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Inverse source problem for an equation of mixed parabolic-hyperbolic type with the time fractional derivative in a cylindrical domain

D. K. Durdiev

[1] Bukhara Branch of the Institute of Mathematics
named after V. I. Romanovskiy at the Academy of Sciences
of the Republic of Uzbekistan,
11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.
[2] Bukhara State University,
11, Muhammad Igbol st., Bukhara, 705018, Uzbekistan.

Abstract

This article is devoted to the study of an inverse source problem for a mixed type equation with a fractional diffusion equation in the parabolic part and a wave equation in the hyperbolic part of a cylindrical domain. The solution is obtained in the form of Fourier–Bessel series expansion using an orthogonal set of Bessel functions. The theorems of uniqueness and existence of a solution are proved.

Keywords: inverse problem, equation of mixed type, Fourier–Bessel series, Mittag–Leffler function, uniqueness and existence.

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1. Formulation of Problem

The importance of considering equations of mixed type, when an equation of parabolic type is given on one part of the domain and an equation of hyperbolic type on the other, was first pointed out by I. M. Gelfand in 1959 [1]. The study of electrical oscillations in wires leads to a problem for a mixed parabolic-hyperbolic type of equations. In a homogeneous medium, in the case of its low conductivity, the strength of the electromagnetic field satisfies the wave equation, but in the

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Author's Details:
Durdimurod K. Durdiev https://orcid.org/0000-0002-6054-2827
Dr. Phys. & Math. Sci.; Head of Branch[1]; Professor; Dept. of Differential Equations[2];
e-mail: durdiev65@mail.ru

case of relatively high conductivity, when displacement currents can be neglected in comparison with conduction currents, the mentioned value satisfies the heat equation (see [2, pp. 443–447]). Problems of this kind are also encountered in studying the motion of a fluid in a channel surrounded by a porous medium; so, in a channel, the hydrodynamic pressure of a liquid satisfies the wave equation, and in a porous medium it satisfies the filtration equation, which in this case coincides with the diffusion equation [3]. In this case, some matching conditions are satisfied at the channel boundary. Such equations arise in a number of other areas of natural science.

Direct problems for mixed parabolic-hyperbolic equation types were studied in [4–8]. Inverse problems about determining the right side or the initial function in the initial-boundary value problems for the equation of mixed parabolic-hyperbolic type in a rectangular domain were considered in the monograph [9] (see also references there). On the basis of the spectral method, criteria for uniqueness and existence are established.

In this paper, we study direct and inverse problems related to finding a solution to an initial-boundary value problem for a mixed equation, when on one part of the domain the fractional diffusion equation and on another part the wave equation are given, and the unknown right-hand side of this equation in a cylindrical domain.

Consider in a cylinder $G := \{(x, y, t) : 0 < r < 1, -a < t < b\}$, $r = \sqrt{x^2 + y^2}$ the equation of mixed type

$$Lu = \begin{cases} \partial_t^\alpha u - \Delta u = f(r), & t > 0, \\ u_{tt} - \Delta u = f(r), & t < 0, \end{cases} \tag{1}$$

where a, b are given positive numbers, $\partial_t^\alpha u$ is the Gerasimov–Caputo fractional derivative of order α $(0 < \alpha \leqslant 1)$ in the time variable and it is defined by formula (see [10, p. 90]):

$$\partial_t^\alpha g(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau, & 0 < \alpha < 1, \\ g'(t), & \alpha = 1; \end{cases}$$

Δ is the Laplace operator in variables x and y.

We pose the following problem: find in the domain G the functions $u(x, y, t)$ and $f(r)$ satisfying the equation (1) and conditions

$$\big((x, y), \nabla u\big)\big|_{r=0} = u\big|_{r=1} = 0, \quad -a \leqslant t \leqslant b, \tag{2}$$

$$u\big|_{t=-a} = \varphi(r), \quad 0 \leqslant r \leqslant 1, \tag{3}$$

$$u\big|_{t=b} = \psi(r), \quad 0 \leqslant r \leqslant 1. \tag{4}$$

Here $\big((x, y), \nabla u\big)$ is scalar product of vectors (x, y) and ∇u; $\varphi(\,\cdot\,)$ and $\psi(\,\cdot\,)$ are given sufficiently smooth functions.

Denote $G_+ = G \cap \{t > 0\}$, $G_- = G \cap \{t < 0\}$.

Definition 1. The solution of problem (1)–(4) are the functions $u(x, y, t)$ and $f(r)$ from the classes $C^{2,\alpha}_{x,y;t}(G_+ \cup \{t = b\}) \cap C^2(G_- \cup \{t = -a\})$ and $C[0, 1]$, respectively, satisfying relations (1)–(4) and the following conjugation conditions:

$$u(x,y,+0) = u(x,y,-0), \quad \lim_{t\to+0} \partial_t^\alpha u(x,y,t) = u_t(x,y,-0), \quad r \in (0,1). \tag{5}$$

Here $C^{2,\alpha}_{x,y;t}(\Omega) := \{v(x,y,t) : v \in C(\Omega), (\Delta v, \partial_t^\alpha v) \in C(\Omega)\}$.

If $\alpha = 1$, then conditions (5) mean the continuity of the solution $u(x,y,t)$ and its derivative with respect to t on the line of change of equation type $t = 0$.

In the parabolic part of the domain, the function $u(x,y,t)$ satisfies the fractional diffusion equation (1). Fractional differential equations become an important tool in mathematical modeling many problems arising in applications. The time fractional diffusion equations can be used to describe superdiffusion and subdiffusion phenomena [11–13] (see also references there). Direct problems, i.e. well-posed initial value problems (Cauchy problem), initial boundary value problems for one time-fractional diffusion equations and various inverse problems, have attracted much more attention in recent years. For instance, on some uniqueness and existence results we refer readers to works [14–17] on direct and inverse source problems (see also references in [17]), and on direct and inverse coefficient problems to [18–23].

The paper organized as follows. Section 2 provides some definitions and known results that will be used later in this article. In Section 3, by using the Fourier method a formal solution of the inverse problem is obtained. In Section 4, the existence and uniqueness of a solution to the inverse problem are proved. Finally, a conclusion and a list of references are given.

2. Preliminaries

In this section, we provide some definitions and results that will be used later in this article.

The classical Mittag–Leffler $E_\alpha(z)$ function with one parameter is defined by the following series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where α, $z \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. This function and its generalizations play an important role in describing solutions to fractional-order differential equations. The Mittag–Leffler function has been studied by many authors who have proposed and studied various generalizations and applications. A very interesting work that has received many results on this function is due to Haubold et al. [24].

If $z = \lambda t^\alpha$, with $\lambda > 0$ and $t > 0$, then

$$\partial_t^\alpha E_\alpha(-\lambda t^\alpha) = -\lambda E_\alpha(-\lambda t^\alpha).$$

Moreover, The Mittag–Leffler function $E_\alpha(-\lambda t^\alpha)$ is bounded [24]:

$$0 < E_\alpha(-\lambda t^\alpha) < 1. \tag{6}$$

Here and throughout this article, M denotes a positive constant.

In studying the problem under consideration, we also need the Bessel function and the conditions for the convergence of the Fourier–Bessel series. The linear differential Bessel equation (or the equation of cylindrical functions) with a parameter λ of order or index $\nu \geqslant 0$ with respect to the function z of the real variable x has the form [25, ch. 8]:

$$z'' + \frac{1}{x}z' + \Big(\lambda^2 - \frac{\nu^2}{x^2}\Big)z = 0. \tag{7}$$

The solution of Equation (7), except for very particular values ν, is not expressed in terms of elementary functions (in the final form) and leads to the so-called Bessel functions, which have large applications in the natural sciences [26]. When ν is an integer number, then Equation (7) has the following solution:

$$z(x) = C_1 J_\nu(\lambda x) + C_2 Y_\nu(\lambda x),$$

where J_ν and Y_ν are the Bessel functions of the first and second kind of order ν, respectively. Bessel functions of the second kind are not bounded near the point $x = 0$, so for a bounded solution near zero it is necessary that $C_2 = 0$, i.e. solution (7) has the following form:

$$z(x) = CJ_\nu(\lambda x).$$

Furthermore, if the boundary condition $z(1) = 0$ is imposed, then the parameter λ must satisfy $J_\nu(\lambda) = 0$, i.e. the values of λ are the zeros of the Bessel function $J_\nu(x)$, which has the following asymptotic representation [25, p. 213]:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\Big(x - \frac{\nu\pi}{2} + \frac{\pi}{4}\Big) + \frac{r_\nu(x)}{x\sqrt{x}},$$

where the function $r_\nu(x)$ is bounded for $x \to \infty$. Therefore, for any large k, the zeros of $J_\nu(x)$ are given by the expression [25, p. 214]:

$$k\pi + \frac{\nu\pi}{2} - \frac{\pi}{4}.$$

We define the Fourier–Bessel expansion of the given function $f(x)$ as follows: for any function $f(x)$, absolutely integrable on $[0, 1]$, one can compose a Fourier series in the system $J_\nu(\lambda_k x)$, $k = 1, 2, \ldots$, or, in briefly, the Fourier–Bessel series

$$f(x) = \sum_{k=1}^{\infty} c_k J_\nu(\lambda_k x), \tag{8}$$

where the constants c_k are determined by the formula:

$$c_k = \frac{2}{J_{\nu+1}^2(\lambda_k)}\int_0^1 x f(x) J_\nu(\lambda_k x)dx, \quad k = 1, 2, \ldots$$

and are called the Fourier–Bessel coefficients.

Let us give without proof the most important criteria for the convergence of the Fourier–Bessel series to the function for which it is composed. These criteria are similar to those known to us for the convergence of trigonometric Fourier series.

THEOREM 1 [25, P. 282]. *If $\nu \geqslant 0$ and for all sufficiently large k, we have the estimate*

$$|c_k| \leqslant \frac{M}{\lambda_k^{1+\varepsilon}},$$

where $\varepsilon > 0$ and $M > 0$ are constants, then series (8) converges absolutely and uniformly on $[0, 1]$.

THEOREM 2 [25, PP. 289–291]. *Let the function $f(x)$ is defined and $2s$ times continuously differentiable on the interval $[0, 1]$ $(s \geqslant 1)$, and*

1) $f(0) = f'(0) = \cdots = f^{(2s-1)}(0) = 0$,
2) $f^{(2s)}(x)$ *is bounded (this derivative may not exist at some points),*
3) $f(1) = f'(1) = \cdots = f^{(2s-2)}(1) = 0$.

Then, for the Fourier–Bessel coefficients of the function $f(x)$, the inequality is valid:

$$|c_k| \leqslant \frac{M}{\lambda_k^{2s-(1/2)}} \quad (M = \text{const}).$$

We now turn to the study of the problem (1)–(4).

3. Formal Construction of the Solution

Note that since the right-hand side of equation (4) and the functions of (6) and (7) depend on the distance r, then $u(x, y, t) = u(r, t)$, i.e. we have an axisymmetric case. Then the operator Laplace on the function $u(x, y, t)$ in polar coordinate systems will not depend on the angle and has the form:

$$\Delta u(x, y, t) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Therefore, equation (4) in these coordinate systems is written as follows:

$$\begin{aligned} \partial_t^\alpha u &= \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + f(r), \quad t > 0, \\ \frac{\partial^2 u}{\partial t^2} &= \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + f(r), \quad t < 0. \end{aligned} \tag{9}$$

Conditions (2)–(4) take the following form:

$$\left[r\frac{\partial}{\partial r}u(r, t)\right]_{r=0} = 0, \quad u\big|_{r=1} = 0, \quad -a \leqslant t \leqslant b, \tag{10}$$

$$u(r, -a) = \varphi(r), \quad 0 \leqslant r \leqslant 1, \tag{11}$$

$$u(r, b) = \psi(r), \quad 0 \leqslant r \leqslant 1. \tag{12}$$

Thus, the inverse problem (2)–(4) is reduced to the problem definitions of the functions $u(r, t)$, $f(r)$ from equalities (9)–(12).

According to the Fourier method, searching partial solutions of equation (9) for the case $f = 0$ in form

$$u(r, t) = R(r)T(t),$$

we get the following relations:

$$\partial_t^\alpha T(t)R(r) = \frac{1}{r}T(t)R'(r) + T(t)R''(r), \quad t > 0,$$

$$T''(t)R(r) = \frac{1}{r}T(t)R'(r) + T(t)R''(r), \quad t < 0.$$

Therefore, separating the variables, we have

$$\frac{\partial_t^\alpha T(t)}{T(t)} = \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{R''(r)}{R(r)} = -\lambda^2, \quad t > 0,$$

$$\frac{T''(t)}{T(t)} = \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{R''(r)}{R(r)} = -\lambda^2, \quad t < 0,$$

where λ is an arbitrary real parameter. Hence, to find the function $R(r)$ we get the problem of the equation

$$R''(r) + \frac{1}{r}R'(r) + \lambda^2 R(r) = 0$$

with boundary conditions

$$\lim_{r\to 0}(rR'(r)) = 0, \quad R(1) = 0, \tag{13}$$

which is a self-adjoint problem.

The solutions of equation (10) are the following zero-order Bessel functions of the first kind:

$$R_k(r) = J_0(\lambda_k r), \quad k = 1, 2, 3, \ldots$$

They also are eigenfunctions. We find the eigenvalues using the second boundary condition of (13) (the validity of the first boundary condition in (13) is obvious), positive roots of the equation $J_0(\lambda_k) = 0$. As noted in the previous section, they look like:

$$\lambda_k = k\pi - \frac{\pi}{4} = (4k-1)\frac{\pi}{4}.$$

Expand now all functions in a Fourier–Bessel series in terms of eigenfunctions $J_0(\lambda_k r)$ i.e.

$$u(r,t) = \sum_{k=1}^{\infty} u_k(t) J_0(\lambda_k r), \tag{14}$$

$$f(r) = \sum_{k=1}^{\infty} f_k J_0(\lambda_k r), \tag{15}$$

where

$$u_k(t) = \frac{2}{J_1^2(\lambda_k)} \int_0^1 r u(r,t) J_0(\lambda_k r) dr, \quad f_k = \frac{2}{J_1^2(\lambda_k)} \int_0^1 r f(r) J_0(\lambda_k r) dr.$$

Substituting (14), (15) into (9), we obtain

$$\partial_t^\alpha u_k(t) = -\lambda_k^2 u_k(t) + f_k, \quad t > 0,$$
$$u_k''(t) = -\lambda_k^2 u_k(t) + f_k, \qquad t < 0.$$

It is not difficult to find that these differential equations have general solutions:

$$
\begin{aligned}
&u_k(t) = c_k E_\alpha(-\lambda_k^2 t^\alpha) + \frac{f_k}{\lambda_k^2}, && t > 0,\\
&u_k(t) = d_k \cos(\lambda_k t) + e_k \sin(\lambda_k t) + \frac{f_k}{\lambda_k^2}, && t < 0,
\end{aligned}
\tag{16}
$$

where $E_\alpha(\,\cdot\,)$ is the Mittag–Leffler function; d_k, e_k, c_k are arbitrary constants.

To find the coefficients d_k, e_k, c_k, we use conditions

$$u_k(0+0) = u_k(0-0), \quad \partial_t^\alpha u_k(0+0) = u_k'(0-0),$$

which follow from conditions (5). In view of this, from (16) we have

$$d_k = c_k, \quad e_k = -\lambda_k c_k.$$

From the initial and additional conditions (11), (12), we get:

$$d_k \cos(\lambda_k a) - e_k \sin(\lambda_k a) + \frac{f_k}{\lambda_k^2} = \varphi_k,$$

$$c_k E_\alpha(-\lambda_k^2 b^\alpha) + \frac{f_k}{\lambda_k^2} = \psi_k,$$

where φ_k, ψ_k are Fourier–Bessel coefficients of functions φ, ψ, respectively:

$$\varphi(r) = \sum_{k=1}^{\infty} \varphi_k J_0(\lambda_k r), \quad \psi(r) = \sum_{k=1}^{\infty} \psi_k J_0(\lambda_k r).$$

Substituting the values d_k, e_k found through c_k, into the previous equations and solving the resulting system with respect to c_k and f_k, we find

$$
\begin{aligned}
&c_k = \frac{\psi_k - \varphi_k}{E_\alpha(-\lambda_k^2 b^\alpha) - \big(\cos(\lambda_k a) + \lambda_k \sin(\lambda_k a)\big)},\\
&f_k = \lambda_k^2 \big(\psi_k - c_k E_\alpha(-\lambda_k^2 b^\alpha)\big).
\end{aligned}
\tag{17}
$$

Introduce the notation

$$A_{ab}(k) = E_\alpha(-\lambda_k^2 b^\alpha) - \big(\cos(\lambda_k a) + \lambda_k \sin(\lambda_k a)\big). \tag{18}$$

4. Existence and Uniqueness of the Solution

We find the values of a and b for which (18) takes values not equal to zero. To do this, we rewrite (18) in the following form:

$$A_{ab}(k) = E_\alpha\Big(-\frac{(4k-1)^2\pi^2}{16} b^\alpha\Big) - \sqrt{1 + \frac{(4k-1)^2\pi^2}{16}} \sin\Big(\frac{4k-1}{4} a\pi + \gamma_k\Big), \tag{18$'$}$$

where $\gamma_k = \arcsin\Big(1/\sqrt{1+\frac{(4k-1)^2\pi^2}{16}}\Big)$. Obtain the values of a for which $A_{ab}(k)=0$. It equals

$$a = \frac{4}{(4k-1)\pi}\left(\arcsin\left(\frac{E_\alpha\big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\big)}{\sqrt{1+\frac{(4k-1)^2\pi^2}{16}}}\right)+\pi n-\gamma_k\right).$$

We now find the values of a and b, for which the following condition is met:

$$|A_{ab}(k)| \geqslant C_0 > 0. \tag{19}$$

For this, we calculate

$$|A_{ab}(k)| = \left|E_\alpha\Big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\Big) - \sqrt{1+\frac{(4k-1)^2\pi^2}{16}}\sin\Big(\frac{4k-1}{4}a\pi+\gamma_k\Big)\right|.$$

If $a=4n$, $n\in\mathbb{N}$, then

$$|A_{ab}(k)| = \left|E_\alpha\Big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\Big) - \sqrt{1+\frac{(4k-1)^2\pi^2}{16}}\sin\Big(\frac{4k-1}{4}a\pi+\gamma_k\Big)\right| \geqslant$$
$$\geqslant \left|E_\alpha\Big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\Big)\pm 1\right| \geqslant \left|1-E_\alpha\Big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\Big)\right|.$$

According to (6) we have $0 < E_\alpha\big(-\frac{(4k-1)^2\pi^2}{16}b^\alpha\big) < 1$, for all $k=1,2,\dots$, then $|A_{ab}(k)| \geqslant C_0 > 0$. As C_0, it can be taken $1-E_\alpha\big(-\frac{9\pi^2}{16}b^\alpha\big)$ as the largest of all possible such constants.

Thus, we have obtained the following uniqueness criterion:

Theorem A. *If there exists a solution to problem (1)–(4), then it is unique for the values $a=4n$, $n\in\mathbb{N}$ for any $b>0$.*

We now investigate the existence of a solution. To this end, we prove the following assertion:

Theorem B. *Assume that $\{\varphi(r),\psi(r)\}\in C^6[0,1]$ and, in addition, condition (19) and the equalities*

$$\varphi^{(i)}(0)=0,\quad \psi^{(i)}(0)=0,\quad i=0,1,\dots,5,$$

$$\varphi^{(i)}(1)=0,\quad \psi^{(i)}(1)=0,\quad j=0,1,\dots,4$$

are satisfied.

Then there is a unique solution to problem (1)–(4), which is defined (20)–(22), where $\varphi^{(i)}$, $\psi^{(i)}$ are $i-$th derivatives of the functions φ, ψ, and φ_k, ψ_k are the Fourier–Bessel coefficients of the functions φ and ψ, respectively.

To prove the theorem, substituting the found values of the coefficients d_k, e_k, c_k in (16), (17), we find $u_k(t)$ and f_k:

$$u_k(t) = \frac{\psi_k-\varphi_k}{A_{ab}(k)}E_\alpha(-\lambda_k^2t^\alpha)+\psi_k-\frac{\psi_k-\varphi_k}{A_{ab}}E_\alpha(-\lambda_k^2b^\alpha),\quad t>0,$$

$$u_k(t) = \frac{\psi_k - \varphi_k}{A_{ab}(k)} \cos(\lambda_k t) - \lambda_k \frac{\psi_k - \varphi_k}{A_{ab}(k)} \sin(\lambda_k t) + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha), \quad t < 0,$$

$$f_k = \lambda_k^2 \Big(\psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha)\Big).$$

Taking into account these relations, from (14) and (15) we obtain the formal solution of problem in the form of series:

$$u(r,t) = \sum_{k=1}^{\infty} u_k(t) J_0(\lambda_k r) = \sum_{k=1}^{\infty} \Big[\frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 t^\alpha) + \\ + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha)\Big] J_0(\lambda_k r), \quad t > 0, \quad (20)$$

$$u(r,t) = \sum_{k=1}^{\infty} u_k(t) J_0(\lambda_k r) = \sum_{k=1}^{\infty} \Big[\frac{\psi_k - \varphi_k}{A_{ab}(k)} \cos(\lambda_k t) - \lambda_k \frac{\psi_k - \varphi_k}{A_{ab}(k)} \sin(\lambda_k t) + \\ + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha)\Big] J_0(\lambda_k r), \quad t < 0, \quad (21)$$

$$f(r) = \sum_{k=1}^{\infty} f_k J_0(\lambda_k r) = \sum_{k=1}^{\infty} \lambda_k^2 \Big[\psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha)\Big] J_0(\lambda_k r). \quad (22)$$

To prove the existence of a solution, we need to show that the series in (20)–(22) and the series obtained as a result of the action of fractional differentiation ∂_t^α, by differentiating with respect to r twice, in domain G_+ and by differentiating twice in r, t, in domain G_-, converge uniformly. To this end, we calculate $\partial_t^\alpha u(r,t)$, $\big(\partial^2/\partial t^2\big)u(r,t)$, $\big(\partial^2/\partial r^2\big)u(r,t)$, by formally performing differentiation under the signs of sums. Using properties of the Bessel functions, namely (see [24]) $J_0'(r) = -J_1(r)$, $2J_1'(r) = J_0(r) - J_2(r)$, from formulas (20), (21) we obtain the following:

$$\partial_t^\alpha u(r,t) = \sum_{k=1}^{\infty} \partial_t^\alpha u_k(t) J_0(\lambda_k r) = \\ = \sum_{k=1}^{\infty} \big[-\lambda_k^2 E_\alpha(-\lambda_k^2 t^\alpha)\big] \frac{(\psi_k - \varphi_k)}{A_{ab}(k)} J_0(\lambda_k r), \quad t > 0, \quad (23)$$

$$\frac{\partial^2 u(r,t)}{\partial t^2} = \sum_{k=1}^{\infty} \frac{\partial^2 u_k(t)}{\partial t^2} J_0(\lambda_k r) = \\ = \sum_{k=1}^{\infty} \big[-\lambda_k^2 \cos(\lambda_k t) + \lambda_k^3 \sin(\lambda_k t)\big] \frac{(\psi_k - \varphi_k)}{A_{ab}(k)} J_0(\lambda_k r), \quad t < 0, \quad (24)$$

$$\frac{\partial^2 u(r,t)}{\partial r^2} = \sum_{k=1}^{\infty} u_k(t) \frac{d^2 J_0(\lambda_k r)}{dr^2} = \sum_{k=1}^{\infty} \Big[\frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 t^\alpha) + \\ + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha) \Big] \frac{\lambda_k^2}{2} \big(J_2(\lambda_k r) - J_0(\lambda_k r) \big), \quad t > 0, \quad (25)$$

$$\frac{\partial^2 u(r,t)}{\partial r^2} = \sum_{k=1}^{\infty} u_k(t) \frac{d^2 J_0(\lambda_k r)}{dr^2} = \sum_{k=1}^{\infty} \Big[\frac{\psi_k - \varphi_k}{A_{ab}(k)} \cos(\lambda_k t) - \lambda_k \frac{\psi_k - \varphi_k}{A_{ab}(k)} \sin(\lambda_k t) + \\ + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha) \Big] \frac{\lambda_k^2}{2} \big(J_2(\lambda_k r) - J_0(\lambda_k r) \big), \quad t < 0. \quad (26)$$

Let the functions $\varphi(r)$ and $\psi(r)$ satisfy the conditions of Theorem 2 with some $s \geqslant 1$ (we define the number s later). Then for the Fourier–Bessel coefficients of these functions are true the following estimates:

$$|\varphi_k| \leqslant \frac{M_1}{\lambda_k^{2s-(1/2)}}, \quad |\psi_k| \leqslant \frac{M_1}{\lambda_k^{2s-(1/2)}}.$$

Now we will evaluate the expressions at Bessel functions on the right-hand sides of equalities (20)–(26). In this case, the expressions in (20), (21) are estimated as follows:

$$\Big| \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 t^\alpha) + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha) \Big| \leqslant \\ \leqslant M \Big(\frac{M_1}{\lambda_k^{2s-(1/2)}} + \frac{M_2}{\lambda_k^{2s-(1/2)}} \Big) \leqslant \frac{N_1}{\lambda_k^{2s-(1/2)}}, \quad t < 0,$$

$$\Big| \frac{\psi_k - \varphi_k}{A_{ab}(k)} \cos(\lambda_k t) - \lambda_k \frac{\psi_k - \varphi_k}{A_{ab}(k)} \sin(\lambda_k t) + \psi_k - \frac{\psi_k - \varphi_k}{A_{ab}(k)} E_\alpha(-\lambda_k^2 b^\alpha) \Big| \leqslant \\ \leqslant N_2 \frac{\lambda_k + 1}{\lambda_k^{2s-(1/2)}}, \quad t > 0,$$

where M, M_1, M_2, N_1, N_2 are positive constants.

Similarly, it is established that the expressions in (22), (23), (25) are less than $N_3 \frac{\lambda_k^2}{\lambda_k^{2s-(1/2)}}$, and the expressions in (24), (26) are less than $N_4 \frac{\lambda_k^2 + \lambda_k^3}{\lambda_k^{2s-(1/2)}}$, N_3, N_4 are positive constants.

It follows from these estimates that if $s = 3$, then, according to Theorem 1, the series in (23)–(26) and the series obtained as a result of the action of fractional differentiation ∂_t^α, by differentiating with respect to r twice, in domain G_+ and by differentiating twice in r, t, in domain G_-, converge uniformly. Thus, Theorem B is proved.

5. Conclusion

This paper concerns the existence and uniqueness of a solution to the inverse source problem for a mixed-type equation with a fractional diffusion equation in the parabolic part and a wave equation in the hyperbolic part of a cylindrical domain. The solution is obtained in the form of Fourier-Bessel series expansion using an orthogonal set of Bessel functions.

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Обратная задача об источнике для уравнения смешанного параболо-гиперболического типа с дробной производной по времени в цилиндрической области

Д. К. Дурдиев

[1] Бухарское отделение Института математики им. В. И. Романовского АН Республики Узбекистан, Узбекистан, 705018, Бухара, ул. Мухаммад Икбол, 11.

[2] Бухарский государственный университет, Узбекистан, 705018, Бухара, ул. Мухаммад Икбол, 11.

Аннотация

Исследуется обратная задача об источнике для уравнения смешанного типа с дробным уравнением диффузии в параболической части и волновым уравнением в гиперболической части цилиндрической области. Решение задачи получено в виде ряда Фурье–Бесселя с использованием ортогонального множества функций Бесселя. Доказаны теоремы единственности и существования решения.

Ключевые слова: обратная задача, уравнение смешанного типа, ряд Фурье–Бесселя, функция Миттаг–Леффлера, единственность и существование.

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Сведения об авторе

Дурдимурод Каландарович Дурдиев https://orcid.org/0000-0002-6054-2827
доктор физико-математических наук, профессор; заведующий отделением[1]; проф. кафедры дифференциальных уравнений[2]; e-mail: durdiev65@mail.ru