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Inverse kernel determination problem for a class of pseudo-parabolic integro-differential equations

D. K. Durdiev[1,2], *H. B. Elmuradova*[2], *A. A. Rahmonov*[1,2]

[1] Bukhara Branch of the Institute of Mathematics named after V. I. Romanovskiy at the Academy of Sciences of the Republic of Uzbekistan, 11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.
[2] Bukhara State University, 11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.

Abstract

This study investigates an inverse problem involving the determination of the kernel function in a multidimensional integrodifferential pseudo-parabolic equation of the third order. The study begins with an analysis of the direct problem, where we examine an initial-boundary value problem with homogeneous boundary conditions for a known kernel. Employing the Fourier method, we construct the solution as a series expansion in terms of eigenfunctions of the Laplace operator with Dirichlet boundary conditions. A crucial component of our analysis involves deriving a priori estimates for the series coefficients in terms of the kernel function norm, which play a fundamental role in our subsequent treatment of the inverse problem.

For the inverse problem, we introduce an overdetermination condition specifying the solution value at a fixed spatial point (pointwise measurement). This formulation leads to a Volterra-type integral equation of the second kind. By applying the Banach fixed-point principle within the framework of continuous functions equipped with an exponentially weighted norm, we establish the global existence and uniqueness of solutions to the inverse problem. Our results demonstrate the well-posedness of the problem under consideration.

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Authors' Details:

Durdimurod K. Durdiev https://orcid.org/0000-0002-6054-2827
Dr. Phys. & Math. Sci., Professor; Head of Branch[1]; Professor, Dept. of Differential Equations[2]; e-mail: d.durdiev@mathinst.uz

Hilola B. Elmuradova https://orcid.org/0000-0003-4306-2589
Teacher; PhD Student; Dept. of Differential Equations[2]; e-mail: helmuradova@mail.ru

Askar A. Rahmonov https://orcid.org/0000-0002-7641-9698
Cand. Phys. & Math. Sci., Associate Professor; Senior Researcher[1]; Associate Professor, Dept. of Differential Equations[2]; e-mail: araxmonov@mail.ru

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Introduction. There are numerous cases where practical applications lead to challenges in determining the coefficients, the right-hand side of the differential equation, and the kernel of integrodifferential equations. Such problems are referred to as inverse problems of mathematical physics.

Inverse problems currently represent a rapidly developing branch of modern mathematics. Various inverse problems for second-order hyperbolic and parabolic equations, as well as first-order systems, are discussed in the monographs [1–5] (see also the extensive bibliographies therein). The recently published monograph [6] investigates a new class of inverse problems involving the determination of the convolution kernel in second-order hyperbolic integrodifferential equations.

Water filtration in double-porosity media, moisture transfer in soil, and similar natural phenomena often lead to boundary value problems involving pseudo-parabolic equations (see, e.g., [7, 8]). When such processes occur in viscoelastic media, Volterra operators — representing the convolution of a time-dependent viscosity function with a solution operator (typically elliptic) — are incorporated into the right-hand side of the pseudo-differential equations.

The study of inverse problems for pseudo-parabolic equations began in the 1980s. The first significant result, obtained in [9], addressed the inverse identification of an unknown source function. Among recent works, we highlight [10], where the author examined an inverse problem of recovering a space-dependent source coefficient in a third-order pseudo-parabolic equation under a final overdetermination condition (see also references therein).

To the best of our knowledge, the problem of determining the convolution kernel in an integrodifferential pseudo-parabolic equation remains unexplored. However, a series of works [11–20] has investigated inverse problems involving convolution kernel determination for linear parabolic integrodifferential equations. These studies established local existence and global uniqueness theorems, as well as stability estimates for the solutions.

In this study, we employ the Fourier method, integral inequalities, and the fixed-point principle to prove the existence and uniqueness of a solution to the inverse problem of determining the kernel of a multidimensional third-order integrodifferential pseudo-parabolic equation. The problem is supplemented with an additional condition specified at a fixed point for the solution of the first boundary value problem.

Consider the following nonhomogeneous pseudo-parabolic integrodifferential equation:

$$u_t - \Delta u_t - \Delta u = (k * \Delta u)(x,t) + f(x,t), \quad (x,t) \in D, \tag{1}$$

where $D = \Omega \times (0,T]$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Here, Δ denotes the Laplacian, $k(t)$ is the convolution kernel representing the "memory effect" (or viscosity function), $f(x,t)$ is a source function, and $(k * \Delta u)(x,t)$ denotes the Laplace convolution:

$$(k * u)(x,t) := \int_0^t k(t-s)u(x,s)\,ds.$$

In the domain D, we study the following problem for Eq. (1): *Find a function $u(x,t)$ satisfying (1) with the initial condition*

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

and the boundary condition

$$u = 0 \quad on \quad \partial\Omega \times (0,T), \tag{3}$$

where $f(x,t)$ and $\varphi(x)$ are given functions. This problem is commonly referred to as the direct (forward) problem.

A function $u(x,t)$ is called a *classical solution* to problem (1)–(3) if it satisfies the following conditions:

1) $u(x,t)$ is continuous in $\overline{D}$ along with all derivatives appearing in Eq. (1);
2) all given conditions are satisfied in the classical sense.

Based on this direct problem, we now consider the following inverse problem.

Inverse problem. *Determine the kernel $k(t)$, $t > 0$, appearing in equation (1), given that the solution of the direct problem satisfies the additional condition*

$$u(x_0,t) = h(t), \quad x_0 \in \Omega, \ t \in [0,T], \tag{4}$$

where $x_0 \in \Omega$ is a fixed point and $h(t)$ is a given sufficiently smooth function.

1. Investigation of the Direct Problem. This section studies problem (1)–(3). We prove the existence and uniqueness of a classical solution to problem (1)–(3).

1.1. Uniqueness of the Solution. The following uniqueness result holds for (1)–(3).

Theorem 1. *If problem (1)–(3) has a solution, then this solution is unique.*

Proof. Applying the method of separation of variables, we seek a solution to (1)–(3) in the form

$$u(x,t) = U(t)X(x). \tag{5}$$

Substituting (5) into (1) with

$$\int_0^t k(t-\tau)\Delta u(x,\tau)d\tau + f(x,t) = 0,$$

we require that $X(x) \not\equiv 0$ satisfies the spectral problem

$$\begin{cases} \Delta X + \lambda X = 0, & \text{in } \Omega, \\ X = 0, & \text{on } \partial\Omega. \end{cases}$$

It is well-known that the operator $-\Delta$ has only positive real and simple eigenvalues λ_m, which when properly ordered satisfy $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lim\limits_{m\to\infty} \lambda_m = +\infty$. We denote by X_m the eigenfunction corresponding to λ_m, normalized such that $\|X_m\|^2_{L^2(\Omega)} = (X_m, X_m) = 1$, where $(\,\cdot\,,\,\cdot\,)$ denotes the inner scalar product in the Hilbert space $L^2(\Omega)$.

Let $u(x,t)$ be a solution to problem (1)–(3). Consider the scalar product

$$u_m(t) = \big(u(\,\cdot\,,t), X_m\big)_{L^2(\Omega)}. \tag{6}$$

From (6) and using equation (1), we obtain

$$u'_m(t) + \lambda_m u'_m(t) + \lambda_m u_m(t) = -\lambda_m (k * u_m)(t) + f_m(t), \tag{7}$$

where $f_m(t) = (f, X_m)$, $m = 1, 2, \dots$. The initial condition (2) yields

$$\varphi_m := u_m(0) = (\varphi, X_m)_{L^2(\Omega)}, \quad m = 1, 2, \dots. \tag{8}$$

One can verify that problem (7), (8) has a unique solution $u_m(t) \in C^1[0,T]$ given by

$$u_m(t) = \chi_m(t)\varphi_m + \frac{1}{1+\lambda_m}(\chi_m * f_m)(t) - \frac{\lambda_m}{1+\lambda_m}\big(\chi_m * (k * u_m)\big)(t), \tag{9}$$

where $\chi_m(t) = \exp\big\{-\frac{\lambda_m}{1+\lambda_m} t\big\}$.

This implies the uniqueness of the solution to problem (1)–(3), since for $\varphi(x) \equiv 0$ and $f(x,t) \equiv 0$, we obtain $\varphi_m \equiv 0$ and $f_m(t) \equiv 0$. From (9) it follows that $u_m(t) \equiv 0$. By (6), this is equivalent to

$$\big(u(\,\cdot\,,t), X_m\big)_{L^2(\Omega)} = 0.$$

Since the system $\{X_m\}$ is complete in $L^2(\Omega)$, we have $u(x,t) = 0$ almost everywhere in Ω for all $t \in [0,T]$. As $u(x,t)$ is continuous on $\overline{D}$, we conclude that $u(x,t) \equiv 0$ on $\overline{D}$. This completes the proof of uniqueness for problem (1)–(3). □

1.2. Existence of the Classical Solution. This subsection establishes the existence of a solution to problem (1)–(3).

Under appropriate conditions on the functions $\varphi(x)$ and $f(x,t)$, we prove that the function

$$u(x,t) = \sum_{m=1}^{\infty} u_m(t) X_m(x) \tag{10}$$

represents a solution to problem (1)–(3).

Lemma 1. *The following estimates hold for all* $m = 1, 2, \dots$:

$$|u_m(t)| \leqslant \max\{1,T\}\big[|\varphi_m| + \|f_m\|_0\big] e^{\|k\|_0 T^2/2}, \quad t \in [0,T], \tag{11}$$

$$|u'_m(t)| \leqslant \|f_m\|_0 + \\ + \max\{1,T\}(1 + \|k\|_0 T)\big[|\varphi_m| + \|f_m\|_0\big] e^{\|k\|_0 T^2/2}, \quad t \in [0,T], \tag{12}$$

where $\|k\|_0 = \max\limits_{t\in[0,T]} |k(t)|$.

Proof. From (9), we estimate $u_m(t)$ as follows:

$$|u_m(t)| \leqslant |\varphi_m| + t|f_m(t)| + \|k\|_0 \int_0^t (t-s)|u_m(s)|\,ds.$$

Applying Gronwall's lemma for all $t \in [0,T]$, we obtain (11). Furthermore, from (7) and (11), we derive (12). This completes the proof of the lemma. □

Assume the following regularity conditions:

$$\begin{cases} \varphi(x) \in H^{[\frac{n}{2}]+3}(\Omega), \quad f(x,t) \in C([0,T]; H^{[\frac{n}{2}]+3}(\Omega)), \\ \varphi = \Delta\varphi = \cdots = \Delta^{[\frac{n+2}{4}]}\varphi \in H_0^1(\Omega), \\ f(\,\cdot\,,t) = \Delta f(\,\cdot\,,t) = \cdots = \Delta^{[\frac{n+2}{4}]} f(\,\cdot\,,t) \in H_0^1(\Omega), \quad t \in [0,T]. \end{cases} \tag{A1}$$

By the Cauchy–Schwarz inequality and Lemma 1 in [17], the series (10) converges uniformly on $\overline{D}$ in view of (11):

$$\begin{aligned} \sum_{m=1}^{\infty} |u_m(t)X_m(x)| &\leqslant C_1 \Bigg(\sum_{m=1}^{\infty} \frac{X_m^2(x)}{\lambda_m^{[\frac{n}{2}]+1}} \sum_{m=1}^{\infty} \varphi_m^2 \lambda_m^{[\frac{n}{2}]+1} \Bigg)^{1/2} + \\ &\quad + C_2 \Bigg(\sum_{m=1}^{\infty} \frac{X_m^2(x)}{\lambda_m^{[\frac{n}{2}]+1}} \sum_{m=1}^{\infty} \|f_m\|_0^2 \lambda_m^{[\frac{n}{2}]+1} \Bigg)^{1/2} \leqslant \\ &\quad \leqslant \widetilde{C}_1 \int_\Omega \big(\Delta^{[\frac{n}{2}]+1}\varphi\big)^2\,dx + \widetilde{C}_2 \int_\Omega \big(\Delta^{[\frac{n}{2}]+1}\|f(x,\,\cdot\,)\|\big)^2 dx. \end{aligned}$$

Differentiating the series in (10) term-wise, we obtain:

$$u_t = \sum_{m=1}^{\infty} u_m'(t) X_m(x), \tag{13}$$

$$u_{x_i x_i} = \sum_{m=1}^{\infty} u_m(t) \frac{\partial^2 X_m(x)}{\partial x_i^2}, \quad i = 1, 2, \ldots, n, \tag{14}$$

$$u_{x_i x_i t} = \sum_{m=1}^{\infty} u_m'(t) \frac{\partial^2 X_m(x)}{\partial x_i^2}, \quad i = 1, 2, \ldots, n. \tag{15}$$

Obviously, that if either series (14) or (15) converges uniformly, then series (13) also converges uniformly.

For series (14), using (11) and (A1), and applying the Cauchy–Schwarz inequality for $(x,t) \in D$ and $i = 1, 2, \ldots, n$, we have:

$$\left| \sum_{m=1}^{\infty} u_m(t) \frac{\partial^2 X_m(x)}{\partial x_i^2} \right| \leqslant C_3 \left[\int_\Omega \big(\Delta^{[\frac{n}{2}]+3}\varphi\big)^2\,dx + \int_\Omega \big(\Delta^{[\frac{n}{2}]+3}\|f(x,\,\cdot\,)\|\big)^2 dx \right].$$

Consequently, the series (14), as well as (13) and (15), converge uniformly in $\overline{D}$.

These results lead to the following theorem.

Theorem 2. *Let $\varphi(x)$ and $f(x,t)$ satisfy condition (A1), and let $k(t) \in C[0,T]$. Then problem (1)–(3) admits a classical solution $u \in C(\overline{D}) \cap C^{2,1}_{x,t}(D)$ defined by the series (10).*

1.3. A Priori Estimates. This subsection establishes estimates for the solution and its first derivative in the direct problem (1)–(3). In the following section, we will prove that problem (1)–(4) has a unique solution for any $T > 0$. For this purpose, we employ weighted norms: for each $\sigma \geqslant 0$, we define the Bielecki norm

$$\|k\|_\sigma = \max_{t\in[0,T]} \big(e^{-\sigma t}|k(t)|\big).$$

Remark. The weighted norm eliminates restrictions on the maximum value of T. In contrast, using the standard supremum norm would require T to be smaller than some finite quantity depending on the problem's data.

The space $C_\sigma[0,T] := (C[0,T], \|\cdot\|_\sigma)$ forms a Banach space, and the norms $\|\cdot\|_\sigma$ and $\|\cdot\|_0$ are equivalent. Moreover, the convolution operator is both commutative and invariant under multiplication by $e^{-\sigma t}$:

$$(f*g)(t) = (g*f)(t), \quad \text{and} \quad e^{-\sigma t}(f*g)(t) = \big(e^{-\sigma t}f(t)\big) * \big(e^{-\sigma t}g(t)\big).$$

Additionally, we have the estimate

$$\|f*g\|_\sigma \leqslant \frac{1}{\sigma}\|f\|_0\|g\|_\sigma, \quad \sigma > 0 \tag{16}$$

(see [16]).

Let $\widetilde{u}_m$ denote the solution of (7), (8) with coefficients $\widetilde{\varphi}_m$, $\widetilde{f}_m$, and $\widetilde{k}$. From (9), we estimate the difference $u_m - \widetilde{u}_m$ in the Bielecki norm:

$$e^{-\sigma t}|u_m(t) - \widetilde{u}_m(t)| \leqslant |\varphi_m - \widetilde{\varphi}_m| + t\|f_m - \widetilde{f}_m\|_0 + \\ + t^2\|u_m\|_0\|k - \widetilde{k}\|_\sigma + t\|\widetilde{k}\|_\sigma \int_0^t e^{-\sigma s}|u_m - \widetilde{u}_m|(s)\,ds.$$

Applying Gronwall's lemma for all $t \in [0,T]$ and $m \in \mathbb{N}$ yields:

$$\|u_m - \widetilde{u}_m\|_\sigma \leqslant \big(|\varphi_m - \widetilde{\varphi}_m| + T\|f_m - \widetilde{f}_m\|_0 + T^2\|k - \widetilde{k}\|_\sigma\|u_m\|_0\big)e^{T^2\|\widetilde{k}\|_\sigma}. \tag{17}$$

Theorem 2 established that problem (1)–(3) possesses a unique classical solution in $\overline{D}$. Consequently, for all $t \geqslant 0$, u_t belongs to $C(\overline{D})$, and the difference of its Fourier coefficients satisfies:

$$\|u'_m - \widetilde{u}'_m\|_\sigma \leqslant \|u_m - \widetilde{u}_m\|_\sigma + T\|u_m\|_0\|k - \widetilde{k}\|_\sigma + \\ + T\|\widetilde{k}\|_\sigma\|u_m - \widetilde{u}_m\|_\sigma + \|f_m - \widetilde{f}_m\|_0. \tag{18}$$

2. The Existence and Uniqueness Theorem for the Inverse Problem. This section investigates the inverse problem of determining the functions $u(x,t)$ and $k(t)$ from relations (1)–(4). We employ the contraction mapping principle to solve this problem.

Let

$$\mu = \left(\sum_{m=1}^{\infty} \frac{\lambda_m}{1+\lambda_m} \varphi_m X_m(x_0) \right)^{-1} \neq 0. \tag{A2}$$

Under condition (A1), the numerical series

$$\sum_{m=1}^{\infty} \frac{\lambda_m}{1+\lambda_m} \varphi_m X_m(x_0)$$

converges.

Substituting $x = x_0$ into (10) and using (4), we obtain

$$h(t) = \sum_{m=1}^{\infty} u_m(t) X_m(x_0), \quad t \in [0, T]. \tag{19}$$

Replacing $u_m(t)$ in (19) with the right-hand side of (9) and differentiating twice yields the integral equation for $k(t)$:

$$\begin{aligned} k(t) = k_0(t) + \mu \sum_{m=1}^{\infty} \left(\frac{\lambda_m}{1+\lambda_m} \right)^2 (k * u_m)(t) X_m(x_0) - \\ - \mu \sum_{m=1}^{\infty} \left(\frac{\lambda_m}{1+\lambda_m} \right)^3 \big(\chi_m * (k * u_m)(\tau) \big)(t) X_m(x_0) - \\ - \mu \sum_{m=1}^{\infty} \frac{\lambda_m}{1+\lambda_m} (k * u'_m)(t) X_m(x_0), \end{aligned} \tag{20}$$

where

$$\begin{aligned} k_0(t) = -\mu h''(t) + \mu \sum_{m=1}^{\infty} \left(\frac{\lambda_m}{1+\lambda_m} \right)^2 \chi_m(t) \varphi_m X_m(x_0) - \\ - \mu \sum_{m=1}^{\infty} \frac{\lambda_m}{(1+\lambda_m)^2} f_m(t) X_m(x_0) + \mu \sum_{m=1}^{\infty} \frac{1}{1+\lambda_m} f'_m(t) X_m(x_0) + \\ + \mu \sum_{m=1}^{\infty} \frac{\lambda_m^2}{(1+\lambda_m)^3} (\chi_m * f_m)(t) X_m(x_0). \end{aligned}$$

Assume the following regularity conditions:

$$\begin{cases} h \in C^2[0, T], \quad f \in C([0, T]; H^{[\frac{n}{2}]+1}(\Omega)) \cap C^1([0, T]; H^{[\frac{n}{2}]-1}(\Omega)), \\ \varphi(x_0) = h(0), \\ f(\,\cdot\,, t) = \Delta f(\,\cdot\,, t) = \cdots = \Delta^{[\frac{n+4}{4}]} f(\,\cdot\,, t) \in H_0^1(\Omega), \quad t \in [0, T]. \end{cases} \tag{A3}$$

Equation (20) can be expressed as the fixed-point equation

$$k = Ak \tag{21}$$

for the operator A defined by

$$Ak(t) = k_0(t) + \mu \sum_{m=1}^{\infty} \Big(\frac{\lambda_m}{1+\lambda_m}\Big)^2 (k * u_m)(t) X_m(x_0) - \\ - \mu \sum_{m=1}^{\infty} \Big(\frac{\lambda_m}{1+\lambda_m}\Big)^3 \big(\chi_m * (k * u_m)(\tau)\big)(t) X_m(x_0) - \\ - \mu \sum_{m=1}^{\infty} \frac{\lambda_m}{1+\lambda_m} (k * u'_m)(t) X_m(x_0).$$

To establish that A has a fixed point, we first demonstrate that A maps a closed convex set into itself in the space $C[0,T]$ equipped with the Bielecki norm.

LEMMA 2. *Under conditions (A1)–(A3), there exists $\sigma_0 > 0$ such that for all $\sigma \geqslant \sigma_0$, there exists $R > 0$ for which the closed convex ball*

$$K = \{k \in C[0,T] : \|Ak - k_0\|_\sigma \leqslant R\}$$

is invariant under A, i.e., $A(K) \subset K$.

Proof. For any $k \in C[0,T]$, $t \in [0,T]$, and $\sigma > 0$, estimate (16) yields:

$$\|Ak - k_0\|_\sigma \leqslant \max_{t\in[0,T]} \Big| \mu \sum_{m=1}^{\infty} \Big(\frac{\lambda_m}{1+\lambda_m}\Big)^2 e^{-\sigma t} (k * u_m)(t) X_m(x_0) \Big| + \\ + \max_{t\in[0,T]} \Big| \mu \sum_{m=1}^{\infty} \Big(\frac{\lambda_m}{1+\lambda_m}\Big)^3 e^{-\sigma t} \big(\chi_m * (k * u_m)(\tau)\big)(t) X_m(x_0) \Big| + \\ + \max_{t\in[0,T]} \Big| \mu \sum_{m=1}^{\infty} \frac{\lambda_m}{1+\lambda_m} e^{-\sigma t} (k * u'_m)(t) X_m(x_0) \Big| \leqslant \\ \leqslant \frac{\|k\|_\sigma}{\sigma} |\mu| \sum_{m=1}^{\infty} \|u_m\|_0 |X_m(x_0)| + \frac{\|k\|_\sigma}{\sigma} |\mu| \sum_{m=1}^{\infty} \|\chi_m * u'_m\|_0 |X_m(x_0)| \\ + \frac{\|k\|_\sigma}{\sigma} |\mu| \sum_{m=1}^{\infty} \|u'_m\|_0 |X_m(x_0)| := I_1 + I_2 + I_3. \quad (22)$$

For $k \in K$, we have

$$\|k\|_\sigma \leqslant \|k_0\|_0 + R := R_0, \quad (23)$$

since $\|\cdot\|_\sigma \leqslant \|\cdot\|_0$.

Applying Lemma 1 and (A1) to I_1 with (23) gives:

$$I_1 \leqslant \frac{R_0}{\sigma} \max\{1,T\} e^{\|k\|_0 T^2/2} |\mu| \sum_{m=1}^{\infty} \big(|\varphi_m| + \|f_m\|_0\big) |X_m(x_0)| \leqslant \\ \leqslant \frac{R_0}{\sigma} \max\{1,T\} e^{\|k\|_0 T^2/2} |\mu| \Big(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)} + \|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))} \Big) := \frac{\widetilde{\sigma}_1}{\sigma}. \quad (24)$$

Similarly, for I_2:

$$I_2 \leqslant \frac{R_0}{\sigma}T|\mu|\sum_{m=1}^{\infty}\|u_m'\|_0|X_m(x_0)| \leqslant \frac{R_0}{\sigma}T|\mu|\sum_{m=1}^{\infty}\|f_m\|_0|X_m(x_0)| +$$

$$+\frac{R_0}{\sigma}\max\{1,T\}(1+R_0T)Te^{\|k\|_0T^2/2}|\mu|\sum_{m=1}^{\infty}\big(|\varphi_m|+\|f_m\|_0\big)|X_m(x_0)| \leqslant$$

$$\leqslant \frac{R_0}{\sigma}T|\mu|\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}+\frac{R_0}{\sigma}\max\{1,T\}(1+R_0T)\times$$

$$\times Te^{\|k\|_0T^2/2}|\mu|\Big(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)}+\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}\Big) := \frac{\widetilde{\sigma}_2}{\sigma}. \quad (25)$$

For I_3, we obtain:

$$I_3 \leqslant \frac{R_0}{\sigma}|\mu|\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}+\frac{R_0}{\sigma}\max\{1,T\}(1+R_0T)\times$$

$$\times e^{\|k\|_0T^2/2}|\mu|\Big(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)}+\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}\Big) := \frac{\widetilde{\sigma}_3}{\sigma}. \quad (26)$$

Combining (24)–(26) for (22) yields

$$\|Ak-k_0\|_\sigma \leqslant \frac{\widetilde{\sigma}_0}{\sigma},$$

where $\widetilde{\sigma}_0 := \widetilde{\sigma}_1+\widetilde{\sigma}_2+\widetilde{\sigma}_3$. Choosing $\sigma \geqslant \sigma_0 := (1/R)\widetilde{\sigma}_0$ ensures $A(K)\subset K$. □

LEMMA 3. *Under the same conditions as in Lemma 2, the family* $(A(k))_{k\in K}$ *is contractive, i.e., there exists* $q\in[0,1)$ *such that*

$$\|Ak-A\tilde{k}\|_\sigma \leqslant q\|k-\widetilde{k}\|_\sigma$$

for all k, $\widetilde{k}\in K$.

Proof. From the commutative and invariant properties of the convolution operator, we have

$$e^{-\sigma t}v_1*v_2(t)-e^{-\sigma t}\widetilde{v}_1*\widetilde{v}_2(t)=e^{-\sigma t}(v_1-\widetilde{v}_1)*v_2(t)+e^{-\sigma t}\widetilde{v}_1*(v_2-\widetilde{v}_2)(t)$$

and

$$\|v_1*v_2(t)-\widetilde{v}_1*\widetilde{v}_2(t)\|_\sigma \leqslant \frac{1}{\sigma}\big(\|v_1-\widetilde{v}_1\|_\sigma\|v_2\|_0+\|\widetilde{v}_1\|_0\|v_2-\widetilde{v}_2\|_\sigma\big).$$

For any k, $\widetilde{k}\in K$, we estimate

$$\|Ak-A\widetilde{k}\|_\sigma \leqslant \frac{|\mu|}{\sigma}\sum_{m=1}^{\infty}\Big(\frac{\lambda_m}{1+\lambda_m}\Big)^2\big(\|k-\widetilde{k}\|_\sigma\|u_m\|_0+\|\widetilde{k}\|_0\|u_m-\widetilde{u}_m\|_\sigma\big)|X_m(x_0)|+$$

$$+\frac{|\mu|}{\sigma}\sum_{m=1}^{\infty}\Big(\frac{\lambda_m}{1+\lambda_m}\Big)^3\big(\|k-\widetilde{k}\|_\sigma\|\chi_m*u_m\|_0+\|\widetilde{k}\|_0\|\chi_m*(u_m-\widetilde{u}_m)\|_\sigma\big)|X_m(x_0)|+$$

$$+\frac{|\mu|}{\sigma}\sum_{m=1}^{\infty}\frac{\lambda_m}{1+\lambda_m}\big(\|k-\widetilde{k}\|_\sigma\|u'_m\|_0+\|\widetilde{k}\|_0\|u'_m-\widetilde{u}'_m\|_\sigma\big)|X_m(x_0)| := \hat{I}_1+\hat{I}_2+\hat{I}_3. \tag{27}$$

We now estimate each term in (27). Using Lemma 1, (A1), and (17) for $\hat{I}_1$, we obtain

$$\hat{I}_1 \leqslant \frac{|\mu|}{\sigma}\big(1+T^2e^{T^2R_0}\|\tilde{k}\|_0\big)\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\|u_m\|_0|X_m(x_0)| \leqslant$$

$$\leqslant \frac{|\mu|}{\sigma}\big(1+T^2e^{T^2R_0}\|\tilde{k}\|_0\big)\max\{1,T\}e^{\|k\|_0T^2/2}\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\big(|\varphi_m|+\|f_m\|_0\big)|X_m(x_0)| \leqslant$$

$$\leqslant \frac{|\mu|}{\sigma}\big(1+T^2e^{T^2R_0}\|\tilde{k}\|_0\big)\max\{1,T\}e^{\|k\|_0T^2/2}\|k-\tilde{k}\|_\sigma \times$$

$$\times\Big(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)}+\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}\Big) := \frac{\hat{\sigma}_1}{\sigma}\|k-\widetilde{k}\|_\sigma.$$

For $\hat{I}_2$, we have

$$\hat{I}_2 \leqslant \frac{|\mu|}{\sigma}\Big(T+T^2e^{T^2R_0}\frac{\|\widetilde{k}\|_0}{\sigma}\Big)\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\|u_m\|_0|X_m(x_0)| \leqslant$$

$$\leqslant \frac{|\mu|}{\sigma}\Big(T+T^2e^{T^2R_0}\frac{\|\widetilde{k}\|_0}{\sigma}\Big)\max\{1,T\}e^{\|k\|_0T^2/2}\|k-\widetilde{k}\|_\sigma \times$$

$$\times\Big(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)}+\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}\Big) := \frac{\hat{\sigma}_2}{\sigma}\|k-\widetilde{k}\|_\sigma.$$

Similarly, for $\hat{I}_3$, Lemma 1 yields

$$\hat{I}_3 \leqslant \frac{|\mu|}{\sigma}\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\|u'_m\|_0|X_m(x_0)| +$$

$$+\frac{|\mu|}{\sigma}\big(T+T^2e^{T^2R_0}+T^3e^{T^2R_0}\big)\|\tilde{k}\|_0\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\|u_m\|_0|X_m(x_0)| \leqslant$$

$$\leqslant \frac{|\mu|}{\sigma}\|k-\widetilde{k}\|_\sigma\sum_{m=1}^{\infty}\|f_m\|_0|X_m(x_0)| +$$

$$+\frac{|\mu|}{\sigma}\|k-\widetilde{k}\|_\sigma\max\{1,T\}\big(1+\|k\|_0T\big)e^{\|k\|_0T^2/2}\sum_{m=1}^{\infty}\big(|\varphi_m|+\|f_m\|_0\big)|X_m(x_0)| +$$

$$+\frac{|\mu|}{\sigma}\big(T+T^2e^{T^2R_0}+T^3e^{T^2R_0}\big)\max\{1,T\}e^{\|k\|_0T^2/2}\|\tilde{k}\|_0\|k-\widetilde{k}\|_\sigma \times$$

$$\times\sum_{m=1}^{\infty}\big(|\varphi_m|+\|f_m\|_0\big)|X_m(x_0)| \leqslant$$

$$\leqslant \frac{|\mu|}{\sigma}\|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))}\|k-\widetilde{k}\|_\sigma+\frac{|\mu|}{\sigma}\max\{1,T\}\big(1+\|k\|_0T\big)e^{\|k\|_0T^2/2} \times$$

$$\times \left(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)} + \|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))} \right) \|k - \widetilde{k}\|_\sigma +$$

$$+ \frac{|\mu|}{\sigma} \big(T + T^2 e^{T^2 R_0} + T^3 e^{T^2 R_0}\big) \max\{1, T\} e^{\|k\|_0 T^2/2} \|\tilde{k}\|_0 \times$$

$$\times \left(\|\varphi\|_{H^{[\frac{n}{2}]+1}(\Omega)} + \|f\|_{C([0,T];H^{[\frac{n}{2}]+1}(\Omega))} \right) \|k - \widetilde{k}\|_\sigma := \frac{\hat{\sigma}_3}{\sigma} \|k - \widetilde{k}\|_\sigma.$$

Choosing $q := \hat{\sigma}_0/\sigma < 1$, where $\hat{\sigma}_0 := \max\{\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3\}$, establishes that A is a contraction on K, completing the proof. □

By the Banach fixed-point theorem, equation (21) has a unique solution for any $T > 0$, yielding:

Theorem 3. *Under assumptions* (A1)–(A3), *for any* $T > 0$, *problem* (1)–(4) *admits a unique solution.*

Conclusion. This study has established the existence and uniqueness of a solution to the inverse problem of determining the kernel of a multidimensional third-order integrodifferential pseudo-parabolic equation. Our approach combines the Fourier method, integral inequalities, and the fixed-point principle, with the solution specified by an additional condition at a fixed point for the first boundary value problem.

All results presented in this article remain valid when the Laplacian operator Δ in (1) is replaced by a more general self-adjoint differential operator L defined in the domain Ω. This operator takes the form:

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial}{\partial x_j} \right] - c(x),$$

where the coefficients satisfy:
- symmetry: $a_{ij}(x) = a_{ji}(x)$ for all i, j;
- uniform ellipticity: $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2$ with $\alpha = \text{const} > 0$;
- non-negativity: $c(x) \geqslant 0$ in Ω.

We additionally assume the coefficients $a_{ij}(x)$ and $c(x)$ satisfy appropriate smoothness conditions (see [17] for details).

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Обратная задача определения ядра для класса псевдопараболических интегро-дифференциальных уравнений

Д. К. Дурдиев[1,2], *Х. Б. Элмурадова*[2], *А. А. Рахмонов*[1,2]

[1] Бухарское отделение Института математики Академии наук Республики Узбекистан, Узбекистан, 705018, Бухара, ул. М. Икбол, 11.

[2] Бухарский государственный университет, Узбекистан, 705018, Бухара, ул. М. Икбол, 11.

Аннотация

Данная работа посвящена исследованию обратной задачи определения ядра в многомерном интегро-дифференциальном псевдопараболическом уравнении третьего порядка. Исследование начинается с анализа прямой задачи с известной функцией ядра при рассмотрении начально-краевой задачи с однородными граничными условиями. Методом Фурье строится решение в виде ряда по собственным функциям задачи Дирихле для оператора Лапласа. Важной частью анализа является получение априорных оценок коэффициентов ряда через норму функции ядра, которые играют ключевую роль при изучении обратной задачи.

Для обратной задачи вводится условие переопределения, задающее значение решения в фиксированной точке пространственной области (точечное измерение). Эта формулировка сводится к интегральному уравнению Вольтерра второго рода. Путем применения принципа сжимающих отображений Банаха в классе непрерывных функций с экспоненциально взвешенной нормой устанавливаются глобальная существование и единственность решения обратной задачи. Полученные результаты демонстрируют корректную разрешимость рассматриваемой проблемы.

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Сведения об авторах

Дурдимурод Каландарович Дурдиев https://orcid.org/0000-0002-6054-2827
доктор физико-математических наук, профессор; заведующий отделением[1]; профессор, каф. дифференциальных уравнений[2]; e-mail: d.durdiev@mathinst.uz

Хилола Ботировна Элмурадова https://orcid.org/0000-0003-4306-2589
преподаватель; базовый докторант; каф. дифференциальных уравнений[2]; e-mail: helmuradova@mail.ru

Аскар Ахмадович Рахмонов https://orcid.org/0000-0002-7641-9698
кандидат физико-математических наук, доцент; старший научный сотрудник[1]; доцент, каф. дифференциальных уравнений[2]; e-mail: araxmonov@mail.ru

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