Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki
[J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 1, pp. 21–36
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu2083

MSC: 35R11, 35R30, 26A33

Solvability of a coefficient recovery problem for a time-fractional diffusion equation with periodic boundary and overdetermination conditions

D. K. Durdiev[1,2]**, *J. J. Jumaev***[1,2]

[1] Bukhara Branch of the Institute of Mathematics
named after V. I. Romanovskiy at the Academy of Sciences
of the Republic of Uzbekistan,
11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.
[2] Bukhara State University,
11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.

Abstract

This article investigates the inverse problem for time-fractional diffusion equations with periodic boundary conditions and integral overdetermination conditions on a rectangular domain. First, the definition of a classical solution to the problem is introduced. Using the Fourier method, the direct problem is reduced to an equivalent integral equation. The existence and uniqueness of the solution to the direct problem are established by employing estimates for the Mittag–Leffler function and generalized singular Gronwall inequalities.

In the second part of the work, the inverse problem is examined. This problem is reformulated as an equivalent integral equation, which is then solved using the contraction mapping principle. Local existence and global uniqueness of the solution are rigorously proven. Furthermore, a stability estimate for the solution is derived.

The study contributes to the theory of inverse problems for fractional differential equations by providing a framework for analyzing problems with periodic boundary conditions and integral overdetermination. The methods developed in this work can be applied to a wide range of problems in mathematical physics and engineering, where time-fractional diffusion models are increasingly used to describe complex phenomena.

Differential Equations and Mathematical Physics
Research Article

Please cite this article in press as:
Durdiev D. K., Jumaev J. J. Solvability of a coefficient recovery problem for a time-fractional diffusion equation with periodic boundary and overdetermination conditions, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 1, pp. 21–36. EDN: YZQBWZ. DOI: 10.14498/vsgtu2083.

Authors' Details:

Durdimurod K. Durdiev https://orcid.org/0000-0002-6054-2827
Dr. Phys. & Math. Sci., Professor; Head of Branch[1]; Professor, Dept. of Differential Equations[2]; e-mail: d.durdiev@mathinst.uz

Jonibek J. Jumaev https://orcid.org/0000-0001-8496-1092
Cand. Phys. & Math. Sci., Associate Professor; Senior Researcher[1]; Associate Professor, Dept. of Differential Equations[2]; e-mail: jonibekjj@mail.ru

Keywords: time-fractional diffusion equation, periodic boundary conditions, inverse problem, integral equation.

Received: 15th February, 2024 / Revised: 19th November, 2024 / Accepted: 21st February, 2025 / First online: 14th March, 2025

1. Introduction. Periodic boundary conditions (PBCs) are a set of boundary conditions often chosen to approximate a large (infinite) system by using a small part called a unit cell. PBCs are widely used in computer simulations and mathematical models. The topology of two-dimensional PBCs is analogous to that of a world map in some video games; the geometry of the unit cell satisfies perfect two-dimensional tiling, and when an object passes through one side of the unit cell, it reappears on the opposite side with the same velocity (see [1–3]).

PBCs arise in many important applications in heat transfer and life sciences [4–8]. In these studies, the existence, uniqueness, and continuous dependence of the solution on the data were proven, and numerical solutions to the diffusion problem with periodic boundary conditions were developed.

Various formulations of inverse problems for determining thermal coefficients in the one-dimensional heat equation have been studied in [8–11]. It is important to note that in [8, 9], the time-dependent thermal coefficient is determined from a nonlocal overdetermination condition. Additionally, in [12–14], the coefficients of the heat equation are determined for cases involving nonlocal boundary conditions.

The studies [15–19] investigated the inverse problem of finding diffusion coefficients in one- and multi-dimensional time-fractional equations. Under certain assumptions about the data, the existence, uniqueness, and continuous dependence of the solution on the data were established.

The problem of determining the kernel $k(t)$ of the integral term in an integro-differential heat equation has been extensively studied in numerous publications [20–28]. These studies address both one- and multidimensional inverse problems with classical initial and initial-boundary conditions. Theorems on the existence and uniqueness of solutions to these inverse problems have been proven.

In the present study, the determination of the coefficient in the time-fractional diffusion equation is considered under initial and periodic boundary conditions. The existence and uniqueness of a classical solution to the problem (1)–(4) are established using the fixed-point principle via the Fourier method.

2. Formulation of the Problem. We consider the following initial-periodic boundary value problem for the fractional diffusion equation:

$$\partial_t^\alpha u - u_{xx} + a(t)u = f(x,t), \quad (x,t) \in D_T, \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in [0,l], \tag{2}$$

$$u(0,t) = u(l,t),\ u_x(0,t) = u_x(l,t),\ \varphi(0) = \varphi(l),\ \varphi'(0) = \varphi'(l),\ t \in [0,T], \tag{3}$$

where ∂_t^α is the Caputo fractional derivative of order $0 < \alpha \leqslant 1$ in the time variable (see Definition 2), $a(t)$, $t > 0$, is the source control term, $f(x,t)$ is the known source term, $\varphi(x)$ is the initial temperature, T is an arbitrary positive number, and $D_T := \{(x,t) : 0 < x < l, 0 < t \leqslant T\}$.

The problem of determining a function $u(x,t)$, $(x,t) \in D_T$, that satisfies (1)–(3) with given functions $a(t)$, $f(x,t)$, and $\varphi(x)$ will be called the direct problem.

In the inverse problem, it is required to determine the coefficient $a(t)$, $t > 0$, in (1) using overdetermination conditions about the solution of the direct problem (1)–(3):

$$u_x(0,t) = h(t), \quad x \in [0,l], \tag{4}$$

where $h(t)$ is a given function.

Let $u(x,t)$ be a classical solution to the problem (1)–(3), and let f, φ, and h be sufficiently smooth functions.

We perform the following transformation of the inverse problem (1)–(4). For this purpose, denote the second derivative of $u(x,t)$ with respect to x by $\vartheta(x,t)$, i.e., $\vartheta(x,t) := u_{xx}(x,t)$. Differentiating (1) and (2) twice with respect to x, we obtain

$$\partial_t^\alpha \vartheta - \vartheta_{xx} + a(t)\vartheta(x,t) = f_{xx}(x,t), \quad (x,t) \in D_T, \tag{5}$$

and

$$\vartheta(x,0) = \varphi''(x), \quad x \in [0,l]. \tag{6}$$

To obtain boundary conditions for the new function $\vartheta(x,t)$, we note that the second term in (1) is $\vartheta(x,t)$. Assume that $f(0,t) = f(l,t)$ and $f'(0,t) = f'(l,t)$. Then we have the following boundary conditions:

$$\vartheta(0,t) = \vartheta(l,t), \quad \vartheta_x(0,t) = \vartheta_x(l,t). \tag{7}$$

To obtain an additional condition for the function $\vartheta(x,t)$, we differentiate equation (1) with respect to x and, using the equality $u_{xx}(x,t) = \vartheta(x,t)$ together with condition (4), we obtain

$$\vartheta_x(0,t) = a(t)h(t) + \partial_t^\alpha h(t) - f_x(0,t). \tag{8}$$

Under the matching condition $\varphi'(0) = h(0)$, it is straightforward to deduce from (5)–(8) the equations (1)–(4).

We introduce the spaces

$$C^{2,\alpha}(D_T) := \{v(x,t) : v, v_x, v_{xx}, \partial_t^\alpha \vartheta \in C(D_T)\}$$

and

$$C^{1,0}(\overline{D}_T) := \{v(x,t) : v, v_x \in C(\overline{D}_T)\}.$$

Definition 1. The functions $\{u(x,t), a(t)\}$ from the class $C^{2,\alpha}(D_T) \cap C^{1,0}(\overline{D}_T) \times C[0,T]$ are said to be a classical solution of problem (1)–(4) if the functions $u(x,t)$ and $a(t)$ satisfy the following conditions:

(1) The function $u(x,t)$ and its derivatives $\partial_t^\alpha u(x,t)$, $u_{xx}(x,t)$ are continuous in the domain D_T;
(2) The function $a(t)$ is continuous on the interval $[0,T]$;
(3) Equation (1) and conditions (2)–(4) are satisfied in the classical sense.

Throughout this article, the functions φ, f, and h are assumed to satisfy the following conditions:

(A1) $\varphi(x) \in C^4[0,1]$; $\varphi^{(5)}(x) \in L_2(0,l)$; $\varphi(0) = \varphi(l)$; $\varphi'(0) = \varphi'(l)$; $\varphi''(0) = \varphi''(l)$; $\varphi^{(3)}(0) = \varphi^{(3)}(l)$; $\varphi^{(4)}(0) = \varphi^{(4)}(l)$;

(A2) $f(x,t) \in C^{4,1}_{x,t}(\overline{D}_T)$; $f^{(5)}_{xxxxx}(x,t) \in L_2(0,l)$; $f(0,t) = f(l,t)$; $f'_x(0,t) = f'_x(l,t)$; $f''_{xx}(0,t) = f''_{xx}(l,t)$; $f^{(3)}_{xxx}(0,t) = f^{(3)}_{xxx}(l,t)$; $f^{(4)}_{xxxx}(0,t) = f^{(4)}_{xxxx}(l,t)$;

(A3) $h(t) \in C[0,T]$ and $|h(t)| \geqslant h_0 = \text{const} > 0$, where h_0 is a given number, and $\varphi'(0) = h(0)$.

In the next section, we recall basic definitions and notations from fractional calculus, which will be used in the subsequent analysis.

3. Preliminaries. Let us introduce the definition of the fractional derivative of Caputo.

Definition 2 [29, pp. 90–94]. The Caputo time fractional derivative of order $0 < \alpha < 1$ of the integrable function u is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \quad 0 < \alpha < 1,$$

$$\partial_t^1 u(x,t) = \frac{\partial u(x,t)}{\partial t},$$

where $\Gamma(\,\cdot\,)$ is Euler's Gamma function.

3.1. Two-Parameter Mittag–Leffler Function [29, pp. 40–42]. The two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where α, β, $z \in \mathbb{C}$ with $\Re(\alpha) > 0$, and $\Re(\alpha)$ denotes the real part of the complex number α.

Several important properties of the Mittag–Leffler function, which will be utilized in subsequent sections, are presented below.

Proposition 1 [29, pp. 40–45]. Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. Suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then, there exists a constant $C = C(\alpha,\beta,\kappa) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \quad \kappa \leqslant |\arg(z)| \leqslant \pi.$$

Proposition 2 [29, pp. 42–45]. For $0 < \alpha < 1$ and $\eta > 0$, we have

$$0 \leqslant E_{\alpha,\alpha}(-\eta) \leqslant \Gamma^{-1}(\alpha).$$

Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function for $\eta > 0$.

We also require the following auxiliary results.

Lemma 1 [30], [31, pp. 188–210]. *Let $m(t) \in C[t_0,T]$ ($t_0 \in \mathbb{R}_+ = [0,\infty)$, $T \leqslant +\infty$) and suppose that*

$$m(t) \leqslant m_0 + \frac{L}{\Gamma(\gamma)} \int_{t_0}^t (t-s)^{\gamma-1} m(s) ds, \quad t \in [t_0,T].$$

Then, we have

$$m(t) \leqslant m_0 E_{\gamma,1}(L(t-t_0)^\gamma), \quad t \in [t_0, T],$$

where m_0 and L are nonnegative constants, and $\gamma \in (0,1)$.

Theorem 1 [29, pp. 135–144]. *The solution $T(t) \in AC[0,T]$ of the linear nonhomogeneous fractional problem*

$$\partial_t^\alpha T(t) + \lambda T(t) = f(t), \quad \alpha \in (0,1], \quad t \in (0,T], \quad \lambda > 0,$$

$$T(0) = c,$$

where $f \in L^1[0,T]$, is given by the integral expression

$$T(t) = cE_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) f(\tau) d\tau.$$

These results will be employed throughout the article.

4. Direct Problem. The use of the Fourier method for solving problem (1)–(3) leads to the spectral problem for the operator given by the differential expression and boundary conditions

$$\begin{gathered} X_n''(x) + \lambda^2 X_n(x) = 0, \quad x \in (0,l), \\ X_n(0) = X_n(l), \quad X_n'(0) = X_n'(l), \quad n = 0,1,2,\dots. \end{gathered} \tag{9}$$

It is known [32] that the system of eigenfunctions

$$1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, \dots, \cos\lambda_n x, \sin\lambda_n x, \dots$$

where $\lambda_n = 2\pi n/l$, $n = 0,1,\dots$, is the solution to the spectral problem (9) and forms an orthogonal basis of $L_2(0,l)$. Therefore, we shall seek the classical solution $u(x,t)$ of the problem (5)–(7) in the form

$$\vartheta(x,t) = \sum_{n=0}^\infty \vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^\infty \vartheta_{2n}(t)\sin\lambda_n x, \quad \lambda_n = \frac{2\pi n}{l}, \tag{10}$$

where

$$\vartheta_{10}(t) = \frac{1}{\sqrt{l}}\int_0^l \vartheta(x,t)dx, \quad \vartheta_{1n}(t) = \sqrt{\frac{2}{l}}\int_0^l \vartheta(x,t)\cos\lambda_n x dx,$$

$$\vartheta_{2n}(t) = \sqrt{\frac{2}{l}}\int_0^l \vartheta(x,t)\sin\lambda_n x dx.$$

Then, applying the formal scheme of the Fourier method for determining the unknown coefficients $\vartheta_{10}(t)$ and $\vartheta_{in}(t)$ ($i = 1,2$; $n = 1,2,\dots$) of the function $\vartheta(x,t)$ from (5) and (6), we obtain

$$\partial^\alpha \vartheta_{10}(t) = -a(t)\vartheta_{10}(t) + f_{10}(t), \tag{11}$$

$$\vartheta_{10}(t)|_{t=0} = \varphi_{10}, \tag{12}$$

$$\partial^\alpha \vartheta_{in}(t) + \lambda_n^2 \vartheta_{in}(t) = -a(t)\vartheta_{in}(t) + f_{in}(t), \tag{13}$$

$$\vartheta_{in}(t)|_{t=0} = \varphi_{in}, \quad i = 1, 2, \quad n = 1, 2, \dots, \tag{14}$$

where

$$f_{10}(t) = \frac{1}{\sqrt{l}} \int_0^1 f''_{xx}(x,t)dx, \quad f_{1n}(t) = \sqrt{\frac{2}{l}} \int_0^l f''_{xx}(x,t) \cos \lambda_n x dx,$$

$$f_{2n}(t) = \sqrt{\frac{2}{l}} \int_0^l f''_{xx}(x,t) \sin \lambda_n x dx,$$

$$\varphi_{10} = \frac{1}{\sqrt{l}} \int_0^l \varphi''(x)dx, \quad \varphi_{1n} = \sqrt{\frac{2}{l}} \int_0^l \varphi''(x) \cos \lambda_n x dx,$$

$$\varphi_{2n} = \sqrt{\frac{2}{l}} \int_0^l \varphi''(x) \sin \lambda_n x dx.$$

According to Theorem 1, the solutions of problems (11), (12) and (13), (14) satisfy the following integral equations:

$$\vartheta_{10}(t) = \varphi_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (f_{10}(\tau) - a(\tau)\vartheta_{10}(\tau))d\tau, \tag{15}$$

and

$$\vartheta_{in}(t) = \varphi_{in} E_\alpha(-\lambda^2 t^\alpha) + \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 (t-\tau)^\alpha)(f_{in}(\tau) - a(\tau)\vartheta_{in}(\tau))d\tau. \tag{16}$$

These equations yield that

$$|\vartheta_{10}(t)| \leqslant |\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{10}\| + \frac{\|a\|}{\Gamma(\alpha)} \int_0^t |\vartheta_{10}(\tau)|(t-\tau)^{\alpha-1} d\tau,$$

$$|\vartheta_{in}(t)| \leqslant |\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{in}\| + \frac{\|a\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 (t-\tau)^\alpha) |\vartheta_{in}(\tau)| d\tau,$$

where $\|a\| = \max_{t\in[0,T]} |a(t)|$. Applying Gronwall's type inequality from Lemma 1 to the last relations leads to the following estimates:

$$|\vartheta_{10}(t)| \leqslant \left(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{10}\|\right) E_\alpha(\|a\| t^\alpha), \tag{17}$$

$$|\vartheta_{in}(t)| \leqslant \left(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{in}\|\right) E_\alpha(\|a\| t^\alpha). \tag{18}$$

Using equalities (11), (13) and (17), (18), we obtain estimates for $\partial^\alpha \vartheta_{10}(t)$ and $\partial^\alpha \vartheta_{in}(t)$:

$$|\partial^\alpha \vartheta_{10}(t)| \leqslant \|a\| \left(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{10}\|\right) E_\alpha(\|a\| t^\alpha) + \|f_{10}\|,$$

$$|\partial^\alpha \vartheta_{in}(t)| \leqslant (\lambda^2 + \|a\|)\left(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{in}\|\right) E_\alpha(\|a\| t^\alpha) + \|f_{in}\|.$$

Thus, we have proved the following lemma:

Lemma 2. *For any $t \in [0,T]$, the following estimates are valid:*

$$|\vartheta_{10}(t)| \leqslant \left(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{10}\|\right) E_\alpha(\|a\| t^\alpha),$$

$$|\vartheta_{in}(t)| \leqslant \left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|f_{in}\|\right) E_\alpha(\|a\| T^\alpha),$$

$$|\partial^\alpha \vartheta_{10}(t)| \leqslant \|a\|\left(|\varphi_{10}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|f_{10}\|\right) E_\alpha(\|a\| T^\alpha) + \|f_{10}\|,$$

$$|\partial^\alpha \vartheta_{in}(t)| \leqslant (\lambda^2 + \|a\|)\left(|\varphi_{in}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|f_{in}\|\right) E_\alpha(\|a\| T^\alpha) + \|f_{in}\|.$$

Formally, from (10) by term-by-term differentiation, we compose the series

$$\partial^\alpha_{t,0+}\vartheta(x,t) = \sum_{n=0}^{\infty} \partial^\alpha_{0+}\vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} \partial^\alpha_{0+}\vartheta_{2n}(t)\sin\lambda_n x, \tag{19}$$

$$\vartheta_{xx}(x,t) = -\sum_{n=0}^{\infty} \lambda_n^2 \vartheta_{1n}(t)\cos\lambda_n x - \sum_{n=1}^{\infty} \lambda_n^2 \vartheta_{2n}(t)\sin\lambda_n x. \tag{20}$$

In view of Lemma 2, if the following series converges, then the series (10), (19), and (20) will converge for any $(x,t) \in D_T$:

$$C_4 \sum_{n=1}^{\infty} (\lambda_n^2 |\varphi_{in}| + \lambda_n^2 \|f_{in}\|), \tag{21}$$

where the constant C_4 depends only on T, α, and $\|a\|$.

We now state the following auxiliary lemma:

Lemma 3. *If conditions (A1) and (A2) are valid, then the following equalities hold:*

$$\varphi_{in} = \frac{1}{\lambda_n^3}\varphi_{in}^{(3)}, \quad f_{in}(t) = \frac{1}{\lambda_n^3} f_{in}^{(3)}, \quad i = 1,2, \tag{22}$$

where

$$\varphi_{1n}^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(5)}(x)\sin\lambda_n x dx, \quad \varphi_{2n}^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(5)}(x)\cos\lambda_n x dx,$$

$$f_{1n}^{(3)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxxxx}^{(5)}(x,t)\sin\lambda_n x dx, \quad f_{2n}^{(3)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxxxx}^{(5)}(x,t)\cos\lambda_n x dx,$$

with the following estimate:

$$\sum_{n=1}^{\infty} |\varphi_{in}^{(3)}|^2 \leqslant \|\varphi^{(3)}\|^2_{L_2(0,l)}, \quad \sum_{n=1}^{\infty} |f_{in}^{(3)}(t)|^2 \leqslant \|f_{xxx}^{(3)}\|^2_{L_2(0,l)\times C[0,T]}, \quad i = 1,2. \tag{23}$$

If the functions $\varphi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 3, then due to representations (22) and (23), the series (10), (19), and (20) converge uniformly in the rectangle D_T. Therefore, the function $\vartheta(x,t)$ satisfies relations (5)-(7).

Using the above results, we obtain the following assertion:

Lemma 4. *Let $a(t) \in C[0,T]$, and suppose that conditions (A1) and (A2) are satisfied. Then, there exists a unique solution to the direct problem (5)–(7) such that $\vartheta(x,t) \in C^{2,\alpha}(D_T) \cap C^{1,0}(\overline{D}_T)$.*

Let us derive an estimate for the norm of the difference between the solution of the original integral equations (15), (16) and the solution of these equations with perturbed functions $\widetilde{a}$, $\widetilde{\varphi}_{in}$, and $\widetilde{f}_{in}$. Let $\widetilde{\vartheta}_{in}(t)$ $(i = 0,1,2)$ be solutions of the integral equation (15), (16) corresponding to the functions $\widetilde{a}$, $\widetilde{\varphi}_{in}$, and $\widetilde{f}_{in}$; i.e.,

$$\widetilde{\vartheta}_{10}(t) = \widetilde{\varphi}_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\widetilde{f}_{10}(\tau) - \widetilde{a}(\tau)\widetilde{\vartheta}_{10}(\tau)) d\tau, \tag{24}$$

$$\widetilde{\vartheta}_{in}(t) = \widetilde{\varphi}_{in} E_\alpha(-\lambda^2 t^\alpha) + \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 (t-\tau)^\alpha)(\widetilde{a}(\tau)\widetilde{\vartheta}_{in}(\tau) + \widetilde{f}_{in}(\tau)) d\tau. \tag{25}$$

Composing the difference $\vartheta_{in} - \widetilde{\vartheta}_{in}$ with the help of equations (15), (24), (16), and (25), and introducing the notations $\vartheta_{in} - \widetilde{\vartheta}_{in} = \overline{\vartheta}_{in}$, $a - \widetilde{a} = \overline{a}$, and $f_{in} - \widetilde{f}_{in} = \overline{f}_{in}$, we obtain the integral equation

$$\overline{\vartheta}_{10}(t) = \overline{\varphi}_{10} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \overline{f}_{10}(\tau) d\tau - \\ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \big(\overline{a}(\tau)\vartheta_{10}(\tau) + \widetilde{a}(\tau)\overline{\vartheta}_{10}(\tau)\big) d\tau;$$

$$\overline{\vartheta}_{in}(t) = \overline{\varphi}_n E_{\alpha,1}(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \overline{f}_{in}(\tau) d\tau - \\ - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \overline{a}(\tau)\vartheta_{in}(\tau) d\tau - \\ - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \widetilde{a}(\tau)\overline{\vartheta}_{in}(\tau) d\tau.$$

From these, we derive the following linear integral inequalities for $|\overline{\vartheta}_{10}(t)|$ and $|\overline{\vartheta}_{in}(t)|$:

$$|\overline{\vartheta}_{10}(t)| \leqslant |\overline{\varphi}_{10}| + \frac{t^\alpha \|\overline{f}_{10}\|}{\Gamma(\alpha+1)} + \frac{\|\overline{a}\| t^\alpha}{\Gamma(\alpha+1)} \Big(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{10}\|\Big) E_\alpha(\|a\| t^\alpha) + \\ + \frac{\|\widetilde{a}\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\overline{\vartheta}_{10}(\tau)| d\tau,$$

$$|\overline{\vartheta}_{in}(t)| \leqslant |\overline{\varphi}_{in}| + \frac{t^\alpha \|\overline{f}_{in}\|}{\Gamma(\alpha+1)} + \frac{\|\overline{a}\| t^\alpha}{\Gamma(\alpha+1)} \Big(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{in}\| \Big) E_\alpha(\|a\| t^\alpha) + \\ + \frac{\|\widetilde{a}\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |\overline{\vartheta}_{in}(\tau)| d\tau.$$

Using Lemma 1 from the last inequalities, we arrive at the following estimates:

$$|\overline{\vartheta}_{10}(t)| \leqslant \Big\{ |\overline{\varphi}_{10}| + \frac{t^\alpha \|\overline{f}_{10}\|}{\Gamma(\alpha+1)} + \\ + \frac{\|\overline{a}\| t^\alpha}{\Gamma(\alpha+1)} \Big(|\varphi_{10}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{10}\| \Big) E_\alpha(\|a\| t^\alpha) \Big\} E_\alpha(\|\widetilde{a}\| t^\alpha), \quad (26)$$

$$|\overline{\vartheta}_{in}(t)| \leqslant \Big\{ |\overline{\varphi}_{in}| + \frac{t^\alpha \|\overline{f}_{in}\|}{\Gamma(\alpha+1)} + \\ + \frac{\|\overline{a}\| t^\alpha}{\Gamma(\alpha+1)} \Big(|\varphi_{in}| + \frac{t^\alpha}{\Gamma(\alpha+1)} \|f_{in}\| \Big) E_\alpha(\|a\| t^\alpha) \Big\} E_\alpha(\|\widetilde{a}\| t^\alpha). \quad (27)$$

In the next section, we study the inverse problem, which involves determining the function $a(t)$ from relations (5)–(8) using the contraction mapping principle.

5. Solvability of the Inverse Problem. First, by differentiating (10) with respect to x, we obtain the following equality:

$$\vartheta_x(x,t) = -\sum_{n=0}^{\infty} \lambda_n \vartheta_{1n}(t) \sin \lambda_n x + \sum_{n=1}^{\infty} \lambda_n \vartheta_{2n}(t) \cos \lambda_n x. \quad (28)$$

Setting $x = 0$ in (28) and using the additional condition (8), after straightforward manipulations, we obtain the following integral equation for determining $a(t)$:

$$a(t) = a_0(t) + \frac{1}{h(t)} \sum_{n=1}^{\infty} \lambda_n \vartheta_{2n}(t;a), \quad (29)$$

where

$$a_0(t) = \frac{1}{h(t)} \big(f_x(0,t) - \partial^\alpha h(t) \big),$$

and $\vartheta_{2n}(t;a)$ denotes that the solution of the integral equation (16) depends on $a(t)$.

The main result of this study is presented as follows:

THEOREM 2. *Let conditions (A1)–(A3) be satisfied. Then, there exists a number $T^* \in (0,T)$ such that the inverse problem (5)–(8) has a unique solution $a(t)$.*

Proof. We consider the operator equation

$$g = \Lambda[g], \quad (30)$$

where $g := a(t)$ is the unknown function, and Λ is defined by the right-hand side of (29):

$$\Lambda[g](t) = g_{01}(t) - \frac{1}{h(t)} \sum_{n=1}^{\infty} \lambda_{2n} \vartheta_{2n}(t;g), \quad (31)$$

where

$$g_{01}(t) = \frac{1}{h(t)}\big(f_x(0,t) - \partial^\alpha h(t)\big).$$

Consider the functional space of vector functions $g \in C[0,T]$ with the norm given by

$$\|g\| = \max_{t\in[0,T]} |g(t)|.$$

Fix a number $\rho > 0$ and consider the ball

$$\Phi^T(g_0,\rho) := \{g : \|g - g_0\|_{C[0,T]} \leqslant \rho\}.$$

We will prove that for sufficiently small $T > 0$, the operator Λ maps the ball $\Phi^T(g_0,\rho)$ into itself. Using the estimates (17) and (18) for ϑ_{1n} and ϑ_{2n}, we obtain the following estimates:

$$\|\Lambda[g](t) - g_{01}(t)\| \leqslant \frac{1}{h_0}\left|\sum_{n=1}^{\infty}\lambda_n\vartheta_{2n}(T;g_1)\right| \leqslant$$

$$\leqslant \frac{1}{h_0}E_\alpha(\|g\|T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_{2n}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big).$$

According to Lemmas 1 and 2, the above series is convergent. Note that the functions occurring on the right-hand side of these inequalities are monotonically increasing with respect to T, and the fact that the function $g(t)$ belongs to the ball $\Phi^T(g_0,\rho)$ implies the inequality

$$\|g\| \leqslant \rho + \|g_0\|. \tag{32}$$

Therefore, we only strengthen the inequality if we replace $\|g\|$ in these inequalities with $\rho + \|g_0\|$. Performing these replacements, we obtain the estimate

$$\|\Lambda_1[g](t) - g_{01}(t)\| \leqslant \frac{1}{h_0}E_\alpha((\rho + \|g_0\|)T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_{2n}| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big).$$

Let T_1 be a positive root of the equation

$$m_1(T) = \frac{1}{h_0}E_{\alpha,1}((\rho + \|q_0\|)T^\alpha)\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_{2n}| + \frac{T^\alpha\|f_{2n}\|}{\alpha\Gamma(\alpha)}\Big) = \rho.$$

Then, for $T \in [0,T_1]$, we have $\Lambda[g](t) \in \Phi^T(g_0,\rho)$.

Now consider two functions $g(t)$ and $\widetilde{g}(t)$ belonging to $\Phi^T(g_0,\rho)$ and estimate the distance between their images $\Lambda[g](t)$ and $\Lambda[\widetilde{g}](t)$ in the space $C[0,T]$. The function $\widetilde{\vartheta}_{in}(t)$ corresponding to $\widetilde{g}(t)$ satisfies the integral equation (24), (25) with the functions $\varphi_{2n} = \widetilde{\varphi}_{2n}$ and $f_{2n} = \widetilde{f}_{2n}$. Composing the difference $\Lambda[g](t) - \Lambda[\widetilde{g}](t)$ with the help of equations (15), (16), (24), and (25), and then estimating its norm, we obtain

$$\|\Lambda[g](t) - \Lambda[\widetilde{g}](t)\| \leqslant \frac{1}{h_0}\sum_{n=0}^{\infty}\lambda_n\|\overline{\vartheta}_{2n}(T;g)\| \leqslant$$

$$\leqslant \frac{1}{h_0}E_\alpha(\|\widetilde{g}\|t^\alpha)\sum_{n=0}^{\infty}\lambda_n\Big\{|\overline{\varphi}_{2n}| + \frac{t^\alpha\|\overline{f}_{2n}\|}{\Gamma(\alpha+1)} +$$

$$+ \frac{\|\overline{g}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{2n}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big)E_\alpha(\|g\|t^\alpha)\Big\}.$$

Using inequalities (17), (18), and the estimate (26) with $\varphi_{2n} = \widetilde{\varphi}_{2n}$ and $f_n = \widetilde{f}_n$, we continue the previous inequality as follows:

$$\|\Lambda[g](t) - \Lambda[\widetilde{g}](t)\| \leqslant \frac{1}{h_0}\sum_{n=0}^{\infty}\lambda_n\|\overline{\vartheta}_{2n}(T;g)\| \leqslant$$

$$\leqslant \frac{1}{h_0}E_\alpha(\|\widetilde{g}\|t^\alpha)\sum_{n=0}^{\infty}\lambda_n\frac{\|\overline{g}\|t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{2n}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big)E_\alpha(\|g\|t^\alpha).$$

Since $g(t)$ and $\widetilde{g}(t)$ belong to the ball $\Phi^T(g_0,\rho)$, the inequality (32) holds for these functions. Note that the functions on the right-hand side of inequality (30) are monotonically increasing with respect to $\|g\|$, $\|\widetilde{g}\|$, and T. Consequently, replacing $\|g\|$ and $\|\widetilde{g}\|$ in inequality (30) with $\rho + \|g_0\|$ will only strengthen the inequality. Thus, we obtain

$$\|\Lambda[g](t) - \Lambda[\widetilde{g}](t)\| \leqslant \frac{1}{h_0}\sum_{n=0}^{\infty}\lambda_n\|\overline{\vartheta}_{2n}(T;g)\| \leqslant$$

$$\leqslant \frac{1}{h_0}\left(E_\alpha\big((\rho+\|g_0\|)t^\alpha\big)\right)^2\sum_{n=0}^{\infty}\lambda_n\frac{t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{2n}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big)\|\overline{g}\| \leqslant$$

$$\leqslant m_2(T)\|\overline{g}\|.$$

Let T_2 be a positive root of the equation

$$\frac{1}{h_0}\left(E_\alpha\big((\rho+\|g_0\|)t^\alpha\big)\right)^2\sum_{n=0}^{\infty}\lambda_n\frac{t^\alpha}{\Gamma(\alpha+1)}\Big(|\varphi_{2n}| + \frac{t^\alpha}{\Gamma(\alpha+1)}\|f_{2n}\|\Big) = 1.$$

Then, for $T \in [0, T_2)$, the operator Λ contracts the distance between the elements $g(t)$ and $\widetilde{g}(t) \in \Phi^T(g_0,\rho)$. Consequently, if we choose $T^* < \min(T_1, T_2)$, then the operator Λ is a contraction in the ball $\Phi^T(g_0,\rho)$. According to the Banach fixed-point theorem (see [33, p. 87–97]), the operator Λ has a unique fixed point in the ball $\Phi^T(g_0,\rho)$; that is, there exists a unique solution to equation (31). □

Let T, l be fixed positive numbers. Consider the set $\Omega(\chi_0)$ (where $\chi_0 > 0$ is a fixed number) of given functions (φ, f, h) for which all conditions (A1)–(A3) are fulfilled, and

$$\max\Big\{\|\varphi\|_{C^4[0,T]}, \|h\|_{C^1[0,T]}, \|h^{-1}\|_{C[0,T]}, \|f\|_{C^{4,1}(\overline{D}_T)}\Big\} \leqslant \chi_0.$$

We denote by $Q(\chi_1)$ the set of functions $a(t)$ that, for some $T > 0$ and $l > 0$, satisfy the following condition:

$$\|a\|_{C[0,T]} \leqslant \chi_1, \quad \chi_1 > 0.$$

THEOREM 3. *Let* $(\varphi, f, h) \in \Omega(\chi_0)$, $(\widetilde{\varphi}, \widetilde{f}, \widetilde{h}) \in \Omega(\chi_0)$, *and* $a \in Q(\chi_1)$, $\widetilde{a} \in Q(\chi_1)$. *Then, for the solution of the inverse problem* (1)–(4), *the following stability estimate holds*:

$$\|a - \widetilde{a}\|_{C[0,T]} \leqslant d\big(\|\varphi - \widetilde{\varphi}\|_{C^4[0,l]} + \|f - \widetilde{f}\|_{C^{4,1}(\overline{D}_T)} + \|h - \widetilde{h}\|_{C^1[0,T]}\big), \tag{33}$$

where the constant d depends only on T, l, χ_0, χ_1.

Proof. To prove this theorem, using (29), we write down the equations for $\widetilde{a}(t)$ and compose the difference $\overline{a}(t) = a(t) - \widetilde{a}(t)$. Then, after evaluating this expression and using estimates (18) and (27), we obtain the following estimate:

$$\|a(t) - \widetilde{a}(t)\| \leqslant d_0\big(\|\overline{f}\| + \|\overline{\varphi}\| + \|\overline{h}\|\big) + d_1\int_0^t (t-\tau)^{\alpha-1}\|a(\tau) - \widetilde{a}(\tau)\|d\tau, \quad t \in [0,T],$$

where d_0 and d_1 depend only on χ_0, χ_1, T, α, $\Gamma(\alpha)$. From (21), using Lemma 1, we obtain the estimate

$$\|a(t) - \widetilde{a}(t)\| \leqslant d_0\big(\|\overline{f}\| + \|\overline{\varphi}\| + \|\overline{h}\|\big)E_{\alpha,1}(d_1\Gamma(\alpha)t^{\alpha}), \quad t \in [0, T].$$

This inequality implies the estimate (33) if we set $d = d_0E_{\alpha,1}(d_1\Gamma(\alpha)t^{\alpha})$. □

Theorem 3 also implies the following assertion on the uniqueness of the solution to the inverse problem.

Theorem 4. *Let the functions φ, f, h and $\widetilde{\varphi}$, $\widetilde{f}$, $\widetilde{h}$ have the same meaning as in Theorem 3, and let conditions (A1)–(A3) be satisfied. Moreover, if $\varphi = \widetilde{\varphi}$, $f = \widetilde{f}$, and $h = \widetilde{h}$ for $t \in [0, T]$, then $a(t) = \widetilde{a}(t)$ for $t \in [0, T]$.*

Competing interests. The authors declare no conflict of interest regarding the authorship and publication of this article.

Authors' contributions and responsibilities. All authors contributed equally to the development of the article's concept and the writing of the manuscript. The authors take full responsibility for submitting the final version of the manuscript for publication. The final version of the manuscript was approved by all authors.

Funding. The research was conducted without external funding.

References

1. Agarwal R., Sharma U. P., Agarwal R. P. Bicomplex Mittag–Leffler function and associated properties, *J. Nonlinear Sci. Appl.*, 2022, vol. 15, no. 1, pp. 48–60, arXiv: 2103.10324 [math.CV]. DOI: https://doi.org/10.22436/jnsa.015.01.04.
2. Haddouchi F., Guendouz C., Benaicha S. Existence and multiplicity of positive solutions to a fourth-order multi-point boundary value problem, *Matemat. Vesn.*, 2021, vol. 73, no. 1, pp. 25–36, arXiv: 1908.08598 [math.CA].
3. Sharma S. *Molecular Dynamics Simulation of Nanocomposites Using BIOVIA Materials Studio, Lammps and Gromacs.* Amsterdam, Netherlands, Elsevier, 2019, xvi+349 pp. DOI: https://doi.org/10.1016/C2017-0-04396-3.
4. Baglan I. Determination of a coefficient in a quasilinear parabolic equation with periodic boundary condition, *Inverse Probl. Sci. Eng.*, 2015, vol. 23, no. 5, pp. 884-900. DOI: https://doi.org/10.1080/17415977.2014.947479.
5. Kanca F. Inverse coefficient problem of the parabolic equation with periodic boundary and integral overdetermination conditions, *Abstr. Appl. Anal.*, 2013, no. 5, 659804. DOI: https://doi.org/10.1155/2013/659804.
6. Kanca F., Baglan I. An inverse coefficient problem for a quasilinear parabolic equation with nonlocal boundary conditions., *Bound. Value Probl.*, 2013, vol. 2013, 213. DOI: https://doi.org/10.1186/1687-2770-2013-213.
7. Kanca F. The inverse problem of the heat equation with periodic boundary and integral overdetermination conditions, *J. Inequal. Appl.*, 2013, vol. 2013, 108. DOI: https://doi.org/10.1186/1029-242X-2013-108.
8. Ivanchov N. I. Inverse problems for the heat-conduction equation with nonlocal boundary condition, *Ukr. Math. J.*, 1993, vol. 45, no. 8, pp. 1186–1192. DOI: https://doi.org/10.1007/BF01070965.
9. Ivanchov M. I., Pabyrivska N. Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions, *Ukr. Math. J.*, 2001, vol. 53, no. 5, pp. 674–684. DOI: https://doi.org/10.1023/A:1012570031242.
10. Liao W., Dehghan M., Mohebbi A. Direct numerical method for an inverse problem of a parabolic partial differential equation, *J. Comput. Appl. Math.*, 2009, vol. 232, no. 2, pp. 351–360. DOI: https://doi.org/10.1016/j.cam.2009.06.017.

11. Oussaeif T. E., Abdelfatah B. An inverse coefficient problem for a parabolic equation under nonlocal boundary and integral overdetermination conditions, *Int. J. Part. Dif. Equ. Appl.*, 2014, vol. 2, no. 3, pp. 38–43. DOI: https://doi.org/10.12691/ijpdea-2-3-1.
12. Cannon J. R., Lin Y., Wang S. Determination of a control parameter in a parabolic partial differential equation, *J. Aust. Math. Soc., Ser. B*, 1991, vol. 33, no. 2, pp. 149–163. DOI: https://doi.org/10.1017/S0334270000006962.
13. Hazanee A., Lesnic D., Ismailov M. I., Kerimov N. B. Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions, *Appl. Math. Comput.*, 2019, vol. 346, pp. 800–815. DOI: https://doi.org/10.1016/j.amc.2018.10.059.
14. Huzyk N. Nonlocal inverse problem for a parabolic equation with degeneration, *Ukr. Math. J.*, 2013, vol. 65, no. 6, pp. 847–863. DOI: https://doi.org/10.1007/s11253-013-0822-6.
15. Durdiev D. K., Jumaev J. J. Inverse coefficient problem for a time-fractional diffusion equation in the bounded domain, *Lobachevskii J. Math.*, 2023, vol. 44, no. 2, pp. 548–557. DOI: https://doi.org/10.1134/S1995080223020130.
16. Durdiev D. K., Rahmonov A. A., Bozorov Z. R. A two-dimensional diffusion coefficient determination problem for the time-fractional equation, *Math. Methods Appl. Sci.*, 2021, vol. 44, no. 13, pp. 10753–10761. DOI: https://doi.org/10.1002/mma.7442.
17. Ionkin N. I. Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, *Differ. Equ.*, 1977, vol. 13, no. 2, pp. 204–211.
18. Subhonova Z. A., Rahmonov A. A. Problem of determining the time dependent coefficient in the fractional diffusion-wave equation, *Lobachevskii J. Math.*, 2021, vol. 42, no. 15, pp. 3747–3760. DOI: https://doi.org/10.1134/S1995080222030209.
19. Sultanov M. A., Durdiev D. K., Rahmonov A. A. Construction of an explicit solution of a time-fractional multidimensional differential equation, *Mathematics*, 2021, vol. 9, no. 17, 2052. DOI: https://doi.org/10.3390/math9172052.
20. Colombo F. An inverse problem for a parabolic integrodifferential model in the theory of combustion, *Phys. D*, 2007, vol. 236, no. 2, pp. 81–89. DOI: https://doi.org/10.1016/j.physd.2007.07.012.
21. Durdiev D. K., Nuriddinov J. Z. Determination of a multidimensional kernel in some parabolic integro-differential equation, *J. Sib. Fed. Univ., Math. Phys.*, 2021, vol. 14, no. 1, pp. 117–127. EDN: RMPPXU. DOI: https://doi.org/10.17516/1997-1397-2021-14-1-117-127.
22. Durdiev D. K., Zhumaev Zh. Zh. Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor, *Math. Methods Appl. Sci.*, 2022, vol. 45, no. 14, pp. 8374–8388. DOI: https://doi.org/10.1002/mma.7133.
23. Durdiev D. K., Zhumaev Zh. Zh. One-dimensional inverse problems of finding the kernel of the integro-differential heat equation in a bounded domain, *Ukr. Math. J.*, 2022, vol. 73, no. 11, pp. 1723–1740. DOI: https://doi.org/10.1007/s11253-022-02026-0.
24. Durdiev D. K., Zhumaev Zh. Zh. Problem of determining a multidimensional thermal memory in a heat conductivity equation, *Methods Funct. Anal. Topol.*, 2019, vol. 25, no. 3, pp. 219–226.
25. Durdiev D. K., Zhumaev Zh. Zh. Problem of determining the thermal memory of a conducting medium, *Differ. Equ.*, 2020, vol. 56, no. 6, pp. 785–796. DOI: https://doi.org/10.1134/S0012266120060117.
26. Dyatlov G. V. Determination for the memory kernel from boundary measurements on a finite time interval, *J. Inverse Ill-Posed Probl.*, 2003, vol. 11, no. 1, pp. 59–66. DOI: https://doi.org/10.1515/156939403322004937.
27. Janno J., Wolfersdorf L. Inverse problems for identification of memory kernels in heat flow, *Ill-Posed Probl.*, 1996, vol. 4, no. 1, pp. 39–66. DOI: https://doi.org/10.1515/jiip.1996.4.1.39.
28. Lorenzi A., Paparoni E. Direct and inverse problems in the theory of materials with memory, *Rend. Semin. Mat. Univ. Padova*, 1992, vol. 87, pp. 105–138.

29. Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Amsterdam, Elsevier, 2006, xx+523 pp. DOI: https://doi.org/10.1016/s0304-0208(06)x8001-5.
30. Kang B., Koo N. A note on generalized singular Gronwall inequalities, *J. Chungcheong Math. Soc.*, 2018, vol. 31, no. 1, pp. 161–167. DOI: https://doi.org/10.14403/jcms.2018.31.1.161.
31. Lakshmikantham V., Leela S., Vasundhara Devi J. *Theory of Fractional Dynamic Systems*. Cambridge, Cambridge Scientific Publ., 2009, vi+170 pp.
32. Budak B. M., Samarskii A. A., Tikhonov A. N. *Sbornik zadach po matematicheskoi fizike* [A collection of Problems on Mathematical Physics]. Moscow, Nauka, 1979, 685 pp. (In Russian)
33. Kolmogorov A. N., Fomin S. V. *Elementy teorii funktsii i funktsional'nogo analiza* [Elements of Function Theory and Functional Analysis]. Moscow, Nauka, 1972, 496 pp. (In Russian)

Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. 2025. Т. 29, № 1. С. 21–36
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu2083
EDN: YZQBWZ

УДК 517.968.7

Разрешимость задачи восстановления коэффициентов в дробно-временном уравнении диффузии с периодическими граничными и переопределенными условиями

Д. К. Дурдиев[1,2], *Ж. Ж. Жумаев*[1]

[1] Бухарское отделение Института математики Академии наук Республики Узбекистан, Узбекистан, 705018, Бухара, ул. М. Икбол, 11.

[2] Бухарский государственный университет, Узбекистан, 705018, Бухара, ул. М. Икбол, 11.

Аннотация

Исследуется обратная задача для уравнений дробно-временной диффузии с периодическими граничными условиями и интегральными условиями переопределения на прямоугольной области. Сначала вводится определение классического решения задачи. Затем с использованием метода Фурье прямая задача сводится к эквивалентному интегральному уравнению. Существование и единственность решения прямой задачи устанавливаются с помощью оценок для функции Миттаг–Леффлера и обобщенных сингулярных неравенств Гронвалля.

Во второй части работы рассматривается обратная задача, которая переформулируется в виде эквивалентного интегрального уравнения, а затем решается с использованием принципа сжимающих отображений. Строго доказываются локальное существование и глобальная единственность решения. Кроме того, получена оценка устойчивости решения.

Данное исследование вносит вклад в теорию обратных задач для дробных дифференциальных уравнений, предоставляя основу для анализа задач с периодическими граничными условиями и интегральными

Дифференциальные уравнения и математическая физика
Научная статья

Образец для цитирования
Durdiev D. K., Jumaev J. J. Solvability of a coefficient recovery problem for a time-fractional diffusion equation with periodic boundary and overdetermination conditions, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 1, pp. 21–36. EDN: YZQBWZ. DOI: 10.14498/vsgtu2083.

Сведения об авторах

Дурдимурод Каландарович Дурдиев https://orcid.org/0000-0002-6054-2827
доктор физико-математических наук, профессор; заведующий отделением[1]; профессор, каф. дифференциальных уравнений[2]; e-mail: d.durdiev@mathinst.uz

Жонибек Жамолович Жумаев https://orcid.org/0000-0001-8496-1092
кандидат физико-математических наук, доцент; старший научный сотрудник[1]; доцент, каф. дифференциальных уравнений[2]; e-mail: jonibekjj@mail.ru

условиями переопределения. Разработанные методы могут быть применены к широкому кругу задач в математической физике и инженерии, где дробно-временные модели диффузии все чаще используются для описания сложных явлений.

Ключевые слова: уравнение дробно-временной диффузии, периодические граничные условия, обратная задача, интегральное уравнение.

Получение: 15 февраля 2024 г. / Исправление: 19 ноября 2024 г. / Принятие: 21 февраля 2025 г. / Публикация онлайн: 14 марта 2025 г.

Конкурирующие интересы. Авторы заявляют об отсутствии конфликта интересов в отношении авторства и публикации данной статьи.

Авторский вклад и ответственность. Все авторы внесли равный вклад в разработку концепции статьи и написание рукописи. Авторы несут полную ответственность за предоставление окончательной версии рукописи в печать. Окончательная версия рукописи была одобрена всеми авторами.

Финансирование. Исследование выполнено без привлечения внешнего финансирования.