transform is more efficient and useful in dealing with fractional differential equations.

Keywords: fractional differential equations, Khalouta transform, Riemann–Liouville derivative, Liouville–Caputo derivative, Caputo–Fabrizio derivative, Atangana–Baleanu–Riemann–Liouville derivative, Atangana–Baleanu–Caputo derivative, exact solution.

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1. Introduction. Nowadays, the field of fractional calculus is one of the most vital fields for many researchers and scientists from all over the world, where fractional calculus plays an important role in modeling and describing many phenomena in many fields including quantum mechanics, plasma physics, chemistry, biology, psychology, electromagnetic theory and other different fields of science and engineering, see the papers, [1–5]. Due to the increasing applications of fractional calculus, there has been great interest in solving fractional differential equations.

Over the past decades, mathematicians and physicists have devoted great efforts to finding robust and stable methods for solving fractional differential equations representing real physical problems. Among these methods, the integral transform method is considered one of the most effective methods to solve this kind of equations.

There are many integral transforms that are used in different fields of science such as physics, engineering, astronomy, etc. In particular, for solving fractional differential equations, integral transforms are widely used and many research works are carried out on the theory and applications of the Laplace transform, Fourier transform, and Mellin transform. The most common integral transform with an exponential-type kernel is the Laplace transform. Laplace transform has proven its dominancy in engineering and applied science applications. In recent years, many integral transforms with an exponential type kernel have been introduced. In 1993, Watugula [6] presented the Sumudu transform. The natural transform was developed by Khan [7] in 2008. In 2011, Elzaki invented the Elzaki transform [8]. Atangana and Kiliçman [9] in 2013, defined the novel transform. In 2015, Srivastava et al. [10] introduced the M-transform. In 2016, many transforms were proposed, such as the ZZ transform by Zafar [11], Ramadan Group transform [12], a polynomial transform by Barnes [13], also, a new integral transform was presented by Yang [14]. In 2017, other transforms were introduced, such as the Aboodh transform [15] and the Rangaig transform [16], while the Shehu transform [17] was created in 2019, by Maitama and Zhao.

In 2023, the author proposed a new integral transform called the Khalouta transform [18], which is a new efficient technique to solve differential equations with real applications in applied physical sciences and engineering. The advantages of this new integral transform lie in the following:

1) this transform covers those existing transforms such as Laplace, Aboodh, Elzaki, Sumudu, natural, Shehu, and ZZ transforms for different values of the transform variables;
2) this method transforms the differential problem into an algebraic problem that can be easily solved;
3) the Khalouta transform method finds the solution without any discretization, transformation or restrictive assumptions;
4) the Khalouta transform method can be used to solve a large number of differential equations with minimal computational effort.

The main objective of this paper is to study the relationship between the Khalouta transform and five different fractional derivative operators and then to use the results obtained to solve fractional differential equations.

The outline of the paper is as follows. In Sect. 2, we explain some of the basic concepts and properties of fractional calculus theory. In Sect. 3, we present the definition of the Khalouta transform and some of its important properties that we need in our work. In Sect. 4, we prove the results and examine the relationship between the Khalouta transform with the Riemann–Liouville fractional derivative, Liouville–Caputo fractional derivative, Caputo–Fabrizio fractional derivative, Atangana–Baleanu–Riemann–Liouville derivative, and Atangana–Baleanu–Caputo derivative, as well as some new results. In Sect. 5, we provide various numerical examples to illustrate the precision of the results of the previous sections. Finally, the conclusion is given in Sect. 6.

2. Preliminary Concepts. In this section, we present some essential concepts of fractional calculus necessary to prove our main results.

Definition 1 [19]. The Riemann–Liouville fractional integral with order $\alpha > 0$ for a function $u \in L^1(\mathbb{R}^+)$ is defined by

$$\mathbb{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \tag{1}$$

where $\Gamma(\,\cdot\,)$ is the Euler gamma function.

Definition 2 [19]. The Riemann–Liouville fractional derivative with order $\alpha > 0$ for a function $u \in L^1(\mathbb{R}^+)$ is defined by

$$\mathbb{D}^\alpha u(t) = \frac{d^n}{dt^n} \mathbb{I}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} u(\tau) d\tau, \tag{2}$$

where $n-1 < \alpha \leqslant n$, $n = [\alpha]+1$ with $[\alpha]$ being the integer part of α.

Definition 3 [19]. The Liouville–Caputo fractional derivative with order $\alpha > 0$ for a function u is defined by

$$D^\alpha u(t) = \mathbb{I}^{n-\alpha} \frac{d^n}{dt^n} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, \tag{3}$$

where $n-1 < \alpha \leqslant n$, $n = [\alpha]+1$ with $[\alpha]$ being the integer part of α.

Definition 4 [19]. The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+\beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}. \tag{4}$$

If $\beta = 1$, equation (4) reduced to the one-parameter Mittag–Leffler function as follows

$$E_\alpha(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

Now, for $n = 1$ in equation (3), if transformations happen as follows

$$(t-\tau)^{-\alpha} \longrightarrow \exp\Big(-\frac{\alpha(t-\tau)}{1-\alpha}\Big) \quad \text{and} \quad \frac{1}{\Gamma(1-\alpha)} \longrightarrow \frac{\mathcal{CF}(\alpha)}{1-\alpha},$$

the new definition of fractional derivative operator is expressed by Caputo and Fabrizio.

Definition 5 [20]. The Caputo–Fabrizio fractional derivative with order α when $0 < \alpha \leqslant 1$ for a function $u \in H^1(\mathbb{R}^+)$ is defined by

$$\mathcal{D}^\alpha u(t) = \frac{\mathcal{CF}(\alpha)}{1-\alpha} \int_0^t u'(\tau) \exp\Big(-\frac{\alpha(t-\tau)}{1-\alpha}\Big) d\tau,$$

where $\mathcal{CF}(\alpha)$ is a normalization function that satisfies $\mathcal{CF}(0) = \mathcal{CF}(1) = 1$.

The above Caputo–Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto as

Definition 6 [21]. The Caputo-Fabrizio fractional derivative with order α when $0 < \alpha \leqslant 1$ for a function $u \in H^1(\mathbb{R}^+)$ is defined by

$$\mathcal{D}^\alpha u(t) = \frac{(2-\alpha)\mathcal{CF}(\alpha)}{2(1-\alpha)} \int_0^t u'(\tau) \exp\Big(-\frac{\alpha(t-\tau)}{1-\alpha}\Big) d\tau. \tag{5}$$

For $\mathcal{CF}(\alpha) = 2/(2-\alpha)$ in equation (5), we have

$$\mathcal{D}^\alpha u(t) = \frac{1}{1-\alpha} \int_0^t u'(\tau) \exp\Big(-\frac{\alpha(t-\tau)}{1-\alpha}\Big) d\tau. \tag{6}$$

Definition 7 [21]. The Caputo–Fabrizio fractional derivative with order $\alpha+n$ when $0 < \alpha \leqslant 1$ and $n \geqslant 1$ is defined by

$$\mathcal{D}^{\alpha+n} u(t) = \mathcal{D}^\alpha\big(\mathcal{D}^n u(t)\big). \tag{7}$$

Definition 8 [22]. Let a function $u(t) \in H^1(\mathbb{R}^+)$ and $n-1 < \alpha < n$, $n \in \mathbb{N}^*$, then the Atangana–Baleanu–Riemann–Liouville fractional derivative with order α is defined by

$${}^{ABR}\mathfrak{D}^\alpha u(t) = \frac{\mathcal{AB}(\alpha)}{1-\alpha} \frac{d^n}{dt^n} \int_0^t u(\tau) E_\alpha\Big(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\Big) d\tau, \tag{8}$$

and the Atangana–Baleanu–Caputo fractional derivative with order α is defined by

$${}^{ABC}\mathfrak{D}^\alpha u(t) = \frac{\mathcal{AB}(\alpha)}{1-\alpha} \int_0^t u^{(n)}(\tau) E_\alpha\Big(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\Big) d\tau, \tag{9}$$

where $\mathcal{AB}(\alpha)$ represents the normalization function that satisfies the conditions $\mathcal{AB}(0) = \mathcal{AB}(1) = 1$ and $E_\alpha(\,\cdot\,)$ is the Mittag–Leffler function for one-parameter.

3. Khalouta transform. Recently, the author introduced a new integral transform, called the Khalouta transform, which is applied to solve ordinary and partial differential equations. For more details, see [18].

Definition 9. The Khalouta transform of the function $u(t)$ of exponential order is defined over the set of functions

$$\mathcal{S} = \big\{u(t) : \exists\, K, \vartheta_1, \vartheta_2 > 0, |u(t)| < K \exp(\vartheta_j |t|),\ \text{if } t \in (-1)^j \times [0,\infty)\big\},$$

by the following integral

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta) = \frac{s}{\gamma\eta} \int_0^\infty \exp\Big(-\frac{st}{\gamma\eta}\Big) u(t) dt, \tag{10}$$

where s, γ, $\eta > 0$ are the Khalouta transform variables.

Some basic properties of the Khalouta transform are given as follows.

PROPERTY 1. The Khalouta transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathbb{KH}[\lambda u(t) \pm \mu v(t)] = \lambda \mathbb{KH}[u(t)] \pm \mu \mathbb{KH}[v(t)].$$

PROPERTY 2. If $u^{(n)}(t)$ is the n-th derivative of the function $u(t) \in \mathcal{S}$ with respect to "t" then its Khalouta transform is given by

$$\mathbb{KH}[u^{(n)}(t)] = \frac{s^n}{\gamma^n \eta^n}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k} u^{(k)}(0).$$

PROPERTY 3 (CONVOLUTION PROPERTY). Suppose $\mathcal{K}_1(s,\gamma,\eta)$ and $\mathcal{K}_2(s,\gamma,\eta)$ are the Khalouta transforms of $u_1(t)$ and $u_2(t)$, respectively, both defined in the set $\mathcal{S}$. Then the Khalouta transform of their convolution is given by

$$\mathbb{KH}[(u_1 * u_2)(t)] = \frac{\gamma\eta}{s}\mathcal{K}_1(s,\gamma,\eta)\mathcal{K}_2(s,\gamma,\eta),$$

where $u_1 * u_2$ is convolution of two functions defined by

$$(u_1 * u_2)(t) = \int_0^t u_1(\tau)u_2(t-\tau)d\tau = \int_0^t u_1(t-\tau)u_2(\tau)d\tau.$$

PROPERTY 4. Khalouta transform of some basic functions

$$\begin{aligned}
\mathbb{KH}[1] &= 1,\\
\mathbb{KH}[t] &= \frac{\gamma\eta}{s},\\
\mathbb{KH}\Big[\frac{t^n}{n!}\Big] &= \frac{\gamma^n\eta^n}{s^n}, \quad n = 0,1,2,\dots,\\
\mathbb{KH}\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big] &= \frac{\gamma^\alpha\eta^\alpha}{s^\alpha}, \quad \alpha > -1,\\
\mathbb{KH}[\exp(at)] &= \frac{s}{s - a\gamma\eta}.
\end{aligned}$$

4. Main results. In this section, we prove new theorems related to the Khalouta transform of different fractional deriative operators, namely Riemann–Liouville fractional derivative, Liouville–Caputo fractional derivative, Caputo–Fabrizio fractional derivative, Atangana–Baleanu–Riemann–Liouville fractional derivative, and Atangana–Baleanu–Caputo fractional derivative. Moreover, we prove a new and important results in solving fractional differential equations.

THEOREM 1. *If $\mathcal{K}(s,\gamma,\eta)$ is the Khalouta transform of the function $u(t)$, then the Khalouta transform of the Riemann–Liouville fractional integral for $u(t)$ with order α, is given by*

$$\mathbb{KH}[\mathbb{I}^\alpha u(t)] = \frac{\gamma^\alpha\eta^\alpha}{s^\alpha}\mathcal{K}(s,\gamma,\eta).$$

Proof. Taking the Khalouta transform of both sides of equation (1) and using Properties 3 and 4, we get

$$\mathbb{KH}[\mathbb{I}^{\alpha}u(t)] = \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau\right] =$$
$$= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * u(t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{KH}[u(t)] =$$
$$= \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathcal{K}(s,\gamma,\eta).$$

The proof is complete. □

THEOREM 2. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ such that $n-1 < \alpha \leqslant n$ and $\mathcal{K}(s,\gamma,\eta)$ is the Khalouta transform of the function $u(t)$, then the Khalouta transform of the Riemann–Liouville fractional derivative of $u(t)$ with order α, is given by*

$$\mathbb{KH}[\mathbb{D}^{\alpha}u(t)] = \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{k+1}\mathbb{D}^{\alpha-k-1}u(0).$$

Proof. According to the definition of the Riemann–Liouville fractional derivative (2)

$$\mathbb{D}^{\alpha}u(t) = \frac{d^n}{dt^n}\mathbb{I}^{n-\alpha}u(t),$$

let

$$v(t) = \mathbb{I}^{n-\alpha}u(t), \tag{11}$$

then

$$\mathbb{D}^{\alpha}u(t) = \frac{d^n}{dt^n}v(t) = v^{(n)}(t). \tag{12}$$

Taking the Khalouta transform of both sides of equation (11) and using Theorem 1, we get

$$\mathcal{V}(s,\gamma,\eta) = \mathbb{KH}[v(t)] = \mathbb{KH}[\mathbb{I}^{n-\alpha}u(t)] = \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\mathcal{K}(s,\gamma,\eta), \tag{13}$$

where $\mathcal{V}(s,\gamma,\eta)$ is the Khalouta transform of the function $v(t)$.

Applying the Khalouta transform on both sides of equation (12) and using Property 2, we get

$$\mathbb{KH}[\mathbb{D}^{\alpha}u(t)] = \mathbb{KH}[v^{(n)}(t)] = \frac{s^n}{\gamma^n\eta^n}\mathcal{V}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}v^{(k)}(0) =$$
$$= \frac{s^n}{\gamma^n\eta^n}\mathcal{V}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{k+1}v^{(n-k-1)}(0). \tag{14}$$

From equation (11), we have

$$v^{(n-k-1)}(0) = \frac{d^{n-k-1}}{dt^{n-k-1}}v(0) = \frac{d^{n-k-1}}{dt^{n-k-1}}\mathbb{I}^{n-\alpha}u(0) = \mathbb{D}^{\alpha-k-1}u(0). \tag{15}$$

Thus, by replacing equations (13) and (15) in equation (14), we obtain

$$\mathbb{KH}[\mathbb{D}^{\alpha}u(t)] = \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{k+1}\mathbb{D}^{\alpha-k-1}u(0).$$

The proof is complete. □

THEOREM 3. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ such that $n-1 < \alpha \leqslant n$ and $\mathcal{K}(s,\gamma,\eta)$ is the Khalouta transform of the function $u(t)$, then the Khalouta transform of the Liouville–Caputo fractional derivative of $u(t)$ with order α, is given by*

$$\mathbb{KH}[D^{\alpha}u(t)] = \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(0).$$

Proof. We put

$$v(t) = u^{(n)}(t). \tag{16}$$

Then, according to the definition of the Liouville–Caputo fractional derivative in equation (3), we have

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau =$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}v(\tau)d\tau = \mathbb{I}^{n-\alpha}v(t). \tag{17}$$

Taking the Khalouta transform of both sides of equation (17) and using Theorem 1, we get

$$\mathbb{KH}[D^{\alpha}u(t)] = \mathbb{KH}[\mathbb{I}^{n-\alpha}v(t)] = \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\mathcal{V}(s,\gamma,\eta), \tag{18}$$

where $\mathcal{V}(s,\gamma,\eta)$ is the Khalouta transform of the function $v(t)$.

Applying the Khalouta transform on both sides of equation (16) and using Property 2, we get

$$\begin{aligned} \mathbb{KH}[v(t)] &= \mathbb{KH}[u^{(n)}(t)], \\ \mathcal{V}(s,\gamma,\eta) &= \frac{s^{n}}{\gamma^{n}\eta^{n}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(0). \end{aligned}$$

Therefore, equation (18) becomes

$$\mathbb{KH}[D^{\alpha}u(t)] = \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\left(\frac{s^{n}}{\gamma^{n}\eta^{n}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(0)\right) =$$

$$= \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(0).$$

The proof is complete. □

Theorem 4. *Let $\mathcal{K}(s,\gamma,\eta)$ be the Khalouta transform of the function $u(t)$, then the Khalouta transform of the Caputo–Fabrizio fractional derivative of $u(t)$ with order $\alpha+n$ when $0<\alpha\leqslant 1$ and $n\in\mathbb{N}\cup\{0\}$, is given by*

$$\mathbb{KH}[\mathcal{D}^{\alpha+n}u(t)]=\frac{s}{s-\alpha(s-\gamma\eta)}\left(\frac{s^n}{\gamma^n\eta^n}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\frac{s^{n-k}}{\gamma^{n-k}\eta^{n-k}}u^{(k)}(0)\right).$$

Proof. According to the definition of the Caputo–Fabrizio fractional derivative in equation (6) and using relation (7), we get

$$\begin{aligned}\mathbb{KH}[\mathcal{D}^{\alpha+n}u(t)]&=\mathbb{KH}[\mathcal{D}^{\alpha}(\mathcal{D}^{n}u(t))]=\\&=\frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^\infty\exp\Big(-\frac{st}{\gamma\eta}\Big)\left(\int_0^t u^{(n+1)}(\tau)\exp\Big(-\frac{\alpha(t-\tau)}{1-\alpha}\Big)d\tau\right)dt=\\&=\frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^\infty\exp\Big(-\frac{st}{\gamma\eta}\Big)\left(u^{(n+1)}(t)*\exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\right)dt=\\&=\frac{1}{1-\alpha}\mathbb{KH}\Big[u^{(n+1)}(t)*\exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\Big].\end{aligned}$$

Using Properties 2, 3, and 4, we have

$$\begin{aligned}\mathbb{KH}[\mathcal{D}^{\alpha+n}u(t)]&=\frac{1}{1-\alpha}\frac{\gamma\eta}{s}\mathbb{KH}[u^{(n+1)}(t)]\mathbb{KH}\Big[\exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\Big]=\\&=\frac{\gamma\eta}{s(1-\alpha)+\alpha\gamma\eta}\left(\frac{s^{n+1}}{\gamma^{n+1}\eta^{n+1}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\Big(\frac{s}{\gamma\eta}\Big)^{n-k+1}u^{(k)}(0)\right)=\\&=\frac{s}{s-\alpha(s-\gamma\eta)}\left(\frac{s^n}{\gamma^n\eta^n}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\frac{s^{n-k}}{\gamma^{n-k}\eta^{n-k}}u^{(k)}(0)\right).\end{aligned}$$

The proof is complete. □

Theorem 5. *Let α, $\beta>0$, $a\in\mathbb{R}$, and $|a|<\dfrac{s^\alpha}{\gamma^\alpha\eta^\alpha}$, then*

$$\mathbb{KH}[t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)]=\frac{s^{\alpha-\beta+1}\gamma^{\beta-1}\eta^{\beta-1}}{s^\alpha+a\gamma^\alpha\eta^\alpha}.$$

Proof. Taking the Khalouta transform of the function $t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)$, we obtain

$$\begin{aligned}\mathbb{KH}[t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)]&=\frac{s}{\gamma\eta}\int_0^\infty\exp\Big(-\frac{st}{\gamma\eta}\Big)t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)dt=\\&=\frac{s}{\gamma\eta}\int_0^\infty\exp\Big(-\frac{st}{\gamma\eta}\Big)t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-at^\alpha)^k}{\Gamma(k\alpha+\beta)}dt=\\&=\sum_{k=0}^{\infty}\frac{s}{\gamma\eta}\frac{(-a)^k}{\Gamma(k\alpha+\beta)}\int_0^\infty\exp\Big(-\frac{st}{\gamma\eta}\Big)t^{\alpha k+\beta-1}dt.\qquad(19)\end{aligned}$$

Now, by integration by parts, we have

$$\int_0^{\infty} \exp\Big(-\frac{st}{\gamma\eta}\Big) t^{\alpha k+\beta-1} dt = \Big(\frac{\gamma\eta}{s}\Big)^{\alpha k+\beta} \Gamma(k\alpha+\beta). \tag{20}$$

Substituting equation (20) into equation (19), we get

$$\mathbb{KH}[t^{\beta-1}E_{\alpha,\beta}(-at^{\alpha})] = \sum_{k=0}^{\infty} \frac{s}{\gamma\eta} \frac{(-a)^k}{\Gamma(k\alpha+\beta)} \Big(\frac{\gamma\eta}{s}\Big)^{\alpha k+\beta} \Gamma(k\alpha+\beta) =$$

$$= \sum_{k=0}^{\infty} \Big(\frac{\gamma\eta}{s}\Big)^{\alpha k+\beta-1} (-a)^k = \Big(\frac{\gamma\eta}{s}\Big)^{\beta-1} \sum_{k=0}^{\infty} \Big(\frac{-a\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\Big)^k =$$

$$= \Big(\frac{\gamma\eta}{s}\Big)^{\beta-1} \frac{1}{1-(\frac{-a\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}})} = \Big(\frac{\gamma\eta}{s}\Big)^{\beta-1} \frac{s^{\alpha}}{s^{\alpha}+a\gamma^{\alpha}\eta^{\alpha}} =$$

$$= \frac{s^{\alpha-\beta+1}\gamma^{\beta-1}\eta^{\beta-1}}{s^{\alpha}+a\gamma^{\alpha}\eta^{\alpha}}, \quad \Big|\frac{a\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\Big| < 1.$$

The proof is complete. □

THEOREM 6. *Let $\mathcal{K}(s,\gamma,\eta)$ be the Khalouta transform of the function $u(t)$. Then the Khalouta transform of the Atangana–Baleanu–Riemann–Liouville fractional derivative is expressed as*

$$\mathbb{KH}[{}^{ABR}\mathfrak{D}^{\alpha}u(t)] = \Big(\frac{s^{\alpha+n-1}\mathcal{AB}(\alpha)}{s^{\alpha}\gamma^{n-1}\eta^{n-1} - \alpha(s^{\alpha}\gamma^{n-1}\eta^{n-1} - \gamma^{\alpha+n-1}\eta^{\alpha+n-1})}\Big)\mathcal{K}(s,\gamma,\eta).$$

Proof. Using the definition of the Khalouta transform (10) and the Atangana–Baleanu–Riemann–Liouville fractional derivative (8), we get

$$\mathbb{KH}\Big[{}^{ABR}\mathfrak{D}^{\alpha}u(t)\Big] = \mathbb{KH}\Big[\frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{d^n}{dt^n}\int_0^t u(\tau)E_{\alpha}\Big[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\Big]d\tau\Big].$$

Applying the properties of the Khalouta transform 2 and 3, we get

$$\mathbb{KH}[{}^{ABR}\mathfrak{D}^{\alpha}u(t)] = \frac{\mathcal{AB}(\alpha)}{1-\alpha}\mathbb{KH}\Big[\frac{d^n}{dt^n}\Big(u(t) * E_{\alpha}\Big(-\frac{\alpha t^{\alpha}}{1-\alpha}\Big)\Big)\Big] =$$

$$= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\Big(\frac{s^n}{\gamma^n\eta^n}\mathbb{KH}\Big[u(t)*E_{\alpha}\Big(-\frac{\alpha t^{\alpha}}{1-\alpha}\Big)\Big] - \sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k} D^k\big(\mathbb{KH}[u(0)*E_{\alpha}(0)]\big)\Big) =$$

$$= \frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{s^n}{\gamma^n\eta^n}\frac{\gamma\eta}{s}\Big(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta)\Big) =$$

$$= \Big(\frac{s^{\alpha+n-1}\mathcal{AB}(\alpha)}{s^{\alpha}\gamma^{n-1}\eta^{n-1} - \alpha(s^{\alpha}\gamma^{n-1}\eta^{n-1} - \gamma^{\alpha+n-1}\eta^{\alpha+n-1})}\Big)\mathcal{K}(s,\gamma,\eta).$$

The proof is complete. □

THEOREM 7. *Let $\mathcal{K}(s,\gamma,\eta)$ be the Khalouta transform of the function $u(t)$. Then the Khalouta transform of the Atangana–Baleanu–Caputo fractional derivative is expressed as*

$$\mathbb{KH}[{}^{ABC}\mathfrak{D}^{\alpha}u(t)] = \\ = \Big(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}\Big)\Big(\frac{s^{n-1}}{\gamma^{n-1}\eta^{n-1}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k-1}u^{(k)}(0)\Big).$$

Proof. Using the definition of the Khalouta transform (10) and the Atangana–Baleanu–Caputo fractional derivative (9), we get

$$\mathbb{KH}[{}^{ABC}\mathfrak{D}^{\alpha}u(t)] = \mathbb{KH}\Big[\frac{\mathcal{AB}(\alpha)}{1-\alpha}\int_0^t u^{(n)}(\tau)E_{\alpha}\Big(-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\Big)d\tau\Big].$$

Applying the properties of the Khalouta transform 2 and 3, we get

$$\mathbb{KH}[{}^{ABC}\mathfrak{D}^{\alpha}u(t)] = \frac{\mathcal{AB}(\alpha)}{1-\alpha}\mathbb{KH}\Big[\Big(u^{(n)}(t)*E_{\alpha}\Big(-\frac{\alpha t^{\alpha}}{1-\alpha}\Big)\Big)\Big] = \\ = \frac{\mathcal{AB}(\alpha)}{1-\alpha}\Big(\frac{\gamma\eta}{s}\mathbb{KH}[u^{(n)}(t)]\mathbb{KH}\Big[E_{\alpha}\Big(-\frac{\alpha t^{\alpha}}{1-\alpha}\Big)\Big]\Big) = \\ = \frac{\mathcal{AB}(\alpha)}{1-\alpha}\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}\gamma^{\alpha}\eta^{\alpha}}\frac{\gamma\eta}{s}\Big(\frac{s^{n}}{\gamma^{n}\eta^{n}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k}u^{(k)}(0)\Big) = \\ = \frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}(1-\alpha)+\alpha\gamma^{\alpha}\eta^{\alpha}}\Big(\frac{s^{n-1}}{\gamma^{n-1}\eta^{n-1}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k-1}u^{(k)}(0)\Big) = \\ = \Big(\frac{s^{\alpha}\mathcal{AB}(\alpha)}{s^{\alpha}-\alpha(s^{\alpha}-\gamma^{\alpha}\eta^{\alpha})}\Big)\Big(\frac{s^{n-1}}{\gamma^{n-1}\eta^{n-1}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{n-k-1}u^{(k)}(0)\Big).$$

The proof is complete. □

5. Applications. In this section, we demonstrate the simplicity and applicability of the Khalouta transform with different fractional derivative operators to solve fractional differential equations.

EXAMPLE 1. Consider the following Riemann–Liouville fractional differential equation

$$\mathbb{D}^{1/2}u(t)+u(t)=0, \tag{21}$$

with the initial condition

$$\mathbb{D}^{-1/2}u(0)=2. \tag{22}$$

Taking the Khalouta transform of both sides of equation (21) and using Theorem 2, we get

$$\frac{s^{1/2}}{\gamma^{1/2}\eta^{1/2}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{k+1}\mathbb{D}^{1/2-k-1}u(0)+\mathcal{K}(s,\gamma,\eta)=0. \tag{23}$$

Substituting the initial condition (22) into equation (23), we get

$$\Big(\frac{s^{1/2}}{\gamma^{1/2}\eta^{1/2}}+1\Big)\mathcal{K}(s,\gamma,\eta)-\frac{2s}{\gamma\eta}=0.$$

So

$$\mathcal{K}(s,\gamma,\eta)=\frac{2s\gamma^{-1/2}\eta^{-1/2}}{s^{1/2}+\gamma^{1/2}\eta^{1/2}}.$$

According to Theorem 5, when $\alpha=1/2$, $\beta=1/2$, and $a=1$, we have

$$\mathbb{KH}[u(t)]=\mathcal{K}(s,\gamma,\eta)=\frac{2s\gamma^{-1/2}\eta^{-1/2}}{s^{1/2}+\gamma^{1/2}\eta^{1/2}}=\mathbb{KH}\big[2t^{-1/2}E_{1/2,1/2}(-t^{1/2})\big]. \tag{24}$$

Taking the inverse Khalouta transform of both sides of equation (24), to obtain

$$u(t)=2t^{-1/2}E_{1/2,1/2}(-t^{1/2}).$$

This is the exact solution of equations (21) and (22), which is the same result as obtained using the natural transform [23].

Example 2. Consider the following Riemann–Liouville fractional differential equation

$$\mathbb{D}^{\alpha}u(t)-\lambda u(t)=f(t),\quad t>0,\quad n-1<\alpha\leqslant n, \tag{25}$$

with the initial conditions

$$\mathbb{D}^{\alpha-k-1}u(0)=a_k,\quad k=0,1,2,\dots, \tag{26}$$

where λ and a_k are constants.

Taking the Khalouta transform of both sides of equation (25) and using Theorem 2, we get

$$\frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{k+1}\mathbb{D}^{\alpha-k-1}u(0)-\lambda\mathcal{K}(s,\gamma,\eta)=\mathcal{F}(s,\gamma,\eta), \tag{27}$$

where $\mathcal{F}(s,\gamma,\eta)$ is the Khalouta transform of the function $f(t)$.

Substituting the initial conditions (26) into equation (27), we get

$$\Big(\frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}-\lambda\Big)\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n-1}\Big(\frac{s}{\gamma\eta}\Big)^{k+1}b_k=\mathcal{F}(s,\gamma,\eta).$$

So

$$\mathcal{K}(s,\gamma,\eta)=\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}-\lambda\gamma^{\alpha}\eta^{\alpha}}\mathcal{F}(s,\gamma,\eta)+\sum_{k=0}^{n-1}\frac{s^{k+1}\gamma^{\alpha-k-1}\eta^{\alpha-k-1}}{s^{\alpha}-\lambda\gamma^{\alpha}\eta^{\alpha}}b_k=$$

$$=\frac{\gamma\eta}{s}\frac{s\gamma^{\alpha-1}\eta^{\alpha-1}}{s^{\alpha}-\lambda\gamma^{\alpha}\eta^{\alpha}}\mathcal{F}(s,\gamma,\eta)+\sum_{k=0}^{n-1}\frac{s^{k+1}\gamma^{\alpha-k-1}\eta^{\alpha-k-1}}{s^{\alpha}-\lambda\gamma^{\alpha}\eta^{\alpha}}b_k.$$

According to Theorem 5 and Property 3, we have

$$\mathbb{KH}[u(t)]=\mathcal{K}(s,\gamma,\eta)=$$

$$= \frac{\gamma\eta}{s}\mathbb{KH}[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})]\mathcal{F}(s,\gamma,\eta) + \sum_{k=0}^{n-1} b_k \mathbb{KH}[t^{\alpha-k-1}E_{\alpha,\alpha-k}(\lambda t^{\alpha})] =$$

$$= \mathbb{KH}\Big[\big[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) * f(t)\big] + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1}E_{\alpha,\alpha-k}(\lambda t^{\alpha})\Big] =$$

$$= \mathbb{KH}\Big[\int_0^{\infty}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\tau)^{\alpha}\big)f(\tau)d\tau + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1}E_{\alpha,\alpha-k}(\lambda t^{\alpha})\Big]. \tag{28}$$

Taking the inverse Khalouta transform of both sides of equation (28), to obtain

$$u(t) = \int_0^{\infty}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\tau)^{\alpha}\big)f(\tau)d\tau + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1}E_{\alpha,\alpha-k}(\lambda t^{\alpha}).$$

This is the exact solution of equations (21) and (22) which is the same result as obtained using the Sumudu transform [24].

Example 3. Consider the following Liouville–Caputo fractional differential equation

$$D^{\alpha}u(t) = u(t) + 1, \quad 0 < \alpha \leqslant 1, \tag{29}$$

with the initial condition

$$u(0) = 0. \tag{30}$$

Taking the Khalouta transform of both sides of equation (29) and using Theorem 3, we get

$$\frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) = \mathcal{K}(s,\gamma,\eta) + 1.$$

So

$$\mathcal{K}(s,\gamma,\eta) = \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha} - \gamma^{\alpha}\eta^{\alpha}}.$$

According to Theorem 5, when $\beta = \alpha + 1$, and $a = -1$, we have

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta) = \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha} - \gamma^{\alpha}\eta^{\alpha}} = \mathbb{KH}[t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha})]. \tag{31}$$

Taking the inverse Khalouta transform of both sides of equation (31), to obtain

$$u(t) = t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha}).$$

This is the exact solution of equations (29) and (30) which is the same result as obtained using the Aboodh transform [25].

Example 4. Consider the following Liouville–Caputo fractional Bagley–Torvik equation

$$u''(t) + D^{3/2}u(t) + u(t) = 1 + t, \tag{32}$$

with the initial conditions

$$u(0) = u'(0) = 1. \tag{33}$$

Taking the Khalouta transform of both sides of equation (32) and using Properties 2, 4 and Theorem 3, we get

$$\frac{s^2}{\gamma^2\eta^2}\mathcal{K}(s,\gamma,\eta) - \frac{s^2}{\gamma^2\eta^2}u(0) - \frac{s}{\gamma\eta}u'(0) + \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}}\mathcal{K}(s,\gamma,\eta) - \\ - \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}}u(0) - \frac{s^{1/2}}{\gamma^{1/2}\eta^{1/2}}u'(0) + \mathcal{K}(s,\gamma,\eta) = 1 + \frac{\gamma\eta}{s}. \quad (34)$$

Substituting the initial conditions (33) into equation (34), we get

$$\Big(\frac{s^2}{\gamma^2\eta^2} + \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}} + 1\Big)\mathcal{K}(s,\gamma,\eta) = \\ = 1 + \frac{\gamma\eta}{s} + \frac{s^2}{\gamma^2\eta^2} + \frac{s}{\gamma\eta} + \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}} + \frac{s^{1/2}}{\gamma^{1/2}\eta^{1/2}}. \quad (35)$$

Then, equation (35) becomes

$$\Big(\frac{s^2}{\gamma^2\eta^2} + \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}} + 1\Big)\mathcal{K}(s,\gamma,\eta) = \Big(1 + \frac{\gamma\eta}{s}\Big)\Big(\frac{s^2}{\gamma^2\eta^2} + \frac{s^{3/2}}{\gamma^{3/2}\eta^{3/2}} + 1\Big).$$

So

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta) = 1 + \frac{\gamma\eta}{s}. \quad (36)$$

Taking the inverse Khalouta transform of both sides of equation (36), to obtain

$$u(t) = 1 + t.$$

This is the exact solution of equations (32) and (33), which is the same result as obtained using the Shehu transform [26].

Example 5. Consider the following Caputo–Fabrizio fractional differential equation

$$\mathcal{D}^{\alpha}u(t) = t, \quad 0 < \alpha \leqslant 1, \quad (37)$$

with the initial condition

$$u(0) = c, \quad c \in \mathbb{R}. \quad (38)$$

Taking the Khalouta transform of both sides of equation (37) and using Theorem 4, we get

$$\frac{1}{s - \alpha(s - \gamma\eta)}[s\mathcal{K}(s,\gamma,\eta) - su(0)] = \mathbb{KH}[t], \quad (39)$$

Substituting the initial condition (38) into equation (39) and using Property 4, we get

$$\frac{s}{s - \alpha(s - \gamma\eta)}[\mathcal{K}(s,\gamma,\eta) - c] = \frac{\gamma\eta}{s}.$$

So

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta) = \frac{\gamma\eta(s - \alpha(s - \gamma\eta)) + cs^2}{s^2} = c + (1-\alpha)\frac{\gamma\eta}{s} + \alpha\frac{\gamma^2\eta^2}{s^2}. \quad (40)$$

Taking the inverse Khalouta transform of both sides of equation (40), we get

$$u(t) = c + (1-\alpha)t + \alpha\frac{t^2}{2}. \tag{41}$$

Note that, when $\alpha = 1$ in equation (41), we obtain

$$u(t) = c + \frac{t^2}{2}.$$

This is the exact solution of equations (37) and (38) which is the same result as that obtained using the Laplace transform [27].

Example 6. Consider the following Caputo–Fabrizio fractional differential equation

$$\mathcal{D}^{\alpha}u(t) + u(t) = 0, \quad 0 < \alpha \leqslant 1, \tag{42}$$

with the initial condition

$$u(0) = 1. \tag{43}$$

Taking the Khalouta transform of both sides of equation (42) and using Theorem 4, we get

$$\frac{1}{s-\alpha(s-\gamma\eta)}[s\mathcal{K}(s,\gamma,\eta) - su(0)] + \mathcal{K}(s,\gamma,\eta) = 0. \tag{44}$$

Substituting the initial condition (43) into equation (44), we get

$$\frac{1}{s-\alpha(s-\gamma\eta)}[s\mathcal{K}(s,\gamma,\eta) - s] + \mathcal{K}(s,\gamma,\eta) = 0.$$

So

$$\mathbb{KH}[u(t)] = \mathcal{K}(s,\gamma,\eta) = \frac{s}{(2-\alpha)s + \alpha\gamma\eta} = \frac{1}{2-\alpha}\Big(\frac{s}{s+\frac{\alpha}{2-\alpha}\gamma\eta}\Big). \tag{45}$$

Taking the inverse Khalouta transform of both sides of equation (45), we get

$$u(t) = \frac{1}{2-\alpha}\exp\Big(-\frac{\alpha}{2-\alpha}t\Big). \tag{46}$$

Note that, when $\alpha = 1$ in equation (46), we obtain

$$u(t) = \exp(-t).$$

This is the exact solution of equations (42) and (43), which is the same result as that obtained using the Sumudu transform [28].

Example 7. Consider the following Atangana–Baleanu–Riemann–Liouville fractional differential equation

$${}^{ABR}\mathfrak{D}^{\alpha}u(t) + u(t) = f(t), \quad 0 < \alpha \leqslant 1, \tag{47}$$

with the initial condition

$$u(0) = 0. \tag{48}$$

Taking the Khalouta transform of both sides of equation (47) and using Theorem 6, we get

$$\Big(\frac{s^\alpha \mathcal{AB}(\alpha)}{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}\Big)\mathcal{K}(s,\gamma,\eta)+\mathcal{K}(s,\gamma,\eta)=\mathbb{KH}[f(t)]. \tag{49}$$

Simplifying equation (49), then we have

$$\mathcal{K}(s,\gamma,\eta)=\Big(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha \mathcal{AB}(\alpha)+s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}\Big)\mathcal{F}(s,\gamma,\eta). \tag{50}$$

Taking the inverse Khalouta transform of both sides of equation (50), we get

$$u(t)=\mathbb{KH}^{-1}\Big[\Big(\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha \mathcal{AB}(\alpha)+s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}\Big)\mathcal{F}(s,\gamma,\eta)\Big].$$

If $f(t)=\sin(t)$, then equation (47) becomes

$$^{ABR}\mathfrak{D}^\alpha u(t)+u(t)=\sin(t), \quad 0<\alpha\leqslant 1,$$

and the exact solution is

$$u(t)=\mathbb{KH}^{-1}\Big[\frac{s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}{s^\alpha \mathcal{AB}(\alpha)+s^\alpha-\alpha(s^\alpha-\gamma^\alpha\eta^\alpha)}\frac{s\gamma\eta}{s^2+\gamma^2\eta^2}\Big]. \tag{51}$$

Note that, when $\alpha=1$ in equation (51), we obtain

$$\begin{aligned}u(t)&=\mathbb{KH}^{-1}\Big[\frac{\gamma\eta}{s+\gamma\eta}\frac{s\gamma\eta}{s^2+\gamma^2\eta^2}\Big]=\\ &=\mathbb{KH}^{-1}\Big[\frac{1}{2}\frac{s}{s+\gamma\eta}-\frac{1}{2}\frac{s^2}{s^2+\gamma^2\eta^2}+\frac{1}{2}\frac{s\gamma\eta}{s^2+\gamma^2\eta^2}\Big]=\\ &=\frac{1}{2}\big(\exp(-t)-\cos(t)+\sin(t)\big).\end{aligned}$$

This is the exact solution of equations (47) and (48), which is the same result as obtained using the Shehu transform [29].

Example 8. Consider the following Atangana–Baleanu–Caputo fractional differential equation

$$^{ABC}\mathfrak{D}^\alpha u(t)=u(t), \quad 0<\alpha\leqslant 1, \tag{52}$$

with the initial condition

$$u(0)=1. \tag{53}$$

Taking the Khalouta transform of both sides of equation (52) and using Theorem 7, we get

$$\frac{s^\alpha \mathcal{AB}(\alpha)}{s^\alpha(1-\alpha)+\alpha\gamma^\alpha\eta^\alpha}\big(\mathcal{K}(s,\gamma,\eta)-u(0)\big)=\mathcal{K}(s,\gamma,\eta). \tag{54}$$

Substituting the initial condition (53) into equation (54), we get

$$\frac{s^\alpha \mathcal{AB}(\alpha)}{s^\alpha(1-\alpha)+\alpha\gamma^\alpha\eta^\alpha}\big(\mathcal{K}(s,\gamma,\eta)-1\big)=\mathcal{K}(s,\gamma,\eta).$$

So

$$\mathcal{K}(s,\gamma,\eta)=\frac{\mathcal{AB}(\alpha)}{\mathcal{AB}(\alpha)-1+\alpha-\alpha\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}}. \tag{55}$$

Equation (55) can be rewritten as

$$\mathbb{KH}[u(t)]=\mathcal{K}(s,\gamma,\eta)=\frac{\mathcal{AB}(\alpha)}{(\mathcal{AB}(\alpha)-1+\alpha)}\Big(1-\frac{\alpha}{\mathcal{AB}(\alpha)-1+\alpha}\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\Big)^{-1}. \tag{56}$$

Taking the inverse Khalouta transform of both sides of equation (56) and using Property 4, we get

$$u(t)=\frac{\mathcal{AB}(\alpha)}{(\mathcal{AB}(\alpha)-1+\alpha)}E_{\alpha}\Big(\frac{\alpha}{\mathcal{AB}(\alpha)-1+\alpha}t^{\alpha}\Big). \tag{57}$$

Note that, when $\alpha=1$ in equation (57), we obtain

$$u(t)=E_1(t)=\exp(t).$$

This is the exact solution of equations (52) and (53), which is the same result as obtained using the ZZ transform [30].

6. Conclusion. In this paper, we have studied the application of the Khalouta transform method to obtain exact solutions of homogeneous and inhomogeneous linear fractional differential equations using different fractional derivative operators. Various examples have been used to illustrate the effectiveness of this technique. The results obtained have shown that the Khalouta transform is a powerful tool and an efficient method for solving initial value problems in the fields of applied mathematics and engineering. In the future, we hope to extend the Khalouta transform method to solve initial value problems by considering other fractional-order differential equations that have not yet been solved analytically.

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Преобразование Халуты, осуществляемое с использованием различных операторов дробной производной

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Аннотация

Недавно автором было введено и разработано новое интегральное преобразование, которое обобщает множество известных интегральных преобразований. Цель этой работы — расширение данного интегрального преобразования (преобразование Халуты) различными операторами дробной производной. Рассматриваются дробные производные в смысле Римана–Лиувилля, Лиувилля–Капуто, Капуто–Фабрицио, Атанганы–Балеану–Римана–Лиувилля и Атанганы–Балеану–Капуто. Доказаны теоремы, касающиеся свойств преобразования Халуты для решения дробных дифференциальных уравнений с использованием указанных операторов дробной производной. Приведено несколько примеров для проверки надежности и эффективности предложенной техники. Результаты показывают, что преобразование Халуты является эффективным инструментом при работе с дробными дифференциальными уравнениями.

Ключевые слова: дробные дифференциальные уравнения, преобразование Халуты, производная Римана–Лиувилля, производная Лиувилля–Капуто, производная Капуто–Фабрицио, производная Атанганы–Балеану–Римана–Лиувилля, производная Атанганы–Балеану–Капуто, точное решение.

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