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Exact solutions to the Navier–Stokes equations describing stratified fluid flows

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Abstract

The paper considers the necessity of constructing exact solutions to the equations of dynamics of a viscous fluid stratified in terms of several physical characteristics, with density and viscosity taken as an example. The application of the families of exact solutions constructed for stratified fluids to modeling various technological processes dealing with moving viscous fluid media is discussed. Based on Lin's exact solutions, linear in some coordinates, a class of exact solutions to the Navier–Stokes equations is constructed for viscous multilayer media in a mass force field. The class is then extended to the case of the arbitrary relation of kinetic force fields to all three Cartesian coordinates and time. The issues of overdetermination and solvability of the reduced (based on the families under study) Navier–Stokes equation system supplemented by the incompressibility equation are discussed. The case of isobaric shearing flow outside the mass force field is considered in detail as an illustration. Three approaches to obtaining consistency conditions for the overdetermined reduced system of motion equations are discussed, and their interrelation is shown.

Keywords: Navier–Stokes equations, exact solution, stratified fluid, mass force field, overdetermined reduced system.

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Introduction. When exact solutions to the Navier–Stokes equations are sought, the main attention is given to homogeneous incompressible fluids with a constant density [1–12]. These theoretical results describe a very wide class of hydrodynamic phenomena for various time and spatial scales and enable viscometers and chemical engineering devices to be designed [13–18]. The construction of mathematical models of film flows [19–23], the analysis of processes in chemical engineering devices [19, 24, 25], and the solution of problems in astrophysics, aerophysics and geophysical hydrodynamics are based on the use of stratified fluids [26–32].

Multilayer structures in isothermal flows of viscous incompressible fluids arise from density stratification. Fluid incompressibility means that density is a Lagrangian invariant [32–35]. In other words, the characteristics of velocity field distribution vary in time and space. The most essential variation of fluid flow at a constant temperature manifests itself with respect to the vertical (transverse) coordinate. Density stratification along the vertical coordinate affects the dynamics of large structures and the energy exchange between vortices of different magnitudes, and it generates the appearance of internal waves [32, 36–39]. Note that the presence of density gradients with respect to the horizontal (longitudinal) coordinates induces various convections [1, 4–7]. Undoubtedly, the density stratification is determined by a continuous time and coordinate function. However, this dependence may prove unknown, or its value may be obtained rather approximately, and this may eventually lead to studying ill-defined problems of hydrodynamics and mathematical physics. In this case, researchers use models of a step density function; i.e., density and, strictly speaking, the coefficient of dynamic viscosity are specified for each fluid flow layer. This approach is applied, e.g., to study the instability of large-scale circulation. The study of hydrodynamic instability is in this case associated with the substitution of continuous stratification by multilayer models, two-layer ones being the most widespread [36, 40–44]. Another example of using two-layer and three-layer fluids is their application to the description of equatorial flows [31, 32, 45, 46]. The discussion found in [47–58] is based on the numerical integration of motion equations.

Thus, it seems urgent to construct a class of exact solutions to the Navier–Stokes problems for describing stratified fluid flows with a step density function. This paper constructs several families of exact solutions.

1. Problem statement. We consider a flowing fluid consisting of n layers (Fig. 1). Each i-th layer is characterized by density ρ_i, dynamic viscosity η_i, and thickness h_i. The motion equations for each layer can then be written in the invariant form as

$$\rho_i \frac{d\vec{V}^{(i)}}{dt} = -\nabla P^{(i)} + \eta_i \vartriangle \vec{V}^{(i)} + \vec{F}^{(i)}, \tag{1}$$

$$\nabla \cdot \vec{V}^{(i)} = 0. \tag{2}$$

Here, the velocity vector $\vec{V}^{(i)}$ has the coordinates $\vec{V}^{(i)} = \left(V_x^{(i)}, V_y^{(i)}, V_z^{(i)}\right)$, and the mass force vector $\vec{F}^{(i)}$ has the coordinates $\vec{F}^{(i)} = \left(F_x^{(i)}, F_y^{(i)}, F_z^{(i)}\right)$. The differential

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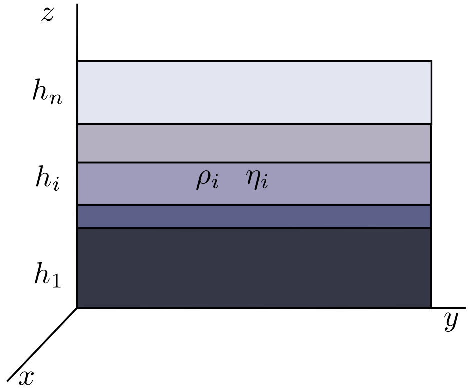

Figure 1. Schematic flow of a stratified fluid

operator $\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{V}^{(i)} \cdot \nabla\right)$. Note that the equation system (1), (2) is closed since the number of equations in it coincides with the number of required functions which here are the projections of the velocity vector $\vec{V}^{(i)}$ and pressure $P^{(i)}$. Note that the function $P^{(i)}$ is introduced for the convenience of the analytical and numerical integration of the system (1), (2), which consists of the Navier–Stokes equation (1) and the incompressibility equation (2).

2. The families of exact solutions for describing three-dimensional flows. To study the properties of stratified fluid flows, it is necessary to have a store of exact solutions to the Navier–Stokes equations satisfying the incompressibility equation. The exact solution to the system (1), (2) will be sought within the Lin–Sidorov–Aristov family [1]. For each layer ($i = 1, \ldots, n$) the velocity field is representable as

$$\begin{aligned} V_x^{(i)} &= U^{(i)}(z,t) + u_1^{(i)}(z,t)x + u_2^{(i)}(z,t)y, \\ V_y^{(i)} &= V^{(i)}(z,t) + v_1^{(i)}(z,t)x + v_2^{(i)}(z,t)y, \\ V_z^{(i)} &= w^{(i)}(z,t). \end{aligned} \tag{3}$$

Note that Eqs. (3) can be treated as a Taylor series expansion of the velocity vector components, restricted to only linear terms.

The structure of Eq. (1) suggests that the pressure $P^{(i)}$ and some projections of the mass force vector $\vec{F}^{(i)}$ needs to be treated as quadratic forms of the same spatial coordinates,

$$\begin{aligned} P^{(i)} = P_0^{(i)}(z,t) + P_1^{(i)}(z,t)x + P_2^{(i)}(z,t)y + \\ + P_{11}^{(i)}(z,t)\frac{x^2}{2} + P_{12}^{(i)}(z,t)xy + P_{22}^{(i)}(z,t)\frac{y^2}{2}, \end{aligned}$$

$$\begin{aligned} F_x^{(i)} &= A_0^{(i)}(z,t) + A_1^{(i)}(z,t)x + A_2^{(i)}(z,t)y, \\ F_y^{(i)} &= B_0^{(i)}(z,t) + B_1^{(i)}(z,t)x + B_2^{(i)}(z,t)y, \end{aligned} \tag{4}$$

$$F_z^{(i)} = C_0^{(i)}(z,t) + C_1^{(i)}(z,t)x + C_2^{(i)}(z,t)y + \\ + C_{11}^{(i)}(z,t)\frac{x^2}{2} + C_{12}^{(i)}(z,t)xy + C_{22}^{(i)}(z,t)\frac{y^2}{2}.$$

Substitute the representations (3), (4) in turns into each scalar equation of the system (1), (2), starting with the projection of the Navier–Stokes equation (1) onto the Ox axis:

$$\rho_i\Big(\frac{\partial V_x}{\partial t} + V_x^{(i)}\frac{\partial V_x^{(i)}}{\partial x} + V_y^{(i)}\frac{\partial V_x^{(i)}}{\partial y} + V_z^{(i)}\frac{\partial V_x^{(i)}}{\partial z}\Big) = \\ = -\frac{\partial P^{(i)}}{\partial x} + \eta_i\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)V_x^{(i)} + F_x^{(i)}.$$

Substituting the expressions for velocity, pressure, and mass forces into the latter equation, we obtain

$$\rho_i\Big[\frac{\partial(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y)}{\partial t} + \big(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y\big)\frac{\partial(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y)}{\partial x} + \\ + \big(V^{(i)} + v_1^{(i)}x + v_2^{(i)}y\big)\frac{\partial(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y)}{\partial y} + w^{(i)}\frac{\partial(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y)}{\partial z}\Big] = \\ = -\frac{\partial(P_0^{(i)} + P_1^{(i)}x + P_2^{(i)}y + P_{11}^{(i)}\frac{x^2}{2} + P_{12}^{(i)}xy + P_{22}^{(i)}\frac{y^2}{2})}{\partial x} + \\ + \eta_i\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)\big(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y\big) + \big(A_0^{(i)} + A_1^{(i)}x + A_2^{(i)}y\big).$$

Then, by calculating the partial derivatives involved in this equation, we can simplify the latter relationship as follows:

$$\rho_i\Big[\Big(\frac{\partial U^{(i)}}{\partial t} + \frac{\partial u_1^{(i)}}{\partial t}x + \frac{\partial u_2^{(i)}}{\partial t}y\Big) + \big(U^{(i)} + u_1^{(i)}x + u_2^{(i)}y\big)u_1^{(i)} + \\ + \big(V^{(i)} + v_1^{(i)}x + v_2^{(i)}y\big)u_2^{(i)} + w^{(i)}\Big(\frac{\partial U^{(i)}}{\partial z} + \frac{\partial u_1^{(i)}}{\partial z}x + \frac{\partial u_2^{(i)}}{\partial z}y\Big)\Big] = \\ = -\big(P_1^{(i)} + P_{11}^{(i)}x + P_{12}^{(i)}y\big) + \eta_i\Big(\frac{\partial^2 U^{(i)}}{\partial z^2} + \frac{\partial^2 u_1^{(i)}}{\partial z^2}x + \frac{\partial^2 u_2^{(i)}}{\partial z^2}y\Big) + \\ + \big(A_0^{(i)} + A_1^{(i)}x + A_2^{(i)}y\big).$$

Note that both the left- and right-hand sides of this equation are linear forms in the x- and y-coordinates. Taking into account that these coordinates are independent parameters and applying the principle of undetermined coefficients, we arrive at the following partial differential system:

$$\rho_i\Big(\frac{\partial U^{(i)}}{\partial t} + U^{(i)}u_1^{(i)} + V^{(i)}u_2^{(i)} + w^{(i)}\frac{\partial U^{(i)}}{\partial z}\Big) = -P_1^{(i)} + \eta_i\frac{\partial^2 U^{(i)}}{\partial z^2} + A_0^{(i)},$$

$$\rho_i\Big(\frac{\partial u_1^{(i)}}{\partial t} + (u_1^{(i)})^2 + v_1^{(i)}u_2^{(i)} + w^{(i)}\frac{\partial u_1^{(i)}}{\partial z}\Big) = -P_{11}^{(i)} + \eta_i\frac{\partial^2 u_1^{(i)}}{\partial z^2} + A_1^{(i)}, \qquad (5)$$

$$\rho_i\Big(\frac{\partial u_2^{(i)}}{\partial t}+u_2^{(i)}u_1^{(i)}+v_2^{(i)}u_2^{(i)}+w^{(i)}\frac{\partial u_2^{(i)}}{\partial z}\Big)=-P_{12}^{(i)}+\eta_i\frac{\partial^2 u_2^{(i)}}{\partial z^2}+A_2^{(i)}.$$

The functions involved in the equations of system (5) depend only on the vertical variable z and time t, with respect to which the differentiation is performed. In the case of steady-state flows, system (5) becomes an ordinary differential system preserving nonlinearity and inhomogeneity.

Acting the same way, we obtain the following consequences from the second and third Navier–Stokes equations (1). Having substituted Eqs. (3) and (4) into these equations and calculated the corresponding derivatives, we arrive at the following two equations:

$$\begin{aligned}\rho_i\Big[\Big(\frac{\partial V^{(i)}}{\partial t}+\frac{\partial v_1^{(i)}}{\partial t}x+\frac{\partial v_2^{(i)}}{\partial t}y\Big)+\big(U^{(i)}+u_1^{(i)}x+u_2^{(i)}y\big)v_1^{(i)}+\\ +\big(V^{(i)}+v_1^{(i)}x+v_2^{(i)}y\big)v_2^{(i)}+w^{(i)}\Big(\frac{\partial V^{(i)}}{\partial z}+\frac{\partial v_1^{(i)}}{\partial z}x+\frac{\partial v_2^{(i)}}{\partial z}y\Big)\Big]=\\ =-\big(P_2^{(i)}+P_{12}^{(i)}x+P_{22}^{(i)}y\big)+\eta_i\Big(\frac{\partial^2 V^{(i)}}{\partial z^2}+\frac{\partial^2 v_1^{(i)}}{\partial z^2}x+\frac{\partial^2 v_2^{(i)}}{\partial z^2}y\Big)+\\ +\big(B_0^{(i)}+B_1^{(i)}x+B_2^{(i)}y\big),\end{aligned}$$

$$\begin{aligned}\rho_i\Big(\frac{\partial w^{(i)}}{\partial t}+w^{(i)}\frac{\partial w^{(i)}}{\partial z}\Big)=\\ =-\Big(\frac{\partial P_0^{(i)}}{\partial z}+\frac{\partial P_1^{(i)}}{\partial z}x+\frac{\partial P_2^{(i)}}{\partial z}y+\frac{\partial P_{11}^{(i)}}{\partial z}\frac{x^2}{2}+\frac{\partial P_{12}^{(i)}}{\partial z}xy+\frac{\partial P_{22}^{(i)}}{\partial z}\frac{y^2}{2}\Big)+\\ +\eta_i\frac{\partial^2 w^{(i)}}{\partial z^2}+\Big(C_0^{(i)}+C_1^{(i)}x+C_2^{(i)}y+C_{11}^{(i)}\frac{x^2}{2}+C_{12}^{(i)}xy+C_{22}^{(i)}\frac{y^2}{2}\Big).\end{aligned}$$

By equating the coefficients of identical powers of the variables x and y in these equations, as well as their various nonlinear combinations, we arrive at the following relationships:

$$\begin{aligned}\rho_i\Big(\frac{\partial V^{(i)}}{\partial t}+V^{(i)}v_1^{(i)}+V^{(i)}v_2^{(i)}+w^{(i)}\frac{\partial V^{(i)}}{\partial z}\Big)=-P_2^{(i)}+\eta_i\frac{\partial^2 V^{(i)}}{\partial z^2}+B_0^{(i)},\\ \rho_i\Big(\frac{\partial v_1^{(i)}}{\partial t}+u_1^{(i)}v_1^{(i)}+v_1^{(i)}v_2^{(i)}+w^{(i)}\frac{\partial v_1^{(i)}}{\partial z}\Big)=-P_{12}^{(i)}+\eta_i\frac{\partial^2 v_1^{(i)}}{\partial z^2}+B_1^{(i)},\\ \rho_i\Big(\frac{\partial v_2^{(i)}}{\partial t}+u_2^{(i)}v_1^{(i)}+(v_2^{(i)})^2+w^{(i)}\frac{\partial v_2^{(i)}}{\partial z}\Big)=-P_{22}^{(i)}+\eta_i\frac{\partial^2 v_2^{(i)}}{\partial z^2}+B_2^{(i)},\\ \rho_i\Big(\frac{\partial w^{(i)}}{\partial t}+w^{(i)}\frac{\partial w^{(i)}}{\partial z}\Big)=-\frac{\partial P_0^{(i)}}{\partial z}+\eta_i\frac{\partial^2 w^{(i)}}{\partial z^2}+C_0^{(i)};\end{aligned}\tag{6}$$

$$\begin{aligned}\frac{\partial P_1^{(i)}}{\partial z}=C_1^{(i)},\quad \frac{\partial P_2^{(i)}}{\partial z}=C_2^{(i)},\\ \frac{\partial P_{11}^{(i)}}{\partial z}=C_{11}^{(i)},\quad \frac{\partial P_{12}^{(i)}}{\partial z}=C_{12}^{(i)},\quad \frac{\partial P_{22}^{(i)}}{\partial z}=C_{22}^{(i)}.\end{aligned}\tag{7}$$

Consideration of Eqs. (3) and (4) in the incompressibility equation (2) results in the following conditions:

$$u_1^{(i)} + v_2^{(i)} + \frac{\partial w^{(i)}}{\partial z} = 0. \qquad (8)$$

System (5)–(8) contains thirteen equations with respect to thirteen coefficients in Eqs. (3) and (4) for the velocity field and the pressure field in the i-th layer under study. Thus, the reduced system (5)–(8) inherits nonlinearity and closedness from system (1), (2).

Besides, in view of system (7), the linear and nonlinear terms in the representation of pressure in Eq. (4) can be considered unknown functions uniquely determined from the boundary condition for pressure and the boundary conditions at the layer boundaries. The constant term in Eqs. (4) (background pressure $P_0^{(i)}$) is determined by the integration of the first equation in system (7) after the vertical velocity $w^{(i)}$ is found.

The formulae in system (5)–(8) determine the class of exact solutions to the Navier–Stokes equations with the linear dependence of velocities on the spatial coordinates x and y. These variables are often referred to as horizontal in applied research. Recall that the pressure field (4) is a quadratic form. Exact solutions to the Navier–Stokes equations for the velocity field quadratically dependent on two spatial variables were presented in [59] as

$$V_x = U_1(z,t) + xU_2(z,t) + yU_3(z,t) + \frac{x^2}{2}U_4(z,t) + xyU_5(z,t) + \frac{y^2}{2}U_6(z,t),$$
$$V_y = V_1(z,t) + xV_2(z,t) + yV_3(z,t) + \frac{x^2}{2}V_4(z,t) + xyV_5(z,t) + \frac{y^2}{2}V_6(z,t),$$
$$V_z = W_1 + xW_2 + yW_3.$$

In this case, the pressure field and the mass force field are described by a polynomial relationship as

$$\begin{aligned} P = {} & P_1(z,t) + xP_2(z,t) + yP_3(z,t) + \frac{x^2}{2}P_4(z,t) + xyP_5(z,t) + \\ & + \frac{y^2}{2}P_6(z,t) + \frac{x^3}{6}P_7(z,t) + \frac{x^2y}{2}P_8(z,t) + \frac{xy^2}{2}P_9(z,t) + \frac{y^3}{6}P_{10}(z,t) + \\ & + \frac{x^4}{24}P_{11}(z,t) + \frac{x^3y}{6}P_{12}(z,t) + \frac{x^2y^2}{4}P_{13}(z,t) + \frac{xy^3}{6}P_{14}(z,t) + \frac{y^4}{24}P_{15}(z,t), \end{aligned}$$

$$\begin{aligned} A = {} & A_1(z,t) + xA_2(z,t) + yA_3(z,t) + \frac{x^2}{2}A_4(z,t) + xyA_5(z,t) + \frac{y^2}{2}A_6(z,t) + \\ & + \frac{x^3}{6}A_7(z,t) + \frac{x^2y}{2}A_8(z,t) + \frac{xy^2}{2}A_9(z,t) + \frac{y^3}{6}A_{10}(z,t), \end{aligned}$$

$$\begin{aligned} B = {} & B_1(z,t) + xB_2(z,t) + yB_3(z,t) + \frac{x^2}{2}B_4(z,t) + xyB_5(z,t) + \frac{y^2}{2}B_6(z,t) + \\ & + \frac{x^3}{6}B_7(z,t) + \frac{x^2y}{2}B_8(z,t) + \frac{xy^2}{2}B_9(z,t) + \frac{y^3}{6}B_{10}(z,t), \end{aligned}$$

$$C = C_1(z,t) + xC_2(z,t) + yC_3(z,t) + \frac{x^2}{2}C_4(z,t) + xyC_5(z,t) + \frac{y^2}{2}C_6(z,t) + \\ + \frac{x^3}{6}C_7(z,t) + \frac{x^2y}{2}C_8(z,t) + \frac{xy^2}{2}C_9(z,t) + \frac{y^3}{6}C_{10}(z,t) + \\ + \frac{x^4}{24}C_{11}(z,t) + \frac{x^3y}{6}C_{12}(z,t) + \frac{x^2y^2}{4}C_{13}(z,t) + \frac{xy^3}{6}C_{14}(z,t) + \frac{y^4}{24}C_{15}(z,t).$$

The equation system describing the unknown functions consists of thirty-eight equations for the determination of thirty unknown functions [59]. Thus, there is great arbitrariness in the construction of exact solutions to the Navier–Stokes equations. The route of finding "excess" equations when obtaining a partial differential system of the heat-conduction type was shown in [59].

By analogy, other solutions with an arbitrary dependence of the velocity field on the horizontal coordinates can be constructed. System (1), (2) is linear in this case; therefore, the following exact solution is valid:

$$V_x = \sum_{k=0}^{n} U_k, \quad V_y = \sum_{k=0}^{n} V_k, \quad V_z = \sum_{k=0}^{n-1} W_k,$$

$$A = \sum_{k=0}^{n^2-n} A_k, \quad B = \sum_{k=0}^{n^2-n} B_k, \quad C = \sum_{k=0}^{n^2-n+1} C_k, \quad P = \sum_{k=0}^{n^2-n+1} P_k.$$

Here, the forms U_k, V_k, W_k, A_k, B_k, C_k, and P_k are determined by the expressions

$$\begin{aligned} U_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i U_{i(k-i)}(z;t)\, x^i y^{k-i}, & V_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i V_{i(k-i)}(z;t)\, x^i y^{k-i}, \\ W_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i W_{i(k-i)}(z;t)\, x^i y^{k-i}, & A_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i A_{i(k-i)}(z;t)\, x^i y^{k-i}, \\ B_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i B_{i(k-i)}(z;t)\, x^i y^{k-i}, & C_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i C_{i(k-i)}(z;t)\, x^i y^{k-i}, \\ P_k &= \frac{1}{k!}\sum_{i=0}^{k} C_k^i P_{i(k-i)}(z;t)\, x^i y^{k-i}, \end{aligned} \tag{9}$$

where C_k^i is the number of combinations without repetition. Note that, if the crawling (slow) flow of a viscous incompressible fluid is considered, the exact solution is transformed as follows:

$$V_x = \sum_{k=0}^{n} U_k, \quad V_y = \sum_{k=0}^{n} V_k, \quad V_z = \sum_{k=0}^{n-1} W_k,$$

$$A = \sum_{k=0}^{n+1} A_k, \quad B = \sum_{k=0}^{n+1} B_k, \quad C = \sum_{k=0}^{n+1} C_k, \quad P = \sum_{k=0}^{n+1} P_k.$$

3. Exact solutions for shearing flows. Let us now consider shearing flows outside the mass force field, i.e., an important particular case of isothermal flows of stratified fluids defined by Eqs. (3)–(8). Assume further that $V_z^{(i)} = 0$, $P^{(i)} = 0$, $\vec{F}^{(i)} = 0$. System (5)–(8) then becomes

$$\begin{aligned}
&\frac{\partial {u_1}^{(i)}}{\partial t} + ({u_1}^{(i)})^2 + {v_1}^{(i)}{u_2}^{(i)} = \nu_i \frac{\partial^2 {u_1}^{(i)}}{\partial z^2},\\
&\frac{\partial {u_2}^{(i)}}{\partial t} + {u_2}^{(i)}({u_1}^{(i)} + {v_2}^{(i)}) = \nu_i \frac{\partial^2 {u_2}^{(i)}}{\partial z^2},\\
&\frac{\partial {v_1}^{(i)}}{\partial t} + ({u_1}^{(i)} + {v_2}^{(i)}){v_1}^{(i)} = \nu_i \frac{\partial^2 {v_1}^{(i)}}{\partial z^2},\\
&\frac{\partial {v_2}^{(i)}}{\partial t} + {u_2}^{(i)}{v_1}^{(i)} + \left({v_2}^{(i)}\right)^2 = \nu_i \frac{\partial^2 {v_2}^{(i)}}{\partial z^2},\\
&{u_1}^{(i)} + {v_2}^{(i)} = 0;
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\frac{\partial U^{(i)}}{\partial t} + U^{(i)}{u_1}^{(i)} + V^{(i)}u_2 = \nu_i \frac{\partial^2 U^{(i)}}{\partial z^2},\\
&\frac{\partial V^{(i)}}{\partial t} + U^{(i)}{v_1}^{(i)} + V^{(i)}{v_2}^{(i)} = \nu_i \frac{\partial^2 V^{(i)}}{\partial z^2}.
\end{aligned} \tag{11}$$

Here $\nu_i = \eta_i/\rho_i$ is the kinematic viscosity of the i-th layer.

These flows are of interest due to the fact that the equations of system (10), (11) enable one to study the balance of convective and viscous forces. This is why incompressible fluid flows under constant pressure arouse great interest when they occur in large currents [1].

The slope of isobaric surfaces relative to isopotential (sea-level) surfaces generates gradient flows in the global ocean. In a rotating fluid, the Coriolis force makes the gradient flow deviate from the direction of gradient pressure, the direction of this deviation being different in different hemispheres. Thus, we have something like the geostrophic wind studied in meteorology [60]. The study of these flows is necessitated by practical considerations.

Note that changing to shearing flows, on the one hand, facilitates the problem due to a reduced number of unknown functions to be determined and, on the other hand, complicates it since the system of constitutive relations (10), (11) becomes overdetermined. The overdetermination lies in system (10). If the latter system is solved, i.e. if a nontrivial simultaneous solution is found, a single integration of the equations in system (11) will yield the homogeneous components of Eq. (1) for the velocity field.

Let us now discuss three ways of deriving consistency conditions for constructing exact solutions to the Navier–Stokes equations (1) and the incompressibility equation (2).

Approach I is the most general [2]. It allows us to obtain consistency conditions without reference to the solution structure (without using Eqs. (3) and (4). We write the Navier–Stokes equation (1) for isobaric shearing flows in the coordinate form as

$$\frac{\partial {V_x}^{(i)}}{\partial t} + {V_x}^{(i)}\frac{\partial {V_x}^{(i)}}{\partial x} + {V_y}^{(i)}\frac{\partial {V_x}^{(i)}}{\partial y} = \nu_i\Big(\frac{\partial^2 {V_x}^{(i)}}{\partial x^2} + \frac{\partial^2 {V_x}^{(i)}}{\partial y^2} + \frac{\partial^2 {V_x}^{(i)}}{\partial z^2}\Big), \tag{12}$$

$$\frac{\partial {V_y}^{(i)}}{\partial t} + {V_x}^{(i)}\frac{\partial {V_y}^{(i)}}{\partial x} + {V_y}^{(i)}\frac{\partial {V_y}^{(i)}}{\partial y} = \nu_i\Big(\frac{\partial^2 {V_y}^{(i)}}{\partial x^2} + \frac{\partial^2 {V_y}^{(i)}}{\partial y^2} + \frac{\partial^2 {V_y}^{(i)}}{\partial z^2}\Big). \quad (13)$$

Equations (12) and (13) close with the continuity equation

$$\frac{\partial {V_x}^{(i)}}{\partial x} + \frac{\partial {V_y}^{(i)}}{\partial y} = 0. \quad (14)$$

We now differentiate Eqs. (12) and (13) with respect to the variables x and y, respectively, and stratify the obtained expressions. Some simple transformations and the use of the incompressibility equation (14) result in the following relationship (consistency condition):

$$\frac{\partial {V_x}^{(i)}}{\partial x}\frac{\partial {V_y}^{(i)}}{\partial y} = \frac{\partial {V_x}^{(i)}}{\partial y}\frac{\partial {V_y}^{(i)}}{\partial x}. \quad (15)$$

Approach II to the derivation of the consistency condition for the overdetermined system (12)–(14) is also general (i.e. not tied to the selected solution structure) and based on the use of the stream function and Eq. (15) [2]. Let us now consider the scalar stream function $\psi^{(i)} = \psi^{(i)}(x, y, z, t)$ has the following property:

$${V_x}^{(i)} = \frac{\partial \psi^{(i)}}{\partial y}, \quad {V_y}^{(i)} = -\frac{\partial \psi^{(i)}}{\partial x}. \quad (16)$$

Note that, substituting Eqs. (16) into the incompressibility equation (14), due to the commutativity of the derivatives with respect to the spatial variables x, y, we arrive at a correct identity.

The expressions of Eqs. (16) are also substituted into the consistency condition (15),

$$\frac{\partial}{\partial x}\Big(\frac{\partial \psi^{(i)}}{\partial y}\Big)\frac{\partial}{\partial y}\Big(-\frac{\partial \psi^{(i)}}{\partial x}\Big) = \frac{\partial}{\partial y}\Big(\frac{\partial \psi^{(i)}}{\partial y}\Big)\frac{\partial}{\partial x}\Big(-\frac{\partial \psi^{(i)}}{\partial x}\Big).$$

Again, due to the commutativity of the derivatives, the consistency condition (15) acquires the form of the homogeneous Monge–Ampère equation

$$\Big(\frac{\partial^2 \psi^{(i)}}{\partial x \partial y}\Big)^2 = \frac{\partial^2 \psi^{(i)}}{\partial x^2}\frac{\partial^2 \psi^{(i)}}{\partial y^2}. \quad (17)$$

Approach III appeals to the structure of the exact solution (3). We decrease the number of unknowns in system (10), express the spatial acceleration $v_2^{(i)} = -u_1^{(i)}$ from the equation $u_1^{(i)} + v_2^{(i)} = 0$ in system (10) and substitute it into the other equations of this system. Some simple transformations result in the following system:

$$\begin{aligned}
&\frac{\partial {u_1}^{(i)}}{\partial t} + \big({u_1}^{(i)}\big)^2 + {v_1}^{(i)}{u_2}^{(i)} = \nu_i\frac{\partial^2 {u_1}^{(i)}}{\partial z^2},\\
&\frac{\partial {u_1}^{(i)}}{\partial t} - \big({u_2}^{(i)}{v_1}^{(i)} + ({u_1}^{(i)})^2\big) = \nu_i\frac{\partial^2 {u_1}^{(i)}}{\partial z^2},\\
&\frac{\partial {u_2}^{(i)}}{\partial t} = \nu_i\frac{\partial^2 {u_2}^{(i)}}{\partial z^2}, \qquad \frac{\partial {v_1}^{(i)}}{\partial t} = \nu_i\frac{\partial^2 {v_1}^{(i)}}{\partial z^2}.
\end{aligned} \quad (18)$$

The comparison of the first two equations in system (18) infers that

$$\left(u_1{}^{(i)}\right)^2 + v_1{}^{(i)}u_2{}^{(i)} = 0. \tag{19}$$

The algebraic condition (19) is the consistency condition for the nontrivial solutions of the overdetermined system (10). In view of Eq. (19), we rewrite Eqs. (18) in the operator form as

$$L^{(i)}u_1{}^{(i)} = 0 = -L^{(i)}v_2{}^{(i)}, \quad L^{(i)}u_2{}^{(i)} = 0, \quad L^{(i)}v_1{}^{(i)} = 0. \tag{20}$$

Here, the linear operator $L^{(i)}$ is determined by the expression $L^{(i)} = \frac{\partial}{\partial t} - \nu_i \frac{\partial^2}{\partial z^2}$. The solution of system (20) can be written as

$$\begin{aligned} u_1^{(i)} &= u^{(i)}(z,t)\cos\vartheta^{(i)}\sin\vartheta^{(i)}, & u_2^{(i)} &= u^{(i)}(z,t)\cos^2\vartheta^{(i)}, \\ v_1^{(i)} &= -u^{(i)}(z,t)\sin^2\vartheta^{(i)}, & v_2^{(i)} &= -u^{(i)}(z,t)\cos\vartheta^{(i)}\sin\vartheta^{(i)}. \end{aligned} \tag{21}$$

The angle $\vartheta^{(i)}$ is an arbitrary constant, and the function $u^{(i)} = u^{(i)}(z,t)$ satisfies the linear operator equation of the heat-conduction type

$$L^{(i)}u = 0.$$

Note that, if the flow under study is steady-state, the linear operator $L^{(i)}$ degenerates simply into the operation of double differentiation with respect to the variable z, and the function u in the general solution (21) becomes simply a z-linear function with constant coefficients.

Additionally, note that the consistency condition (19) for class (3) can be easily obtained from Eq. (17). To do this, it would suffice to find the form of the stream function $\psi^{(i)}$ for class (3) from Eqs. (16),

$$\begin{aligned} \frac{\partial \psi^{(i)}}{\partial y} &= V_x^{(i)} = U^{(i)} + u_1^{(i)}x + u_2^{(i)}y, \\ \frac{\partial \psi^{(i)}}{\partial x} &= -V_y^{(i)} = -\left(V^{(i)} + v_1^{(i)}x + v_2^{(i)}y\right). \end{aligned} \tag{22}$$

The independent integration of Eqs. (22) yields the following expressions:

$$\psi^{(i)} = U^{(i)}y + xyu_1^{(i)} + \tfrac{y^2}{2}u_2^{(i)} + \Psi_1^{(i)}(x,z),$$
$$\psi^{(i)} = -V^{(i)}x - xyv_2^{(i)} - \tfrac{x^2}{2}v_1^{(i)} + \Psi_2^{(i)}(y,z).$$

Equating these relationships and taking into account the relation $u_1 = -v_2$ between the velocity gradients V_x and V_y, we arrive at a quadratic (in terms of the variables x, y) representation with the coefficients determined by the z, t dependences of an arbitrary form

$$\begin{aligned} \psi^{(i)} = U^{(i)}y + xyu_1^{(i)} + \frac{y^2}{2}u_2^{(i)} - V^{(i)}x - \frac{x^2}{2}v_1^{(i)} = \\ = U^{(i)}y - xyv_2^{(i)} + \frac{y^2}{2}u_2^{(i)} - V^{(i)}x - \frac{x^2}{2}v_1^{(i)}. \end{aligned} \tag{23}$$

Then, Eq. (23) is substituted into the consistency condition (17),

$$\left(u_1{}^{(i)}\right)^2 = (-v_1{}^{(i)})u_2{}^{(i)}.$$

This expression coincides with the consistency condition (19) obtained from absolutely different reasonings. Note that the operator equations (20) are solved by standard methods, e.g., by variable separation. After finding their solution and satisfying the consistency condition (19), the nonlinear equations (11) are integrated.

The class of exact solutions (3) for system (10), (11) can be extended similarly to the results reported in [2]. It can be easily shown that the velocity field

$$V_x^{(i)} = \sum_{k=0}^{n} U_k^{(i)}(z,t)\frac{y^k}{k!}, \quad V_y^{(i)} = V^{(i)}(z,t)$$

satisfies the reduced Navier–Stokes equation system and the incompressibility equation (system (10), (11)). By rotational transformation of the coordinates and velocities

$$x \to x\cos\varphi - y\sin\varphi, \quad y \to x\sin\varphi + y\cos\varphi,$$

$$V_x^{(i)} \to V_x^{(i)}\cos\varphi - V_y^{(i)}\sin\varphi, \quad V_y^{(i)} \to V_x^{(i)}\sin\varphi + V_y^{(i)}\cos\varphi,$$

we obtain a family of exact solutions of the form (9) with a nonlinear dependence on two coordinates.

Conclusion. The paper discusses a family of exact solutions to the Navier–Stokes equations for describing flows of stratified viscous fluids in various force fields. The reported solutions are based on the known Lin–Sidorov–Aristov family of exact solutions, and they enable us to take into account the difference of the physical characteristics of the stratified fluid layers (viscosity, density) from the geometrical ones (thickness). An algorithm for a subsequent extension of the family to the case of arbitrary dependence of the velocity field on the horizontal coordinates has been shown.

A particular case of the family has been separately discussed, namely, the class of solutions for describing shearing isothermal flows of stratified fluids outside the mass force field. It has been demonstrated that the reduced overdetermined Navier–Stokes equation system has a simultaneous solution determined by the integration of a system of operator equations like the nonstationary heat conduction equation.

Besides, the paper has shown the transformation undergone by the discussed families of exact solutions when the coordinate system is rotated. This is a key issue, e.g., in the description of stratified fluid flow in an inclined layer, where gravitation affects the flow structure in all three orthogonal directions determined by the magnitude of the flow surface slope.

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References

1. Ershkov S. V., Prosviryakov E. Y., Burmasheva N. V., Christianto V. Towards understanding the algorithms for solving the Navier–Stokes equations, *Fluid Dyn. Res.*, 2021, vol. 53, no. 4, 044501. https://doi.org/10.1088/1873-7005/ac10f0.
2. Zubarev N. M., Prosviryakov E. Y. Exact solutions for layered three-dimensional nonstationary isobaric flows of a viscous incompressible fluid, *J. Appl. Mech. Techn. Phys.*, 2019, vol. 60, no. 6, pp. 1031–1037. https://doi.org/10.1134/S0021894419060075.
3. Ryzhkov I. I. *Thermal Diffusion in Mixtures: Equations, Symmetries, Solutions and Their Stability*, Thesis of Dissertation (Cand. Phys. & Math. Sci.). Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 2013, 199 pp. (In Russian)
4. Burmasheva N. V., Prosviryakov E. Yu. Convective layered flows of a vertically whirling viscous incompressible fluid. Velocity field investigation, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2019, vol. 23, no. 2, pp. 341–360. https://doi.org/10.14498/vsgtu1670.
5. Burmasheva N. V.,Privalova V. V., Prosviryakov E. Yu. Layered Marangoni convection with the Navier slip condition, *Sãdhanã*, 2021, vol. 46, no. 1, 55. https://doi.org/10.1007/s12046-021-01585-5.
6. Burmasheva N. V., Prosviryakov E. Yu. On Marangoni shear convective flows of inhomogeneous viscous incompressible fluids in view of the Soret effect, *J. King Saud Univ. — Science*, 2020, vol. 32, no. 8, pp. 3364–3371. https://doi.org/10.1016/j.jksus.2020.09.023.
7. Burmasheva N. V., Prosviryakov E. Yu. Exact solution for stable convective concentration flows of a Couette type, *Computational Continuum Mechanics*, 2020, vol. 13, no. 3, pp. 337–349 (In Russian). https://doi.org/10.7242/1999-6691/2020.13.3.27.
8. Burmasheva N. V., Prosviryakov E. Yu. Exact solution of Navier–Stokes equations describing spatially inhomogeneous flows of a rotating fluid, *Trudy Inst. Mat. i Mekh. UrO RAN*, 2020, vol. 26, no. 2, pp. 79–87 (In Russian). https://doi.org/10.21538/0134-4889-2020-26-2-79-87.
9. Burmasheva N. V., Prosviryakov E. Yu. A class of exact solutions for two–dimensional equations of geophysical hydrodynamics with two Coriolis parameters, *The Bulletin of Irkutsk State University. Series Mathematics*, 2020, vol. 32, pp. 33–48 (In Russian). https://doi.org/10.26516/1997-7670.2020.32.33.
10. Burmasheva N. V., Prosviryakov E. Yu. Investigation of a velocity field for the Marangoni shear convection of a vertically swirling viscous incompressible fluid, *AIP Conference Proceedings*, 2018, vol. 2053, no. 1, 040011. https://doi.org/10.1063/1.5084449.
11. Burmasheva N. V., Prosviryakov E. Yu. Exact solution for the layered convection of a viscous incompressible fluid at specified temperature gradients and tangential forces on the free boundary, *AIP Conference Proceedings*, 2017, vol. 1915, no. 1, 040005. https://doi.org/10.1063/1.5017353.
12. Aristov S. N., Prosviryakov E. Y. A new class of exact solutions for three-dimensional thermal diffusion equations, *Theor. Found. Chem. Technol.*, 2016, vol. 50, no. 3, pp. 286–293. https://doi.org/10.1134/S0040579516030027.
13. Davidson J. F., Harrison D. *Fluidised Particles.* New York, Cambridge Univ. Press, 1963, 155 pp.
14. Tanford C. *Physical Chemistry of Macromolecules.* New York, John Wiley and Sons, 1961, xiv+710 pp.

15. Sherman Ph. *Emulsion Science*. New York, Academic Press, 1968, x+496 pp.
16. Barr G. *A Monograph of Viscometry*. London, Oxford Univ. Press, 1931, xiv+318 pp.
17. Malkin A. Ya., Chalykh A. E. *Diffuziia i viazkost' polimerov. Metody izmereniia* [Diffusion and Viscosity of Polymers. Methods of Measurement]. Moscow, Khimiya, 1979, 304 pp. (In Russian)
18. Fuks G. I. *Viazkost' i plastichnost' nefteproduktov* [Viscosity and Plasticity of Petroleum Products]. Moscow, Leningrad, Gostoptekhizdat, 1951, 272 pp. (In Russian)
19. Sokolov V. N., Domanskii I. V. *Gazozhidkostnye reaktory* [Gas-Liquid Reactors]. Leningrad, Mashinostroenie, 1976, 216 pp. (In Russian)
20. Kapitza P. L. Wave flow of thin layers of a viscous liquid, *Zh. Eksperim. Teor. Fiz.*, 1948, vol. 18, no. 1, pp. 3–28 (In Russian); Kapitza P. L. Wave flow of thin layers of a viscous liquid, In: *Collected Papers of P.L. Kapitza*, vol. 2. Oxford, Pergamon Press, 1965, pp. 662–708. https://doi.org/10.1016/B978-0-08-010973-2.50013-6.
21. Gogonin I. I., Shemagin I. A., Budov V. M., Dorokhov A. R. *Teploobmen pri plenochnoy kondensatsii i plenochnom kipenii v elementakh oborudovaniya AES* [Heat Transfer during Film Condensation and Film Boiling in Elements of Equipment at Nuclear Power Plants]. Moscow, Energoizdat, 1993, 208 pp. (In Russian)
22. Hirshburg R. I., Florschuetz L. W. Laminar wavy-film flow: Part II, Condensation and evaporation, *J. Heat Transfer*, 1982, vol. 104, no. 3, pp. 459–464. https://doi.org/10.1115/1.3245115.
23. Trifonov Yu. Ya. Wavy flow of a liquid film in the presence of a cocurrent turbulent gas flow, *J. Appl. Mech. Tech. Phys.*, 2013, vol. 54, no. 5, pp. 762–772. https://doi.org/10.1134/S002189441305009X.
24. Domanskii I. V., Isakov V. P., Ostrovsky G. M., Reshanov A. S., Sokolov V. N. *Mashiny i apparaty khimicheskikh proizvodstv: primery i zadachi* [Machines and Devices for Chemical Production: Examples and Tasks]. Moscow, Mashinostroenie, 1982, 384 pp. (In Russian)
25. Romankov P. G., Kurochkina M. I., Morzherin Yu. Ya., Smirnov N. N. *Protsessy i apparaty khimicheskoi promyshlennosti* [Processes and Devices of Chemical Industry]. Moscow, Khimiya, 1989, 554 pp. (In Russian)
26. Koskov V. N. *Geofizicheskoe issledovanie skvazhin* [Geophysical Well Logging]. Perm, Perm State Techn. Univ., 2004, 122 pp. (In Russian)
27. Vakhromeev G. S., Davydenko A. Yu. *Modelirovanie v razvedochnoi geofizike* [Modeling in Exploration Geophysics]. Moscow, Nedra, 1987, 192 pp. (In Russian)
28. Kostitsyn V. I., Khmelevskoi V. K. *Geofizika* [Geophysics]. Perm, Perm State National Research Univ., 2018, 428 pp. (In Russian)
29. Barrenblatt G. E., Zheltov I. P., Kochina I. N. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata], *J. Appl. Math. Mech.*, 1960, vol. 24, no. 5, pp. 1286–1303. https://doi.org/10.1016/0021-8928(60)90107-6.
30. Dietrich P., Helmig R., Sauter M., Hötzl H., Köngeter J., Teutsch G. *Flow and Transport in Fractured Porous Media*. Berlin, Springer-Verlag, 2005, xviii+447 pp. https://doi.org/10.1007/b138453.
31. Pedlosky J. *Geophysical Fluid Dynamics*. New York, Springer, 1987, xiv+710 pp. https://doi.org/10.1007/978-1-4612-4650-3.
32. Dolzhanskii F. V. *Lektsii po geofizicheskoi gidrodinamike* [Lectures on Geophysical Fluid Dynamics]. Moscow, Inst. Vychisl. Mat. Ross. Akad. Nauk, 2006, 377 pp. (In Russian)
33. Bogolyubov N. N., Shirkov D. V. *Introduction to the Theory of Quantized Fields*, Interscience Monographs in Physics and Astronomy, vol. 3. New York, Interscience Publ., 1959, xvi+720 pp.
34. Ruban V. P. Motion of magnetic flux lines in magnetohydrodynamics, *J. Exp. Theor. Phys.*, 1999, vol. 89, no. 2, pp. 299–310. https://doi.org/10.1134/1.558984.
35. Kochin N. K., Kibel I. A., Roze N. V. *Theoretical Hydromechanics*. New York, John Wiley and Sons, 1964, v+577 pp.

36. Talipova T. G., Pelinovsky E. N., Kurkina O. E., Rouvinskaya E. A., Giniyatullin A. R., Naumov A. A. Nonreflective propagation of internal waves in a channel of variable cross-section and depth, *Fundam. Prikl. Gidrofiz.*, 2013, vol. 6, no. 3, pp. 46–53 (In Russian).
37. Smith N. R. Ocean modeling in a global ocean observing system, *Rev. Geophys.*, 1993, vol. 31, no. 3, pp. 281–317. https://doi.org/10.1029/93RG00134.
38. Lighthill J. *Waves in Fluids*, Cambridge Mathematical Library. New York, Cambridge Univ. Press, 1978, xv+504 pp.
39. Miropolsky Yu. Z. *Dynamics of Internal Gravity Waves in the Ocean*, Atmospheric and Oceanographic Sciences Library, vol. 24. Dordrecht, Kluwer Acad. Publ., 2001, xviii+406 pp. https://doi.org/10.1007/978-94-017-1325-2.
40. Grimshaw R., Pelinovsky E., Talipova T. Fission of a weakly nonlinear interfacial solitary wave at a step, *Geophys. Astrophys. Fluid Dynamics*, 2008, vol. 102, no. 2, pp. 179–194. https://doi.org/10.1080/03091920701640115.
41. Chesnokov A. A. Properties and exact solutions of the rotating shallow-water equations for stratified multilayered flows, *Vestnik of Lobachevsky University of Nizhni Novgorod*, 2011, no. 4 (3), pp. 1252–1254 (In Russian).
42. Pozhalostin A. A., Goncharov D. A. Free axisymmetric oscillations of two-layered liquid with the elastic separator between layers in the presence of surface tension forces, *Engineering Journal. Science and Innovation*, 2013, no. 12 (24), 1147, 8 pp. (In Russian). https://doi.org/10.18698/2308-6033-2013-12-1147.
43. Pozhalostin A. A., Goncharov D. A., Kokushkin V. V. Small oscillations of two-layer liquid in view permeability of separator, *Herald of the Bauman Moscow State Technical University*, 2014, no. 5 (56), pp. 109–116 (In Russian).
44. Shiryaeva S. O., Grigor'ev A. I., Yakovleva L. S. Effect of initial conditions on wave motion in density-stratified three-layer liquid with free surface, *Tech. Phys.*, 2017, vol. 62, no. 3, pp. 374–379. https://doi.org/10.1134/s1063784217030203.
45. Pedlosky J. *Ocean Circulation Theory*. Berlin, Heidelberg, Springer-Verlag, 1996, xi+455 pp. https://doi.org/10.1007/978-3-662-03204-6.
46. Shtokman V. B. *Ekvatorial'nye protivotecheniia v okeanakh. Osnovy teorii* [Theory of the Equatorial Countercurrent. Fundamentals of Theory]. Leningrad, Gidrometeoizdat, 1948, 156 pp. (In Russian)
47. Sarkisyan A. S. *Chislennyi analiz i prognoz morskikh techenii* [Numerical Analysis and Sea Current Prediction]. Leningrad, Gidrometeoizdat, 1977, 182 pp. (In Russian)
48. Zhukov V. T., Feodoritova O. B., Duben A. P., Novikova N. D. Explicit time integration of the Navier–Stokes equations using the local iteration method, *KIAM Preprint*, 2019, no. 12, 32 pp (In Russian). https://doi.org/10.20948/prepr-2019-12.
49. Kozelkov A. S., Meleshkina D. P., Kurkin A. A., Tarasova N. V., Lashkin S. V., Kurulin V. V. Fully implicit method for solution of Navier—Stokes equations for simulation of multiphase flows with free surface, *Vychisl. Tekhn.*, 2016, vol. 21, no. 5, pp. 54–76 (In Russian).
50. Anderson D., Tannehill J. C., Pletcher R. H. *Computational Fluid Mechanics and Heat Transfer*. Washington, DC, Taylor and Francis, 2016, 774 pp. https://doi.org/10.1201/b12884.
51. Temam R. *Navier–Stokes Equations. Theory and Numerical Analysis*, vol. 2, Studies in Mathematics and Its Applications. Amsterdam, North-Holland Publ., 1977, vi+500 pp. https://doi.org/10.1016/s0168-2024(09)x7004-9.
52. Taylor T. D., Ndefo E. Computation of viscous flow in a channel by the method of splitting, In: *Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics*, Lecture Notes in Physics, 8. Berlin, Heidelberg, Springer, 1971, pp. 356–364. https://doi.org/10.1007/3-540-05407-3_51.
53. Roache P. J. *Computational Fluid Dynamics*. Albuquerque, Hermosa, 1972, vii+434 pp.
54. Soboleva E. B. Onset of Rayleigh–Taylor convection in a porous medium, *Fluid Dyn.*, 2021, vol. 56, no. 2, pp. 200–210. https://doi.org/10.1134/S0015462821020105.

55. Demyshev S. G., Evstigneeva N. A., Alekseev D. V., Dymova O. A., Miklashevskaya N. A. Analysis of the dynamic and energy characteristics of water circulation near the Western Crimea coast and in the Sevastopol region based on the observational data assimilation in the numerical model of the Black sea dynamics, *Morskoy Gidrofizicheskiy Zhurnal* [Physical Oceanography], 2021, vol. 37, no. 1, pp. 23–40 (In Russian). https://doi.org/10.22449/0233-7584-2021-1-23-40.
56. Tarasevich S. E., Giniyatullin A. A. CFD investigation of flow behavior and heat transfer in tubes with ribbed twisted tape inserts, *Tepl. Prots. Tekhn.*, 2021, vol. 13, no. 2, pp. 78–84 (In Russian). https://doi.org/10.34759/tpt-2021-13-2-78-84.
57. Belokon A. Yu., Fomin V. V. Simulation of tsunami wave propagation in the Kerch strait, *Fund. Prikl. Gidrofizika*, 2021, vol. 14, no. 1, pp. 67–78 (In Russian). https://doi.org/10.7868/S207366732101007X.
58. Gataulin Ya. A., Smirnov E. M. A flow in the blood vessel with a one-side stenosis: numerical study of the structure and local turbulization, *St. Petersburg State Polytechnical University Journal. Physics and Mathematics*, 2021, vol. 14, no. 1, pp. 72–84. https://doi.org/10.18721/JPM.14105.
59. Prosviryakov E. Yu. New class of exact solutions of Navier–Stokes equations with exponential dependence of velocity on two spatial coordinates, *Theor. Found. Chem. Eng.*, 2019, vol. 53, no. 1, pp. 107–114. https://doi.org/10.1134/S0040579518060088.
60. Stewart R. H. *Introduction to Physical Oceanography*, 2008, viii+346 pp. https://github.com/introocean/introocean-en.

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Точные решения уравнений Навье–Стокса для описания течений многослойных жидкостей

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Аннотация

Статья посвящена рассмотрению вопросов необходимости построения точных решений для уравнений динамики вязкой жидкости, стратифицированной по нескольким физическим характеристикам (на примере плотности и вязкости). Обсуждаются вопросы применения семейств точных решений, построенных для многослойных жидкостей, при моделировании различных технологических процессов, имеющих дело с движущимися вязкими жидкими средами. В работе на основе точных решений Линя, линейных по части координат, построен класс точных решений уравнений Навье–Стокса для вязких многослойных сред в поле массовых сил. Далее производится обобщение приведенного класса на случай произвольной зависимости кинетико-силовых полей от всех трех декартовых координат и времени. Обсуждаются вопросы переопределенности и разрешимости редуцированной (на основе данных семейств) системы уравнений Навье–Стокса, дополненных уравнением несжимаемости. В качестве наглядной иллюстрации подробно разбирается случай изобарических сдвиговых течений вне поля массовых сил. Обсуждаются три подхода к получению условий совместности переопределенной редуцированной системы уравнений движения, показывается их взаимосвязь.

Ключевые слова: уравнения Навье–Стокса, точное решение, многослойная жидкость, поле массовых сил, переопределенная приведенная система.

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