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Exact boundaries for the analytical approximate solution of a class of first-order nonlinear differential equations in the real domain

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Abstract

The paper gives a solution to one of the problems of the analytical approximate method for one class first order nonlinear differential equations with moving singular points in the real domain. The considered equation in the general case is not solvable in quadratures and has movable singular points of the algebraic type. This circumstance requires the solution of a number of mathematical problems.

Previously, the authors have solved the problem of the influence of a moving point perturbation on the analytical approximate solution. This solution was based on the classical approach and, at the same time, the area of application of the analytic approximate solution shrank in comparison with the area obtained in the proved theorem of existence and uniqueness of the solution.

Therefore, the paper proposes a new research technology based on the elements of differential calculus. This approach allows to obtain exact boundaries for an approximate analytical solution in the vicinity of a moving singular point.

New a priori estimates are obtained for the analytical approximate solution of the considered class of equations well in accordance with the known ones for the common area of action. These results complement the previously obtained ones, with the scope of the analytical approximate solution in the vicinity of the movable singular point being significantly expanded.

These estimates are consistent with the theoretical positions, as evidenced by the experiments carried out with a non-linear differential equation having the exact solution. A technology for optimizing a priori error

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estimates using a posteriori estimates is provided. The series with negative fractional powers are used.

Keywords: moving singular points, nonlinear differential equation, Cauchy problem, exact boundaries of a domain, a priori and a posteriori errors, analytical approximate solution.

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1. Introduction

In many areas of the problem: building optimal filters [1, 2], mathematical physics, nonlinear optics [3, 4], theory of evolutionary processes [5–10], the theory of elasticity [11], nonlinear diffusion [12], in the theory of sustainability of building structures' elements and in the analysis of buildings' vitality [13–15] (both of applied nature), are solved with the help of mathematical models, which are differential equations. The latter have a problem in finding a solution associated with the presence of movable singular points, which refer such equations to the class of equations in the general case not solvable in quadratures.

Significant results in solving this problem have been achieved by the Belarusian School of Analytical Theory of Differential Equations [16,17]. A great contribution has been made by such well-known scientists as N. P. Erugin, A. I. Yablonskii, N. A. Lukashevich, as well as their students A. V. Chichurin, A. A. Samodurov, etc. However, it should be noted that the results were obtained only for special cases of solvability in quadratures of nonlinear differential equations, as in the works [18–20]. The lack of exact methods actualizes the development of analytical approximate methods for solving this category of nonlinear differential equations.

In the works [21, 22], the disadvantage of the classical theorem of existence and uniqueness of the solution of differential equations was pointed out, the elimination of which develops a new approach, which made it possible to construct an analytical approximate method for solving nonlinear differential equations. This paper presents a solution to one of six problems of an analytical approximate method for solving one class of nonlinear differential equations of the first order, with a polynomial part of the fourth degree. In [23], the problem was solved — the study of the influence of the perturbation of moving singular points on the analytical approximate solution. As a result, the domain of representation of the analytical approximate solution near the approximate value of the moving singular point was obtained, which significantly decreased in comparison with the result of the previously proved existence and uniqueness theorem for the solution of the considered nonlinear differential equation. The investigations in this work made it possible to significantly expand the area of application of the approximate solution near the moving singular point, due to the constructiveness of the method for obtaining a priori estimates. The results obtained not only supplement the studies in [23], but also allow to obtain the exact boundaries of the application area of the approximate solution near the approximate value of the moving singular point. Theoretical results are illustrated by calculations characterizing their consistency with theoretical studies and adequacy with an exact solution.

2. Methodology

Non-linear differential equation

$$Y'(x) = \sum_{i=0}^{4} a_i(x)Y^i(x),$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$ are functions of a real variable in a certain domain, with the help of some transformations [23], is reduced to the normal form.

Consider the Cauchy problem

$$Y'(x) = Y^4(x) + r(x), \tag{1}$$

$$Y(x_0) = Y_0. \tag{2}$$

The perturbed value $\tilde{x}^*$ of the mobile singular point affects the structure of the analytical approximate solution, which takes the form

$$\tilde{Y}_N(x) = (x - \tilde{x}^*)^{-1/3} \sum_{n=0}^{N} \tilde{C}_n (x - \tilde{x}^*)^{n/3}, \quad \tilde{C}_0 \neq 0, \tag{3}$$

where $\tilde{C}_n$ are the perturbed values of the coefficients [23].

Theorem. *Let us suppose that:*

1) $r(x) \in C^\infty$ *in the area* $\{x : |x - \tilde{x}^*| < r_1\}$, *in which* $\tilde{x}^*$ *is the perturbed value of the movable singular point of solving the Cauchy problem* (1), (2), $\rho_1 = \mathrm{const} > 0$;
2) $\exists M_n : \left|\dfrac{r^{(n)}(\tilde{x}^*)}{n!}\right| \leqslant M_n$, $n = 0, 1, 2, 3, \ldots$, $M_n = \mathrm{const}$;
3) $\tilde{x}^* \leqslant x^*$;
4) *known is the error estimate of the value* $\tilde{x}^*$: $|x^* - \tilde{x}^*| \leqslant \Delta\tilde{x}^*$;
5) $\Delta\tilde{x}^* < 1/\sqrt[4]{3^6(1 + M + \Delta M)^3}$.

Then for the approximate solution (3) *of the problem* (1)–(2) *in the area*

$$F = F_1 \cap F_2 \cap F_3,$$

true is the estimate

$$\Delta\tilde{Y}_N(x) \leqslant \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3,$$

in which

$$\Delta_0 = \frac{\Delta\tilde{x}^*}{\sqrt[3]{3}|x - \tilde{x}_1^*|^{4/3}},$$

in case $N + 1 = 4n$,

$$\Delta_1 \leqslant \frac{3^{(N-3)/4}(M+1)^{(N+1)/4}|x - \tilde{x}^*|^{N/3}}{1 - 3(M+1)|x - \tilde{x}^*|^{4/3}} \times \\ \times \Big(\frac{1}{N+4} + \frac{|x - \tilde{x}^*|^{1/3}}{N+5} + \frac{|x - \tilde{x}^*|^{2/3}}{N+6} + \frac{|x - \tilde{x}^*|}{N+7}\Big);$$

for the variants $N + 1 = 4n + 1$, $N + 1 = 4n + 2$, $N + 1 = 4n + 3$, *respectively*:

$$\Delta_1 \leqslant \frac{3^{N/4-1}(M+1)^{N/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}} \times \\ \times \Big(\frac{1}{N+4} + \frac{|x-\tilde{x}^*|^{1/3}}{N+5} + \frac{|x-\tilde{x}^*|^{2/3}}{N+6} + \frac{9(M+1)|x-\tilde{x}^*|}{N+7}\Big),$$

$$\Delta_1 \leqslant \frac{3^{(N-5)/4}(M+1)^{(N-1)/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}} \times \\ \times \Big(\frac{1}{N+4} + \frac{|x-\tilde{x}^*|^{1/3}}{N+5} + \frac{9(M+1)|x-\tilde{x}^*|^{2/3}}{N+6} + \frac{9(M+1)|x-\tilde{x}^*|}{N+7}\Big),$$

$$\Delta_1 \leqslant \frac{3^{(N-6)/4}(M+1)^{(N-2)/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}} \times \\ \times \Big(\frac{1}{N+6} + \frac{9|x-\tilde{x}^*|^{1/3}}{N+7} + \frac{9(M+1)|x-\tilde{x}^*|^{2/3}}{N+8} + \frac{9(M+1)|x-\tilde{x}^*|}{N+9}\Big);$$

$$\Delta_2 = \frac{3^{-1}\Delta\tilde{x}^*(1+M+\Delta M)^{N/4}|x-\tilde{x}_2^*|^{1/3}}{1-3(1+M+\Delta M)|x-\tilde{x}_2^*|^{4/3}} \times \\ \times \big(1+|x-\tilde{x}_2^*|^{1/3} + |x-\tilde{x}_2^*|^{2/3} + |x-\tilde{x}_2^*|\big);$$

$$\Delta_3 = \frac{\Delta M(1+M+\Delta M)|x-\tilde{x}_2^*|}{1-9(1+M+\Delta M)|x-\tilde{x}_2^*|^{4/3}} \times \\ \times \Big(\frac{1}{21} + \frac{1}{24}|x-\tilde{x}_2^*|^{1/3} + \frac{1}{27}|x-\tilde{x}_2^*|^{2/3} + \frac{1}{30}|x-\tilde{x}_2^*|\Big)$$

under the condition

$$\Delta M = \Delta\tilde{Y}_0 = \sup_{n,G}\Big|\frac{r^{(n+1)}(x)}{n!}\Big|\Delta\tilde{x}^*,$$

$$M = \max\left|\,|y_0|, \sup_n\Big|\frac{r^{(n)}(\tilde{x}^*)}{n!}\Big|\,\right|, \quad \textit{where} \quad n = 0,1,2,3,\ldots,$$

$$F_1 = \{x : \tilde{x}^* - \rho_2 < x < \tilde{x}^*\}, \quad F_2 = \{x : \tilde{x}_1^* - \rho_3 < x < \tilde{x}_1^*\},$$

$$F_3 = \{x : \tilde{x}_2^* - \rho_4 < x < \tilde{x}_2^*\}, \quad \rho_2 = \min\Big\{\rho_1, 1/\sqrt[4]{27(M+1)^3}\Big\},$$

$$\rho_3 = \min\Big\{\rho_1, 1/\sqrt[4]{27(M+\Delta M+1)^3}\Big\},$$

$$\rho_4 = \min\Big\{\rho_1, 1/(3\sqrt[4]{9(M+\Delta M+1)^3})\Big\},$$

$$\tilde{x}_1^* = \tilde{x}^* - \Delta\tilde{x}^*, \quad \tilde{x}_2^* = \tilde{x}^* + \Delta\tilde{x}^*, \quad G = \{x : |x-\tilde{x}^*| \leqslant \Delta\tilde{x}^*\}.$$

Proof. When estimating the error of the analytical approximate solution (3)

$$\Delta\tilde{Y}_N(x) = |Y(x) - \tilde{Y}_N(x)| \leqslant |Y(x) - \tilde{Y}(x)| + |\tilde{Y}(x) - \tilde{Y}_N(x)|,$$

for the expression we shall use the elements of differential calculus [24]:

$$|Y(x)-\tilde{Y}(x)| \leqslant \sup_G \left|\frac{\partial \tilde{Y}(x)}{\partial \tilde{x}^*}\right| \Delta\tilde{x}^* + \sum_{n=0}^{\infty} \left|\frac{\partial \tilde{Y}(x)}{\partial \tilde{C}_n}\right| \Delta\tilde{C}_n,$$

in which $G = \{x : |x-\tilde{x}| \leqslant \Delta\tilde{x}\}$. We denote:

$$\Delta M = \Delta\tilde{Y}_0 = \sup_{n,G}\Big(\frac{r^{(n+1)}(x)}{n!}\Big)\Delta\tilde{x}, \quad M = \max\left| |y_0|, \sup_n \left|\frac{r^{(n)}(\tilde{x})}{n!}\right| \right|,$$

where $n = 0, 1, 2, \ldots$. Further

$$\sup_G \left|\frac{\partial \tilde{Y}(x)}{\partial \tilde{x}^*}\right| = \sup_G \left|\sum_{n=0}^{\infty} \tilde{C}_n \frac{n-1}{3}(x-\tilde{x}^*)^{(n-4)/3}\right| \leqslant$$

$$\leqslant \sum_{n=0}^{\infty}\left|\frac{n-1}{3}\right| \sup_G |\tilde{G}_n| \sup_G \left|(x-\tilde{x}^*)^{(n-4)/3}\right|.$$

In this case, we have:

$$\sup_G \left|(x-\tilde{x}^*)^{(n-4)/3}\right| = \begin{cases} |x-\tilde{x}_1^*|^{(n-4)/3}, & n = 0,1,2,3; \\ |x-\tilde{x}_2^*|^{(n-4)/3}, & n = 4,5,\ldots \end{cases}$$

and

$$\sup_G \left|\frac{\partial \tilde{Y}(x)}{\partial \tilde{C}_n}\right| = \sup_G \left|(x-\tilde{x}^*)^{(n-4)/3}\right| = \begin{cases} |x-\tilde{x}_1^*|^{(n-4)/3}, & n = 0; \\ |x-\tilde{x}_2^*|^{(n-4)/3}, & n = 1,2,3,\ldots, \end{cases}$$

where $\tilde{x}_1^* = \tilde{x}^* - \Delta\tilde{x}^*$, $\tilde{x}_2^* = \tilde{x}^* + \Delta\tilde{x}^*$. As

$$\sup_G |\tilde{C}_n| \leqslant \tilde{C}_n(|A_0+\Delta\tilde{A}_0|, |A_1+\Delta\tilde{A}_1|, |A_2+\Delta\tilde{A}_2|, \ldots) \leqslant \tilde{C}_n(1+M+\Delta M) = \tilde{\vartheta}_n,$$

in which A_n are the expansion coefficients of $r(x)$ function in the regular series, then

$$\left|Y(x)-\tilde{Y}(x)\right| \leqslant$$

$$\leqslant \Delta\tilde{x}^* \sum_{n=0}^{\infty}\left|\frac{n-1}{3}\right| \tilde{\vartheta}^* \sup_G \left|(x-\tilde{x}^*)^{(n-4)/3}\right| + \sum_{n=0}^{\infty} \Delta\tilde{C}^* \sup_G \left|(x-\tilde{x}^*)^{(n-1)/3}\right|.$$

Taking into account

$$C_0 = \tilde{C}_0 = -1/\sqrt[3]{3},$$

$$C_1 = C_2 = C_3 = C_5 = C_6 = \tilde{C}_1 = \tilde{C}_2 = \tilde{C}_3 = \tilde{C}_5 = \tilde{C}_6 = 0,$$

$$\Delta\tilde{C}_n = |C - \tilde{C}_n|,$$

this results in

$$\Delta\tilde{C}_1 = \Delta\tilde{C}_2 = \Delta\tilde{C}_3 = \Delta\tilde{C}_5 = \Delta\tilde{C}_6 = 0.$$

Hence

$$|Y(x)-\tilde{Y}_N(x)|\leqslant \frac{\Delta\tilde{x}^*}{\sqrt[3]{3}|x-\tilde{x}_1^*|^{4/3}}+ \\ +\Delta\tilde{x}^*\sum_{n=4}^{\infty}\Big|\frac{n-1}{3}\Big||x-\tilde{x}_2^*|^{(n-4)/3}+\sum_{n=4}^{\infty}\Delta\tilde{C}_n|x-\tilde{x}_2^*|^{(n-1)/3}$$

or

$$\Delta\tilde{Y}_N(x)=|Y(x)-\tilde{Y}_N(x)|\leqslant \frac{\Delta\tilde{x}^*}{\sqrt[3]{3}|x-\tilde{x}_1^*|^{4/3}}+ \\ +\sum_{n=N+1}^{\infty}|\tilde{C}_n||x-\tilde{x}^*|^{(n-1)/3}+\Delta\tilde{x}^*\sum_{n=4}^{\infty}\left|\frac{n-1}{3}\right||x-\tilde{x}_2^*|^{(n-4)/3}+ \\ +\sum_{n=4}^{\infty}\Delta\tilde{C}_n|x-\tilde{x}_2^*|^{(n-1)/3}=\Delta_0+\Delta_1+\Delta_2+\Delta_3,$$

in which $\Delta_0=\Delta\tilde{x}^*/(\sqrt[3]{3}|x-\tilde{x}_1^*|^{4/3})$. Next, we use the estimate of the coefficients $\tilde{C}_n$ [23]:

$$C_{4n}\leqslant\frac{3^{n-1}(M+1)^n}{4n+3}=\vartheta_{4n},\quad C_{4n+1}\leqslant\frac{3^{n-1}(M+1)^n}{4n+4}=\vartheta_{4n+1},$$

$$C_{4n+2}\leqslant\frac{3^{n-1}(M+1)^n}{4n+5}=\vartheta_{4n+2},\quad C_{4n+3}\leqslant\frac{3^{n-1}(M+1)^n}{4n+6}=\vartheta_{4n+3}.$$

When $N+1=4n$, according to the result of [23], we have

$$\Delta_1\leqslant\frac{3^{(N-3)/4}(M+1)^{(N+1)/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}}\times \\ \times\Big(\frac{1}{N+4}+\frac{|x-\tilde{x}^*|^{1/3}}{N+5}+\frac{|x-\tilde{x}^*|^{2/3}}{N+6}+\frac{|x-\tilde{x}^*|}{N+7}\Big).$$

For cases $N+1=4n+1$, $N+1=4n+2$, and $N+1=4n+3$, respectively:

$$\Delta_1\leqslant\frac{3^{N/4-1}(M+1)^{N/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}}\times \\ \times\Big(\frac{1}{N+4}+\frac{|x-\tilde{x}^*|^{1/3}}{N+5}+\frac{|x-\tilde{x}^*|^{2/3}}{N+6}+\frac{9(M+1)|x-\tilde{x}^*|}{N+7}\Big),$$

$$\Delta_1\leqslant\frac{3^{(N-5)/4}(M+1)^{(N-1)/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}}\times \\ \times\Big(\frac{1}{N+4}+\frac{|x-\tilde{x}^*|^{1/3}}{N+5}+\frac{9(M+1)|x-\tilde{x}^*|^{2/3}}{N+6}+\frac{9(M+1)|x-\tilde{x}^*|}{N+7}\Big),$$

$$\Delta_1 \leqslant \frac{3^{(N-6)/4}(M+1)^{(N-2)/4}|x-\tilde{x}^*|^{N/3}}{1-3(M+1)|x-\tilde{x}^*|^{4/3}} \times \\ \times \Big(\frac{1}{N+6} + \frac{9|x-\tilde{x}^*|^{1/3}}{N+7} + \frac{9(M+1)|x-\tilde{x}^*|^{2/3}}{N+8} + \frac{9(M+1)|x-\tilde{x}^*|}{N+9}\Big).$$

Moving on to the assessment Δ_2:

$$\Delta_2 = \Delta\tilde{x}^* \sum_{n=4}^{\infty} \left|\frac{n-1}{3}\right| \vartheta_n |x-\tilde{x}_2^*|^{(n-4)/3} = \Delta\tilde{x}^* \sum_{n=1}^{\infty} \frac{4n-1}{3} \vartheta_{4n} |x-\tilde{x}_2^*|^{(4n-4)/3} + \\ + \Delta\tilde{x}^* \sum_{n=1}^{\infty} \frac{4n}{3} \vartheta_{4n+1} |x-\tilde{x}_2^*|^{(4n-3)/3} + \Delta\tilde{x}^* \sum_{n=1}^{\infty} \frac{4n+1}{3} \vartheta_{4n+2} |x-\tilde{x}_2^*|^{(4n-2)/3} + \\ + \Delta\tilde{x}^* \sum_{n=1}^{\infty} \frac{4n+2}{3} \vartheta_{4n+3} |x-\tilde{x}_2^*|^{(4n-1)/3}.$$

Or given the expressions for ϑ_{4n}, ϑ_{4n+1}, ϑ_{4n+2}, ϑ_{4n+3} finally we get

$$\Delta_2 \leqslant \frac{\Delta\tilde{x}^*(M+\Delta M+1)}{3(1-3(M+\Delta M+1)|x-\tilde{x}_2^*|^{4/3})} \times \\ \times \big(1 + |x-\tilde{x}_2^*|^{1/3} + |x-\tilde{x}_2^*|^{2/3} + |x-\tilde{x}_2^*|\big).$$

Based on the estimates $\Delta\tilde{C}_n$ [23]

$$|\Delta\tilde{C}_{4n}| \leqslant \frac{3^{n-1}\Delta M(1+M+\Delta M)^n}{4n+3}, \quad |\Delta\tilde{C}_{4n+1}| \leqslant \frac{3^{n-1}\Delta M(1+M+\Delta M)^n}{4n+4},$$

$$|\Delta\tilde{C}_{4n+2}| \leqslant \frac{3^{n-1}\Delta M(1+M+\Delta M)^n}{4n+5}, \quad |\Delta\tilde{C}_{4n+3}| \leqslant \frac{3^{n-1}\Delta M(1+M+\Delta M)^n}{4n+6}$$

we get

$$\Delta_3 = \sum_{n=4}^{\infty} \Delta\tilde{C}_n |x-\tilde{x}_2^*|^{(4n-1)/3} = \\ = \sum_{n=1}^{\infty} \Delta\tilde{C}_{4n} |x-\tilde{x}_2^*|^{(4n-1)/3} + \sum_{n=1}^{\infty} \Delta\tilde{C}_{4n+1} |x-\tilde{x}_2^*|^{(4n)/3} + \\ + \sum_{n=1}^{\infty} \Delta\tilde{C}_{4n+2} |x-\tilde{x}_2^*|^{(4n+1)/3} + \sum_{n=1}^{\infty} \Delta\tilde{C}_{4n+3} |x-\tilde{x}_2^*|^{(4n+2)/3} \leqslant \\ \leqslant \frac{\Delta M(1+M+\Delta M)|x-\tilde{x}_2^*|}{1-9(1+M+\Delta M)|x-\tilde{x}_2^*|^{4/3}} \times \\ \times \Big(\frac{1}{21} + \frac{1}{24}|x-\tilde{x}_2^*|^{1/3} + \frac{1}{27}|x-\tilde{x}_2^*|^{2/3} + \frac{1}{30}|x-\tilde{x}_2^*|\Big).$$

The estimate for Δ_1 is valid in the region $F_1 = \{x : \tilde{x}^* - \rho_2 < x < \tilde{x}^*\}$, where $\rho_2 = \min\{\rho_1, 1/\sqrt[4]{27(M+1)^3}\}$. The estimate for Δ_2 is valid in the region

$F_2 = \{x : \tilde{x}_1^* - \rho_3 < x < \tilde{x}_1^*\}$, where $\rho_3 = \min\{\rho_1, 1/\sqrt[4]{27(M + \Delta M + 1)^3}\}$. And the estimate for Δ_3 is valid in the region $F_3 = \{x : \tilde{x}_2^* - \rho_4 < x < \tilde{x}_2^*\}$, where $\rho_4 = \min\{\rho_1, 1/(3\sqrt[4]{27(M + \Delta M + 1)^3})\}$. Therefore, the estimate for $\Delta\tilde{Y}_N(x)$ is valid in the region $F = F_1 \cap F_2 \cap F_3$, which proves the theorem. □

3. Results

Example 1. We shall consider the Cauchy problem for the equation $Y'(x) = Y^4(x) + r(x)$, in which $Y(1) = 1$, $r(x) = 0$ and find an approximate solution of the problem (1), (2), near the movable singular point. The Cauchy problem has the exact solution $Y = 1/\sqrt[3]{4 - 3x}$. The radius of the vicinity of the movable singular point, given the initial conditions of the Cauchy problem, is $\rho_4 = 0.114432$. The perturbed value of the movable singular point is $\tilde{x}^* = 1.3334$. The disturbance value is $\Delta\tilde{x}^* = 0.0001$. $M = 1$, we shall choose the value of $x = 1.2375$ from the vicinity of the movable singular point, its radius is ρ_4. The value of the argument is considered, for which, when obtaining an estimate of the error of the approximate solution, one can use both the results of this study and the work [23]. The calculations are presented in Table 1.

Table 1

The comparative variant of the approximate solutions' characteristics

x	$Y(x)$	$\tilde{Y}_7(x)$	Δ	Δ'_{11}	Δ''_{21}	Δ''_1
1.2375	1.515144	1.514793	0.000351	0.00456	0.00368	0.0008

Here, $Y(x)$ is the exact solution; $\tilde{Y}_7(x)$ is the analytical approximate solution; Δ is the absolute error; Δ'_{11} is the a priori error obtained by this theorem; Δ''_{12} is the a priori error obtained by the theorem from [23], Δ''_1 is the a posteriori error. The theorem in [23] allows solving the inverse problem of the theory of error, determining the N value for the given accuracy of the approximate solution ε. The case $\varepsilon = 8 \cdot 10^{-4}$ results in the value $N = 17$. For $N = 8 \div 17$ we obtain the clarification of the approximate solution, which in total does not exceed the required accuracy $\varepsilon = 8 \cdot 10^{-4}$.

Thus, we can restrict ourselves to the value of $N = 7$ in the structure of the approximate solution. Thus, we obtain the value of the error for the approximate solution $\tilde{Y}_7(x)$ equal to $\varepsilon = 8 \cdot 10^{-4}$. Note that the a priori estimates obtained by the theorem of this paper and the theorem from [23] have values of the same order.

Example 2. Let us find an approximate solution to the Cauchy problem (1), (2) with the conditions of Example 1 in the case $\tilde{x}^* = 1.33334$, $\Delta\tilde{x}^* = 0.00001$. Calculated value $\rho_4 = 0.114432$. The magnitude of the disturbance does not exceed the value $\varepsilon = 0.000050$. A point is considered, the value of which falls only under the results of this work. The calculations are presented in Table 2.

Table 2

Calculation of the characteristics for the approximate solution of the nonlinear differential equation on the theorem

x	$Y(x)$	$\tilde{Y}_7(x)$	Δ	Δ'_1	Δ''_1
1.21901	1.428613	1.428585	0.000028	0.003564	0.000067

Here, Y is the exact value of the equation's solution; $\tilde{Y}_7$ is the approximate solution; Δ is the absolute error; Δ_1' is the a priori error obtained by the theorem, Δ_1'' is the a posteriori error.

Solving the inverse problem of the theory of error, we determine the value of N for a given accuracy of the approximate solution ε. For the case $\varepsilon = 0.67 \cdot 10^{-4}$, the resulting value is $N = 20$. For $N = 8 \div 20$, we obtain a more accurate approximate solution, which in total does not exceed the required accuracy $\varepsilon = 8.99 \cdot 10^{-4}$.

Limiting in the structure of the approximate solution to the value $N = 7$, we get the error value for the approximate solution $\tilde{Y}_7(x)$ equal to the value $\varepsilon = 0.67 \cdot 10^{-4}$.

4. Discussion

The theorem proved in the study allows to significantly expand the area of application of the analytical approximate solution, which was obtained due to the constructiveness of the proof technology. The presented theoretical provisions of this study supplement the results of [23]. Calculations in Table 1 confirm that in the general area of validity of the proved theorem and work [23] we have values with an error of the same order of magnitude. The research used series with negative fractional powers.

5. Conclusion

The results of this work are the completion of research on the analytical approximate solution of a nonlinear differential equation near a movable singular point of algebraic type in a real domain. The obtained theoretical results are tested on model problems.

Competing interests. We have no competing interests.

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О точных границах области для аналитического приближенного решения одного класса нелинейных дифференциальных уравнений первого порядка в окрестности приближенного значения подвижной особой точки для вещественной области

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Аннотация

Дано решение одной из задач аналитического приближенного метода для одного класса нелинейных дифференциальных уравнений первого порядка с подвижными особыми точками в вещественной области. Рассматриваемое уравнение в общем случае не разрешимо в квадратурах и имеет подвижные особые точки алгебраического типа. Это обстоятельство требует решение ряда математических задач.

Ранее авторами была решена задача влияния возмущения подвижной особой точки на аналитическое приближенное решение. Это решение основывалось на классическом подходе и, при этом, существенно уменьшилась область применения аналитического приближенного решения, по сравнению с областью, полученной в доказанной теореме существования и единственности решения.

Поэтому в статье предлагается новая технология исследования, основанная на элементах дифференциального исчисления. Этот подход позволяет получить точные границы для аналитического приближенного решения в окрестности подвижной особой точки.

Получены новые априорные оценки для аналитического приближенного решения рассматриваемого класса уравнений, хорошо согласующиеся с известными для общей области действия. При этом, представленные результаты дополняют ранее полученные, существенно расширена область применения аналитического приближенного решения в окрестности подвижной особой точки.

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Приведенные расчеты согласуются с теоретическими положениями, о чем свидетельствуют эксперименты, проведенные с нелинейным дифференциальным уравнением, обладающим точным решением. Дана технология оптимизации априорных оценок погрешности с помощью апостериорных оценок. В исследованиях применялись ряды с дробными отрицательными степенями.

Ключевые слова: подвижная особая точка, нелинейное дифференциальное уравнение, задача Коши, точные границы области, априорная и апостериорная погрешности, аналитическое приближенное решение.

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