Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki
[J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 2, pp. 207–219
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu2141

MSC: 45J05, 47G20, 35A22, 34A30

Solution of systems of linear Caputo fractional Volterra integro-differential equations using the Khalouta integral transform method

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Abstract

The Khalouta integral transform is a powerful method for solving various types of equations, including integro-differential equations and integral equations. It can also be applied to initial and boundary value problems associated with ordinary differential equations and partial differential equations with constant coefficients. The main objective of this paper is to derive solutions to systems of linear Caputo fractional Volterra integro-differential equations using the Khalouta integral transform.

To solve such systems using this technique, it is essential to establish and define several key properties of the Khalouta integral transform, which are crucial for deriving the transformation of the Caputo fractional derivative appearing in the systems. Several numerical examples are presented and solved by using the Khalouta integral transform method to demonstrate the applicability of the proposed approach. The results obtained from these numerical examples confirm that the proposed method is highly efficient and provides exact solutions for systems of linear fractional Volterra integro-differential equations in a straightforward manner.

Keywords: Khalouta integral transform, Volterra integro-differential equations, Caputo fractional derivative, exact solution.

Received: 6th January, 2025 / Revised: 9th June, 2025 / Accepted: 18th June, 2025 / First online: 20th June, 2025

Differential Equations and Mathematical Physics
Research Article

Please cite this article in press as:
Khalouta A. Solution of systems of linear Caputo fractional Volterra integro-differential equations using the Khalouta integral transform method, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 2, pp. 207–219. EDN: YHPHPY. DOI: 10.14498/vsgtu2141.

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1. Introduction. Fractional integro-differential equations have wide applications in numerous fields of physics and applied mathematics, including continuum mechanics, geophysics, potential theory, electromagnetism, optimization, renewal theory, kinetic theory, quantum mechanics, radiation, optimal control systems, mathematical economics, communication theory, queuing theory, radiative equilibrium, acoustics, steady-state heat conduction, fluid mechanics, and fracture mechanics. Among these, fractional Volterra integro-differential equations are particularly significant due to their prevalence in approximation theory, computational mathematics, and physical mathematics.

Recent research has explored various methods to analyze fractional integro-differential equations for accurate and reliable solutions. Integral transforms stand out as a prominent approach for solving such mathematical problems. Several integral transforms have proven effective for handling different types of fractional integro-differential equations. For instance:

- In [1], an exact solution for Volterra-type fractional integro-differential equations was proposed using the Elzaki integral transform.
- The Aboodh integral transform was applied in [2] to study analytical solutions of linear and nonlinear dynamical systems of fractional integro-differential equations.
- The Mohand integral transform was employed in [3] to analyze fractional integro-differential equations.
- The Ramadan group integral transform was used in [4] to solve fractional Fredholm-Volterra integro-differential equations.
- The Laplace integral transform facilitated solutions to fourth-order fractional partial integro-differential equations in [5].
- In [6], the Laplace integral transform yielded analytical solutions for a system of Volterra-type integro-fractional differential equations with variable coefficients and multi-time delay.

The objective of this paper is to investigate the efficacy of a novel integral transform method, the Khalouta integral transform, for solving systems of linear fractional Volterra integro-differential equations. The system under consideration is given by:

$$\begin{cases} D^\alpha X_1(t) = f_1(t) + \sum_{j=1}^{n} \int_0^t K_{1j}(t-s) X_j(s) ds, \\ D^\alpha X_2(t) = f_2(t) + \sum_{j=1}^{n} \int_0^t K_{2j}(t-s) X_j(s) ds, \\ \qquad \vdots \\ D^\alpha X_n(t) = f_n(t) + \sum_{j=1}^{n} \int_0^t K_{nj}(t-s) X_j(s) ds, \end{cases} \tag{1}$$

subject to the initial conditions

$$X_1^{(k)}(0) = C_{1k},\ X_2^{(k)}(0) = C_{2k},\ \ldots,\ X_n^{(k)}(0) = C_{nk}, \quad k = 0, 1, \ldots, m-1, \tag{2}$$

where $X_1(t), X_2(t), \ldots, X_n(t)$ are unknown functions to be determined, $K_{ij}(t,s) = K_{ij}(t-s)$ are difference kernels for $i, j = 1, 2, \ldots, n$, $f_i(t)$ are real-valued functions for $i = 1, 2, \ldots, n$, and D^α denotes the Caputo fractional derivative of order α with $m - 1 < \alpha \leqslant m$, $m \geqslant 1$.

The Khalouta integral transform is a novel integral transform recently introduced by the author, generalizing several well-known integral transforms including the Laplace transform [7], Sumudu transform [8], ZZ transform [9], ZMA transform [10], Elzaki transform [11], Aboodh transform [12], natural transform [13], and Shehu transform [14]. The Khalouta integral transform offers several key advantages:

- Unit Preservation: It enables direct problem-solving without frequency-domain conversion, particularly valuable in physical sciences where dimensional consistency is crucial;
- Linear Operator Properties: As a linear operator preserving linear functions, it maintains units and dimensions without modification;
- Handling of Initial Conditions and Singularities: The transform is specifically designed to address initial conditions and singularities commonly encountered in practical engineering models;
- Simplified Inversion: The inverse Khalouta transform avoids complex contour integration, offering a more straightforward solution methodology.

This paper is organized as follows. Section 1 presents the introduction. Section 2 provides essential definitions, properties, theorems, and foundational results on fractional calculus and the Khalouta integral transform, which are utilized throughout subsequent sections. Section 3 develops the solution methodology for systems of linear fractional Volterra integro-differential equations (1) with initial conditions (2) using the Khalouta transform. Section 4 demonstrates numerical applications, and Section 5 concludes the paper.

2. Fundamental Definitions and Theorems. This section presents the essential definitions and theorems of fractional calculus and the Khalouta transform, along with their key properties.

Definition 1 [15]. Let $X(t)$ be a continuous function on the interval $[0, T]$, where $T > 0$. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ is defined as

$$I^\alpha X(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t - \xi)^{\alpha - 1} X(\xi) d\xi, & \alpha > 0, \\ X(t), & \alpha = 0, \end{cases}$$

where $\Gamma(\,\cdot\,)$ denotes the Gamma function.

Definition 2 [15]. Let $X(t)$ be a continuous function on $[0, T]$, $T > 0$. The Caputo fractional derivative operator of order $\alpha > 0$ is defined as

$$D^\alpha X(t) = \begin{cases} I^{m-\alpha} X^{(m)}(t), & m - 1 < \alpha < m, \\ X^{(m)}(t), & \alpha = m, \end{cases}$$

where $m = [\alpha] \in \mathbb{N}$.

An important relationship between these operators is given by:

$$I^\alpha D^\alpha X(t) = X(t) - \sum_{k=0}^{n-1} X^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0.$$

Definition 3 [16]. Let $X(t)$ be a continuous function of exponential order on $[0,T]$, $T>0$. The Khalouta integral transform of $X(t)$ is defined as

$$\mathbb{KH}[X(t)] = \mathcal{K}(s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^{\infty} \exp\Big(-\frac{st}{\gamma\eta}\Big)X(t)dt,$$

where s, γ, $\eta>0$ are real or complex parameters independent of t.

The inverse Khalouta transform is given by:

$$X(t) = \mathbb{KH}^{-1}[\mathcal{K}(s,\eta,\gamma)] = \frac{1}{2\pi i}\int_{\varphi-i\infty}^{\varphi+i\infty} \frac{1}{s}\exp\Big(\frac{st}{\gamma\eta}\Big)\mathcal{K}(s,\eta,\gamma)ds, \tag{3}$$

where φ is a real constant and the integral in (3) is evaluated along $s=\varphi$ in the complex plane $s=x+iy$.

The Khalouta integral transform possesses the following fundamental properties.

1. Linearity Property:

$$\mathbb{KH}[\lambda X(t) \pm \mu Y(t)] = \lambda\mathbb{KH}[X(t)] \pm \mu\mathbb{KH}[Y(t)],$$

where λ and μ are nonzero constants.

2. Differentiation Property:

$$\mathbb{KH}[X^{(m)}(t)] = \Big(\frac{s}{\gamma\eta}\Big)^m\mathbb{KH}[X(t)] - \sum_{k=0}^{m-1}\Big(\frac{s}{\gamma\eta}\Big)^{m-k}X^{(k)}(0),$$

where $X^{(m)}(t)$ denotes the m-th derivative of $X(t)$ with respect to t for $m = 0,1,2,\ldots$

3. Convolution Property:

$$\mathbb{KH}[(X*Y)(t)] = \frac{\gamma\eta}{s}\mathbb{KH}[X(t)]\mathbb{KH}[Y(t)].$$

4. Inverse Transform of Elementary Functions:

$$\begin{aligned}
\mathbb{KH}^{-1}[1] &= 1,\\
\mathbb{KH}^{-1}\Big[\frac{\gamma\eta}{s}\Big] &= t,\\
\mathbb{KH}^{-1}\Big[\Big(\frac{\gamma\eta}{s}\Big)^n\Big] &= \frac{t^n}{n!}, \quad n=0,1,2,\ldots,\\
\mathbb{KH}^{-1}\Big[\Big(\frac{\gamma\eta}{s}\Big)^\alpha\Big] &= \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad \alpha>-1,\\
\mathbb{KH}^{-1}\Big[\frac{s}{s-a\gamma\eta}\Big] &= \exp(at),\\
\mathbb{KH}^{-1}\Big[\frac{as\gamma\eta}{s^2+a^2\gamma^2\eta^2}\Big] &= \sin(at),\\
\mathbb{KH}^{-1}\Big[\frac{s^2}{s^2+a^2\gamma^2\eta^2}\Big] &= \cos(at),
\end{aligned}$$

where a is a constant.

THEOREM 1 [17]. The Khalouta integral transform of the Caputo fractional derivative is given by:

$$\mathbb{KH}[D^{\alpha}X(t)] = \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X(t)] - \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}X^{(k)}(0), \quad m-1<\alpha\leqslant m.$$

REMARK 1. The Khalouta integral transform generalizes several well-known integral transforms through appropriate parameter choices:

1) For $\gamma = \eta = 1$, we recover the Laplace–Carson transform of the Caputo fractional derivative [18]:

$$\mathbb{LC}[D^{\alpha}X(t)] = s^{\alpha}\mathbb{LC}[X(t)] - \sum_{k=0}^{m-1}s^{\alpha-k}X^{(k)}(0), \quad m-1<\alpha\leqslant m;$$

2) For $s = \gamma = 1$, we obtain the Sumudu transform of the Caputo fractional derivative [19]:

$$\mathbb{S}[D^{\alpha}X(t)] = \frac{1}{\eta^{\alpha}}\mathbb{S}[X(t)] - \sum_{k=0}^{m-1}\frac{1}{\eta^{\alpha-k}}X^{(k)}(0), \quad m-1<\alpha\leqslant m;$$

3) For $\gamma = 1$, we derive the ZZ transform of the Caputo fractional derivative [20]:

$$\mathbb{Z}[D^{\alpha}X(t)] = \left(\frac{s}{\eta}\right)^{\alpha}\mathbb{Z}[X(t)] - \sum_{k=0}^{m-1}\left(\frac{s}{\eta}\right)^{\alpha-k}X^{(k)}(0), \quad m-1<\alpha\leqslant m;$$

4) For $s = 1$, we obtain the ZMA transform of the Caputo fractional derivative [10]:

$$\mathbb{Z}_{MA}[D^{\alpha}X(t)] = \frac{1}{(\gamma\eta)^{\alpha}}\mathbb{Z}_{MA}[X(t)] - \sum_{k=0}^{m-1}\frac{1}{(\gamma\eta)^{\alpha-k}}X^{(k)}(0), \quad m-1<\alpha\leqslant m.$$

3. Procedure of Khalouta Integral Transform Method. This section presents the methodology for solving systems of linear fractional Volterra integro-differential equations using the Khalouta integral transform.

Consider the general system of linear fractional Volterra integro-differential equations (1), subject to the initial conditions (2).

Applying the Khalouta integral transform to system (1) and using the convolution property yields:

$$\begin{cases} \mathbb{KH}[D^{\alpha}X_1(t)] = \mathbb{KH}[f_1(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{1j}(t)]\mathbb{KH}[X_j(t)], \\ \mathbb{KH}[D^{\alpha}X_2(t)] = \mathbb{KH}[f_2(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{2j}(t)]\mathbb{KH}[X_j(t)], \\ \qquad\vdots \\ \mathbb{KH}[D^{\alpha}X_n(t)] = \mathbb{KH}[f_n(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{nj}(t)]\mathbb{KH}[X_j(t)]. \end{cases} \tag{4}$$

Applying Theorem 1 to the first term of system (4) yields:

$$\begin{cases} \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_1(t)] - \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}X_1^{(k)}(0) = \\ \qquad = \mathbb{KH}[f_1(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{1j}(t)]\mathbb{KH}[X_j(t)], \\ \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_2(t)] - \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}X_2^{(k)}(0) = \\ \qquad = \mathbb{KH}[f_2(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{2j}(t)]\mathbb{KH}[X_j(t)], \\ \qquad\vdots \\ \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_n(t)] - \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}X_n^{(k)}(0) = \\ \qquad = \mathbb{KH}[f_n(t)] + \sum_{j=1}^{n} \frac{\gamma\eta}{s}\mathbb{KH}[K_{nj}(t)]\mathbb{KH}[X_j(t)]. \end{cases} \tag{5}$$

Substituting the initial conditions (2) into system (5) gives:

$$\begin{cases} \left(\dfrac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_1(t)] - \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{1k} = \\ \qquad = \mathbb{KH}[f_1(t)] + \dfrac{\gamma\eta}{s}\displaystyle\sum_{j=1}^{n}\mathbb{KH}[K_{1j}(t)]\mathbb{KH}[X_j(t)], \\ \left(\dfrac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_2(t)] - \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{2k} = \\ \qquad = \mathbb{KH}[f_2(t)] + \displaystyle\sum_{j=1}^{n}\dfrac{\gamma\eta}{s}\mathbb{KH}[K_{2j}(t)]\mathbb{KH}[X_j(t)], \\ \qquad \vdots \\ \left(\dfrac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_n(t)] - \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{nk} = \\ \qquad = \mathbb{KH}[f_n(t)] + \displaystyle\sum_{j=1}^{n}\dfrac{\gamma\eta}{s}\mathbb{KH}[K_{nj}(t)]\mathbb{KH}[X_j(t)]. \end{cases} \tag{6}$$

Simplifying system (6) results in:

$$\begin{cases} \left(\left(\dfrac{s}{\gamma\eta}\right)^{\alpha} - \dfrac{\gamma\eta}{s}\mathbb{KH}[K_{11}(t)]\right)\mathbb{KH}[X_1(t)] - \displaystyle\sum_{j=2}^{n}\dfrac{\gamma\eta}{s}\mathbb{KH}[K_{1j}(t)]\mathbb{KH}[X_j(t)] = \\ \qquad = \mathbb{KH}[f_1(t)] + \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{1k}, \\ \left(\left(\dfrac{s}{\gamma\eta}\right)^{\alpha} - \dfrac{\gamma\eta}{s}\mathbb{KH}[K_{22}(t)]\right)\mathbb{KH}[X_2(t)] - \displaystyle\sum_{\substack{j=1\\ j\neq 2}}^{n}\dfrac{\gamma\eta}{s}\mathbb{KH}[K_{2j}(t)]\mathbb{KH}[X_j(t)] = \\ \qquad = \mathbb{KH}[f_2(t)] + \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{2k}, \\ \qquad \vdots \\ \left(\left(\dfrac{s}{\gamma\eta}\right)^{\alpha} - \dfrac{\gamma\eta}{s}\mathbb{KH}[K_{nn}(t)]\right)\mathbb{KH}[X_n(t)] - \displaystyle\sum_{j=1}^{n-1}\dfrac{\gamma\eta}{s}\mathbb{KH}[K_{nj}(t)]\mathbb{KH}\,[X_j(t)] = \\ \qquad = \mathbb{KH}[f_n(t)] + \displaystyle\sum_{k=0}^{m-1}\left(\dfrac{s}{\gamma\eta}\right)^{\alpha-k}C_{nk}. \end{cases} \tag{7}$$

The solution of system (7) is obtained through Cramer's rule as follows:

$$\mathbb{KH}[X_i(t)] = \frac{\Delta_i}{\Delta}, \quad i = 1, 2, \dots, n,$$

where Δ is the determinant of the coefficient matrix:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

with matrix elements

$$a_{ij} = \begin{cases} \left(\frac{s}{\gamma\eta}\right)^{\alpha} - \frac{\gamma\eta}{s}\mathbb{KH}[K_{ii}(t)], & i = j, \\ -\frac{\gamma\eta}{s}\mathbb{KH}[K_{ij}(t)], & i \neq j, \end{cases}$$

and Δ_i are the determinants formed by replacing the i-th column of Δ with the column vector

$$\begin{pmatrix} \mathbb{KH}[f_1(t)] + \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k} C_{1k} \\ \mathbb{KH}[f_2(t)] + \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k} C_{2k} \\ \vdots \\ \mathbb{KH}[f_n(t)] + \sum_{k=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k} C_{nk} \end{pmatrix}.$$

Applying the inverse Khalouta transform to $\mathbb{KH}[X_i(t)]$ yields the final solution for each $X_i(t)$, $i = 1, 2, \ldots, n$.

4. Numerical Applications. This section demonstrates the Khalouta integral transform method through a concrete example of fractional integro-differential equations.

Example 1. Consider the system of linear Caputo fractional Volterra integro-differential equations:

$$\begin{cases} D^{\alpha}X_1(t) = 2t^2 + \int_0^t (t-s)X_1(s)ds + \int_0^t (t-s)X_2(s)ds, \\ D^{\alpha}X_2(t) = -3t^2 - \frac{1}{10}t^5 + \int_0^t (t-s)X_1(s)ds - \int_0^t (t-s)X_2(s)ds, \end{cases} \tag{8}$$

where $0 < \alpha \leqslant 1$, with initial conditions

$$X_1(0) = 1, \quad X_2(0) = 1.$$

Applying the Khalouta transform yields:

$$\begin{cases} \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_1(t)] - \left(\frac{s}{\gamma\eta}\right)^{\alpha} = 4\left(\frac{\gamma\eta}{s}\right)^2 + \left(\frac{\gamma\eta}{s}\right)^2\mathbb{KH}[X_1(t)] + \left(\frac{\gamma\eta}{s}\right)^2\mathbb{KH}[X_2(t)], \\ \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}[X_2(t)] - \left(\frac{s}{\gamma\eta}\right)^{\alpha} = \\ \qquad = -6\left(\frac{\gamma\eta}{s}\right)^2 - 12\left(\frac{\gamma\eta}{s}\right)^5 + \left(\frac{\gamma\eta}{s}\right)^2\mathbb{KH}[X_1(t)] - \left(\frac{\gamma\eta}{s}\right)^2\mathbb{KH}[X_2(t)]. \end{cases}$$

Solving the last system gives:

$$X_1(t) = 1 + 6\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}, \quad X_2(t) = 1 - 6\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}.$$

For the integer case ($\alpha = 1$), we recover the exact solution:

$$X_1(t) = 1 + t^3, \quad X_2(t) = 1 - t^3,$$

which agrees with known results obtained via Laplace transform methods [21].

EXAMPLE 2. Consider the following system of linear Caputo fractional Volterra integro-differential equations:

$$\begin{cases} D^\alpha X_1(t) = -t^3 - t^4 + 3\int_0^t X_2(s)ds + 4\int_0^t X_3(s)ds, \\ D^\alpha X_2(t) = 2 + t^2 - t^4 - 2\int_0^t X_1(s)ds + 4\int_0^t X_3(s)ds, \\ D^\alpha X_3(t) = 6t - t^2 + t^3 + 2\int_0^t X_1(s)ds - 3\int_0^t X_2(s)ds, \end{cases} \tag{9}$$

where $1 < \alpha \leqslant 2$, subject to the initial conditions

$$X_1(0) = 0,\ X_1'(0) = 1; \quad X_2(0) = X_2'(0) = 0; \quad X_3(0) = X_3'(0) = 0.$$

Applying the Khalouta integral transform to system (9) yields:

$$\begin{cases} \mathbb{KH}[D^\alpha X_1(t)] = \mathbb{KH}[-t^3 - t^4] + \frac{\gamma\eta}{s}\mathbb{KH}[3]\mathbb{KH}[X_2(t)] + \frac{\gamma\eta}{s}\mathbb{KH}[4]\mathbb{KH}[X_3(t)], \\ \mathbb{KH}[D^\alpha X_2(t)] = \mathbb{KH}[2 + t^2 - t^4] - \frac{\gamma\eta}{s}\mathbb{KH}[2]\mathbb{KH}[X_1(t)] + \frac{\gamma\eta}{s}\mathbb{KH}[4]\mathbb{KH}[X_3(t)], \\ \mathbb{KH}[D^\alpha X_3(t)] = \mathbb{KH}[6t - t^2 + t^3] + \frac{\gamma\eta}{s}\mathbb{KH}[2]\mathbb{KH}[X_1(t)] - \frac{\gamma\eta}{s}\mathbb{KH}[3]\mathbb{KH}[X_2(t)]. \end{cases} \tag{10}$$

Using Theorem 1, we transform system (10) to

$$\begin{cases} \left(\frac{s}{\gamma\eta}\right)^\alpha \mathbb{KH}[X_1(t)] - \left(\frac{s}{\gamma\eta}\right)^{\alpha-1} = \\ \qquad = -6\left(\frac{\gamma\eta}{s}\right)^3 - 12\left(\frac{\gamma\eta}{s}\right)^4 + 3\frac{\gamma\eta}{s}\mathbb{KH}[X_2(t)] + 4\frac{\gamma\eta}{s}\mathbb{KH}[X_3(t)], \\ \left(\frac{s}{\gamma\eta}\right)^\alpha \mathbb{KH}[X_2(t)] = \\ \qquad = 2 + 2\left(\frac{\gamma\eta}{s}\right)^2 - 12\left(\frac{\gamma\eta}{s}\right)^4 - 2\frac{\gamma\eta}{s}\mathbb{KH}[X_1(t)] + 4\frac{\gamma\eta}{s}\mathbb{KH}[X_3(t)], \\ \left(\frac{s}{\gamma\eta}\right)^\alpha \mathbb{KH}[X_3(t)] = \\ \qquad = 6\frac{\gamma\eta}{s} - 2\left(\frac{\gamma\eta}{s}\right)^2 + 6\left(\frac{\gamma\eta}{s}\right)^3 + 2\frac{\gamma\eta}{s}\mathbb{KH}[X_1(t)] - 3\frac{\gamma\eta}{s}\mathbb{KH}[X_2(t)]. \end{cases}$$

Solving the last system gives:

$$X_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad X_2(t) = 2\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad X_3(t) = 6\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

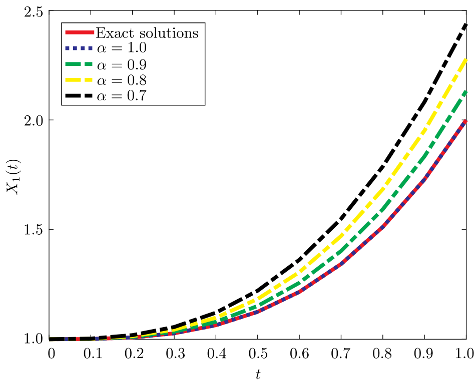

Figure 1. Behavior of $X_1(t)$ for different fractional orders α compared to the exact solution ($\alpha = 1$)

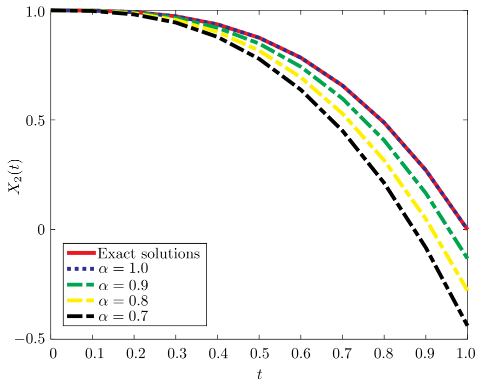

Figure 2. Behavior of $X_2(t)$ for different fractional orders α compared to the exact solution ($\alpha = 1$)

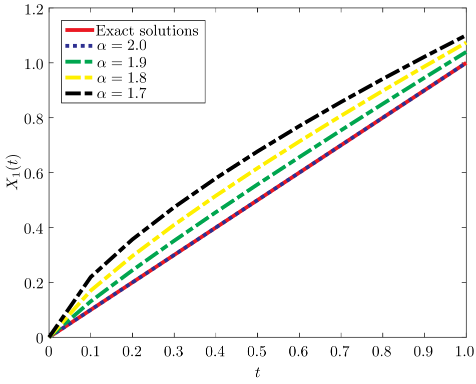

Figure 3. Behavior of $X_1(t)$ for different fractional orders α compared to the exact solution ($\alpha = 2$)

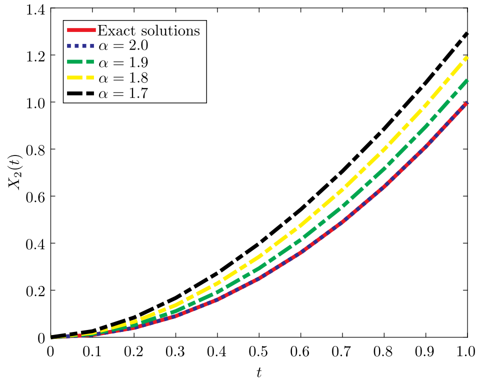

Figure 4. Behavior of $X_2(t)$ for different fractional orders α compared to the exact solution ($\alpha = 2$)

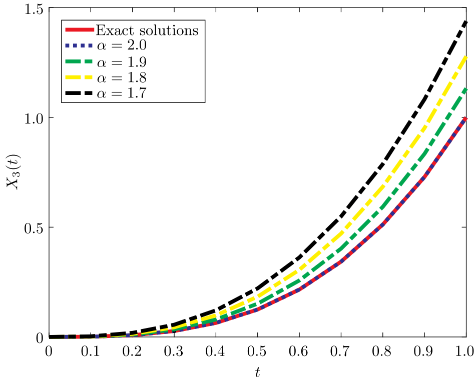

Figure 5. Behavior of $X_3(t)$ for different fractional orders α compared to the exact solution ($\alpha = 2$)

For $\alpha = 2$, we obtain the exact solution:

$$X_1(t) = t, \quad X_2(t) = t^2, \quad X_3(t) = t^3,$$

which matches the known solution obtained via Laplace transform methods [21].

Remark 2. Figures 1–5 demonstrate the behavior of the obtained solutions for different values of the fractional order α, comparing them with exact solutions in two applications of linear Caputo fractional Volterra integro-differential equation systems solved via the Khalouta integral transform method. Numerical simulations confirm that the fractional solutions converge precisely to the exact solutions when α approaches 1 for system (8) and 2 for system (9). All computations were performed using `MATLAB R2016a` for symbolic computation.

5. Conclusion. This study has demonstrated the successful application of the Khalouta integral transform method for solving systems of linear Caputo fractional Volterra integro-differential equations. The numerical applications and accompanying graphical representations have validated both the accuracy and effectiveness of the proposed approach. Our results establish the Khalouta transform as a powerful tool for obtaining solutions to systems of linear fractional integro-differential equations, making a significant contribution to this field of research.

The presented findings not only advance current theoretical understanding but also create new opportunities for applications across various scientific and mathematical disciplines. Future research directions will focus on extending this methodology to nonlinear Caputo–Volterra–Fredholm integro-differential equations with complex kernels, potentially combining the Khalouta integral transform with established techniques such as the Adomian decomposition method, the homotopy perturbation method, the variational iteration method, or the differential transform method to develop more robust solution frameworks for these challenging problems.

Competing interests. The author declares no competing interests.

Authors' contributions. The author is solely responsible for the content and preparation of this manuscript. The final version has been reviewed and approved by the author.

Funding. This research received no external funding.

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Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. 2025. Т. 29, № 2. С. 207–219
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu2141
EDN: YHPHPY

УДК 519.642.2

Решение систем линейных интегро-дифференциальных уравнений Вольтерра дробного порядка с производной Капуто методом интегрального преобразования Халаута

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Аннотация

Интегральное преобразование Халуты представляет собой мощный метод решения различных типов уравнений, включая интегро-дифференциальные уравнения и интегральные уравнения. Оно также может быть применено к начальным и краевым задачам для обыкновенных дифференциальных уравнений и уравнений в частных производных с постоянными коэффициентами. Основная цель данной работы — получение решений систем линейных интегро-дифференциальных уравнений Вольтерра дробного порядка с производной Капуто с использованием интегрального преобразования Халуты.

Для решения таких систем данным методом необходимо установить и определить ключевые свойства интегрального преобразования Халуты, которые играют важнейшую роль при выводе преобразования для дробной производной Капуто, входящей в системы. В работе представлены и решены несколько численных примеров с применением метода интегрального преобразования Халуты, демонстрирующие применимость предложенного подхода. Полученные результаты подтверждают, что данный метод обладает высокой эффективностью и позволяет находить точные решения систем линейных интегро-дифференциальных уравнений Вольтерра дробного порядка прямым способом.

Ключевые слова: интегральное преобразование Халуты, интегро-дифференциальные уравнения Вольтерра, дробная производная Капуто, точное решение.

Получение: 6 января 2025 г. / Исправление: 9 июня 2025 г. / Принятие: 18 июня 2025 г. / Публикация онлайн: 20 июня 2025 г.

Дифференциальные уравнения и математическая физика
Научная статья

Образец для цитирования
Khalouta A. Solution of systems of linear Caputo fractional Volterra integro-differential equations using the Khalouta integral transform method, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2025, vol. 29, no. 2, pp. 207–219. EDN: YHPHPY. DOI: 10.14498/vsgtu2141.

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