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Dynamic equilibria of a nonisothermal fluid

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Abstract

In this paper, stationary dynamic equilibria of the rotating mass of a nonisothermal fluid are discussed within the accuracy limits of the Boussinesq approximation. It is demonstrated that, in this case, a fluid exhibits a finite number of counterflows, higher values of velocities than those specified on the boundary and the formation of zones of positive and negative pressures and temperatures.

Keywords: dynamic equilibrium, rotating fluid flow, exact solution, Boussinesq approximation, counterflows, increased velocities.

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Introduction. The study of the motion of a rotating self-gravitating fluid was started as long ago as by Isaak Newton [1, 2], and it is currently far from being completed. Newton proved to be the first to endeavour to obtain, from the motion equations, a solution simulating the shape of the Earth. Newton's pioneering study gave rise to numerous investigations in this field of mechanics and physics and a great impetus to the development of the mathematics used practically in all the sections of mathematics. The development of the potential

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theory, the creation of the theory of special functions and the formation of the rapidly developing theory of integrable systems should be mentioned first.

Practically all the most prominent scientists, such as Appel, Basset, Betti, Gagen, Helmholtz, Dedekind, Dirichlet, Kirchhoff, Lipschitz, Lame, McLoren, Poincare, Riemann, Chandrasekhar, Jacoby, took part in the construction of the theory [3–13]. This list of discoverers of exact solutions can be extended [3–13]. The history of discovering the solutions can be found in the monographs and surveys [3–13]. It should be noted that the scientific breakthrough in the simulation of rotating equilibrium figures was made by the Russian scientists Steklov, Lyapunov, Ovsyannikov and Zel'dovich.

Note that the development of the theory of motion stability described by systems of ordinary differential equations was initiated by studying the stability of equilibrium figures and the motion of an elastic body partially filled with a fluid [4, 5]. It is the work in these fields that offered a formulation to the well-known Lyapunov theorems of stability and instability, among other things, from the first approximation [4, 5]. Original research has currently been done on the stability of Dirichlet and Jacobi ellipsoids [8].

Ovsyannikov's exact gas-hydrodynamic solution became the first solution simulating gas cloud motion in the class of linear velocities [8]. Afterwards models of Dyson and Fujimoto were proposed [8].

There is another substantial model, which influenced the development of the theory of rotating fluids. It is the generalized solid proposed for research by Arnold [14]. In the survey made by Dolzhansky [15] there is a detailed analysis of this approach with some generalizations for different classes of fluids.

As a rule, dynamic fluid equilibria are simulated with the application of two mathematical models. One model is Lagrangian fluid simulation, i.e. the use of the ordinary differential equations of Lagrange, Hamilton and Jacoby to describe the relative equilibrium (solid-state) of the continuum. The other model consists in using the Euler equations with potential forces to analyze the structures of a relative fluid equilibrium. Thus, in the simulation of dynamic fluid equilibria no account is taken of dissipation and temperature effect. In other words, an isothermal perfect incompressible fluid is dealt with.

It is apparent that this deficiency in studying fluid flows can be compensated if this problem is analyzed with the application of the Oberbeck–Boussinesq system of equations. Note that it is not only the Boussinesq approximation that has not been used before to describe dynamic fluid equilibria, but also the characteristic scale of solutions. This statement needs to be clarified. Despite Chandrasekhar [6] used the Boussinesq approximation to simulate astrophysical objects, he used the virial method of physical field expansion for solving his problems. The application of this body of mathematics enabled one to reduce the Oberbeck–Boussinesq system to a system of ordinary differential equations and to use the classical problem statements when interpreting results. Similar reasoning holds for [14, 15].

There are studies dealing with large-scale fluid or gas motions [6], when it is necessary to take into account the inhomogeneity of the gravitational field, and with scales for which surface tension forces are principally important, e. g. the

Hadamard-Rybczynski problem [16, 17]. In both extreme cases, the principal equilibrium shape is spherical. The consideration of rotation was always viewed as some disturbance of the principal flow, however it was not viewed as a forming factor for equilibrium structures.

Note that the very possibility of the existence of equilibrium figures in the region of intermediate scales, when neither the inhomogeneity of the gravitational field nor the surface tension forces are essential, does not seem to have been studied. At any rate, the authors are unfamiliar with works viewing the problem this way.

Thus, we intend to ascertain the possible existence of equilibrium structures in the region of the so-called intermediate scales, when the Boussinesq approximation is obviously applicable to the simulation of a heat-conducting viscous incompressible fluid.

1. Problem statement. In the Boussinesq approximation [6, 18] in the Cartesian coordinates (the Oz axis is directed upwards), the system of equations describing the heat convection of a viscous incompressible fluid are written as

$$\begin{gathered}\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\nabla P + \nu\boldsymbol{\Delta}\mathbf{V} + \mathbf{g}\beta T\mathbf{k},\\ \frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T = \chi\boldsymbol{\Delta} T,\\ \nabla\cdot\mathbf{V} = \mathbf{0}.\end{gathered} \tag{1}$$

The system of equations (1) involves the following symbols: $\mathbf{V} = (V_x, V_y, V_z)$ is the flow velocity vector; P is the pressure deviation from hydrostatic, which is related to the constant average fluid density ρ; T is the deviation from the average temperature; ν and χ are the coefficients of kinematic viscosity and thermal diffusivity of the fluid, respectively; β is the temperature coefficient of fluid volumetric expansion; $\mathbf{k}$ is the unit vector directed vertically upwards, ∇ is the Hamilton operator, Δ is the Laplace operator (Laplacian).

The solution of system (1) is sought in the approximation to the large scale. For large-scale flows, the characteristic horizontal scale greatly exceeds in magnitude the characteristic dimension during the entire fluid motion. Free surface curvature can in this scale be neglected, the vertical velocity V_z being assumed zero. Thus, system (1) is transformed to the following system:

$$\begin{gathered}\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\Big(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\Big),\\ \frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu\Big(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\Big),\\ \frac{\partial P}{\partial z} = g\beta T,\\ \frac{\partial T}{\partial t} + V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} = \chi\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\Big),\\ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0.\end{gathered} \tag{2}$$

Obviously, system (2) is nonlinear and overdetermined, four unknown functions of velocity and temperature being required from five equations. The hydrodynamic and temperature fields are computed as [18]

$$
\begin{gathered}
V_x = U + xu_1 + yu_2, \quad V_y = V + xv_1 + yv_2, \\
P = P_0 + xP_1 + yP_2 + \frac{x^2}{2}P_{11} + \frac{y^2}{2}P_{22} + xyP_{12}, \\
T = T_0 + xT_1 + yT_2 + \frac{x^2}{2}T_{11} + \frac{y^2}{2}T_{22} + xyT_{12}.
\end{gathered} \tag{3}
$$

For horizontal coordinates, all the functions in the formulae of system (3) depend on the variable z ant time t. Substituting relation (3) into system (2), we arrive at the system

$$
\begin{gathered}
\hat{L}U + Uu_1 + Vu_2 + P_1 = 0, \\
\hat{L}u_1 + u_1^2 + v_1 u_2 + P_{11} = 0, \\
\hat{L}u_2 + u_1 u_2 + u_2 v_2 + P_{12} = 0, \\
\hat{L}V + Uv_1 + Vv_2 + P_2 = 0, \\
\hat{L}v_1 + u_1 v_1 + v_1 v_2 + P_{12} = 0, \\
\hat{L}v_2 + u_2 v_1 + v_2^2 + P_{22} = 0, \\
\hat{M}T_0 + UT_1 + VT_2 - \chi\left(T_{11} + T_{22}\right) = 0, \\
\hat{M}T_1 + UT_{11} + VT_{12} + u_1 T_1 + v_1 T_2 = 0, \\
\hat{M}T_2 + UT_{12} + VT_{22} + u_2 T_1 + v_2 T_2 = 0, \\
\hat{M}T_{11} + 2u_1 T_{11} + 2v_1 T_{12} = 0, \\
\hat{M}T_{22} + 2u_2 T_{12} + 2v_2 T_{22} = 0, \\
\hat{M}T_{12} + u_1 T_{12} + u_2 T_{11} + v_1 T_{22} + v_2 T_{12} = 0, \\
u_1 + v_2 = 0, \\
\frac{\partial P_0}{\partial z} = g\beta T_0, \quad \frac{\partial P_1}{\partial z} = g\beta T_1, \quad \frac{\partial P_2}{\partial z} = g\beta T_2, \\
\frac{\partial P_{11}}{\partial z} = g\beta T_{11}, \quad \frac{\partial P_{12}}{\partial z} = g\beta T_{12}, \quad \frac{\partial P_{22}}{\partial z} = g\beta T_{22}.
\end{gathered} \tag{4}
$$

Here, the following differential parabolic operators in partial derivatives are introduced:

$$
\hat{L} = \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2}, \quad \hat{M} = \frac{\partial}{\partial t} - \chi \frac{\partial^2}{\partial z^2}.
$$

2. Analyzing the solvability of system. In what follows we do not discuss the solvability of system (4) as a whole, but restrict our analysis to one special case. We will seek a solution to the system of equations when the boundaries of the fluid layer rotate. In [19] the validity of equalities for the velocity gradients in the simulation of axisymmetric flows within class (3) was demonstrated,

$$
u_1 = 0, \quad v_2 = 0, \quad u_2 = -v_1. \tag{5}
$$

Next the lower boundary of the layer $z = 0$ is assumed to be immobile. The upper boundary describable by the equation $z = h$ rotates with constant angular velocity Ω. Using equalities (5), we obtain the following solutions to system (4), which determine a number of functions in class (3) for the coordinates x and y:

$$\begin{gathered} u_2 = \frac{\Omega z}{h}, \quad v_1 = -\frac{\Omega z}{h}, \quad P_{12} = 0, \\ P_{11} = P_{22} = -u_2 v_1 = \frac{\Omega^2 z^2}{h^2}, \quad T_{12} = 0, \\ g\beta T_{11} = g\beta T_{22} = \frac{2\Omega^2 z}{h^2}. \end{gathered} \tag{6}$$

For the homogeneous summands in expressions (3), due to solution (6), system (4) yields two closed subsystems. First system has of the form

$$\begin{gathered} \hat{L}U + \frac{\Omega z}{h} V + P_1 = 0, \\ \hat{L}V - \frac{\Omega z}{h} U + P_2 = 0, \\ \hat{M}T_1 + \frac{2\Omega^2 z}{g\beta h^2} U - \frac{\Omega z}{h} T_2 = 0, \\ \hat{M}T_2 + \frac{2\Omega^2 z}{g\beta h^2} V + \frac{\Omega z}{h} T_1 = 0, \\ \frac{\partial P_1}{\partial z} = g\beta T_1, \quad \frac{\partial P_2}{\partial z} = g\beta T_2. \end{gathered} \tag{7}$$

Second system has of the form

$$\begin{gathered} \hat{M}T_0 + UT_1 + VT_2 = \frac{4\chi\Omega^2 z}{g\beta h^2}, \\ \frac{\partial P_0}{\partial z} = g\beta T_0. \end{gathered} \tag{8}$$

Note that the integration of system (8) is possible only after the integration of system (7). In order to find a solution to system (7), we introduce the following complex functions:

$$W = U + iV, \quad S = P_1 + iP_2, \quad \Theta = g\beta\left(T_1 + iT_2\right).$$

Here i is an imaginary unit. Thus, system (7) is in this case transformed to the following form:

$$\begin{gathered} (\hat{L} - i\Omega z)W + S = 0, \\ (\hat{M} + i\Omega z)\Theta + 2\Omega^2 z W = 0, \\ \frac{dS}{dz} = \Theta. \end{gathered} \tag{9}$$

To reduce the order of the system, the following transformation is made:

$$S = S_1 + i\Omega z W, \quad \Theta = \Theta_1 + 2i\Omega W. \tag{10}$$

Substituting (10) into system (9), we obtain the equalities

$$\begin{gathered} \nu \frac{d^2 W}{dz^2} + S_1 = 0, \\ \chi \frac{d^2 \Theta_1}{dz^2} + i\Omega\Big(z\Theta_1 - 2\frac{\chi}{\nu} S_1\Big) = 0, \\ \frac{dS_1}{dz} = \Theta_1 - i\Omega z^2 \frac{d}{dz}\Big(\frac{W}{z}\Big). \end{gathered} \tag{11}$$

For further transformations, we use the differential identity

$$\frac{d^2 W}{dz^2} = \frac{1}{z}\frac{d}{dz}\left(z^2 \frac{d}{dz}\Big(\frac{W}{z}\Big)\right).$$

This identity holds for any twice-differentiable function. Consequently, by differentiating the last equation in system (11), we arrive at a system for determining the functions Θ_1 and S_1,

$$\chi \frac{d^2 \Theta_1}{dz^2} + i\Omega\left(z\Theta_1 - 2\frac{\chi}{\nu} S_1\right) = 0, \quad \nu \frac{d^2 S_1}{dz^2} = i\Omega z S_1 + \nu \frac{d\Theta_1}{dz}. \tag{12}$$

Taking into account that equations (11) offer an explicit expression for W, we obtain an integral of the initial system with automatic order reduction, system (12) being transformed to the operator equation

$$\left[\Big(\nu \frac{d^2}{dz^2} - i\Omega z\Big)\Big(\chi \frac{d^2}{dz^2} + i\Omega z\Big) - 2i\Omega\chi \frac{d}{dz}\right]\Theta_1 = 0.$$

Simplifying the latter equality, we derive the following fourth-order ordinary differential equation with complex coefficients:

$$\left[\nu\chi \frac{d^4}{dz^4} + i\Omega\,(\nu - \chi)\,\frac{d^2}{dz^2} z + \Omega^2 z^2\right]\Theta_1 = 0. \tag{13}$$

3. Solving equation when the dissipative coefficients coincide. We assume in equation (13) that $\nu = \chi$ (the Prandtl number equals one). In this case, equation (13) acquires the form

$$\frac{d^4 \Theta_1}{dz^4} + \left(36\sigma^3 z\right)^2 \Theta_1 = 0, \tag{14}$$

where $\sigma = \frac{1}{6}\sqrt[3]{\frac{\Omega}{6\nu h}}$ is the dispersion relation. For the convenience of computations, in order to illustrate the physical meaning of the solutions and to make the notation concise, we introduce a new variable $Z = \frac{z}{h\sigma}$. The general solution of equation (14) is written as

$$\Theta_1 = {}^3_0F\left[\left\{\frac{1}{2},\frac{2}{3},\frac{5}{6}\right\},-Z^6\right]C_1 + {}^3_0F\left[\left\{\frac{2}{3},\frac{5}{6},\frac{7}{6}\right\},-Z^6\right]C_2Z+ \\ + {}^3_0F\left[\left\{\frac{5}{6},\frac{7}{6},\frac{4}{3}\right\},-Z^6\right]C_3Z^2 + {}^3_0F\left[\left\{\frac{7}{6},\frac{4}{3},\frac{3}{2}\right\},-Z^6\right]C_4Z^3. \quad (15)$$

Here, ${}^3_0F = F\left(a;b;Z\right) = F\left(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;Z\right)$ is a generalized hypergeometric function [20]. The choice of using the generalized hypergeometric function for writing the solutions is dictated by notation conciseness. Note that there is another form of writing the solution of equation (14), which is based on using the Mittag–Leffler functions [20]. Note that each particular solution (15) is a function of a real variable and that the general solution is considered in the complex plane. Thus, at least one integration constant is a complex number.

We now turn to finding the generalized pressure S_1 from the differential equation (11)

$$S_1 = -\frac{i}{2\sigma}\frac{d^2\Theta_1}{dz^2} + \frac{z}{2}\Theta_1. \quad (16)$$

The substitution of solution (15) into equation (16) gives

$$S_1 = {}^3_0F\left[\left\{\frac{1}{2},\frac{2}{3},\frac{5}{6}\right\},-Z^6\right]\frac{C_1}{2}Z + {}^3_0F\left[\left\{\frac{2}{3},\frac{5}{6},\frac{7}{6}\right\},-Z^6\right]\frac{C_2}{2}Z^2+ \\ + {}^3_0F\left[\left\{\frac{5}{6},\frac{7}{6},\frac{4}{3},\right\},Z\right]\sqrt[3]{36\sigma^2}\frac{2\cdot 5C_3}{6!}Z^3 + {}^3_0F\left[\left\{\frac{7}{6},\frac{4}{3},\frac{3}{2}\right\},Z\right]\sigma\frac{2\cdot 5C_4}{6!}Z^4- \\ -\frac{1}{72\sigma}i\left\{{}^3_0F\left[\left\{\frac{5}{6},\frac{7}{6},\frac{4}{3}\right\},Z\right]\sqrt[3]{676\sigma^2}C_3Z^3 - {}^3_0F\left[\left\{\frac{7}{6},\frac{4}{3},\frac{3}{2}\right\},Z\right]6\sigma C_4Z+\right. \\ + {}^3_0F\left[\left\{\frac{5}{6},\frac{7}{6},\frac{4}{3}\right\},Z\right]3\sigma^2C_1Z^4+ \\ + {}^3_0F\left[\left\{\frac{5}{3},\frac{11}{6},\frac{13}{6},\right\},Z\right]\frac{2^4\cdot 3^3\sigma^2}{7!}\sqrt[3]{\frac{3\sigma}{4}}C_2Z^5+ \\ + {}^3_0F\left[\left\{\frac{5}{3},\frac{11}{6},\frac{13}{6},\right\},Z\right]\frac{2^4\cdot 3^4\sigma^2}{7!}\sqrt[3]{6\sigma}C_2Z^5+ \\ + {}^3_0F\left[\left\{\frac{11}{6},\frac{13}{6},\frac{7}{3}\right\},Z\right]\frac{2^2\cdot 3^4\sigma^2}{7!}\sqrt[3]{\frac{9\sigma^2}{2}}C_3Z^6+ \\ + {}^3_0F\left[\left\{\frac{13}{6},\frac{7}{3},\frac{5}{2}\right\},Z\right]\frac{2\cdot 5\cdot 11\sigma^3}{7!}C_4Z^7- \\ - {}^3_0F\left[\left\{\frac{5}{2},\frac{8}{3},\frac{17}{6}\right\},Z\right]\frac{2^7\cdot 3^3\cdot 7\sigma^4}{11!}C_1Z^{10}- \\ - {}^3_0F\left[\left\{\frac{8}{3},\frac{17}{6},\frac{19}{6}\right\},Z\right]\frac{2^9\cdot 3^6\sigma^4}{13!}\sqrt[3]{\frac{3\sigma}{4}}C_2Z^{11}- \\ - {}^3_0F\left[\left\{\frac{17}{6},\frac{19}{6},\frac{10}{3}\right\},Z\right]\frac{2^8\cdot 3^5\cdot 5\sigma^4}{14!}\sqrt[3]{\frac{9\sigma^2}{2}}C_3Z^{12}-$$

$$-{}_0^3F\left[\left\{\frac{19}{6},\frac{10}{3},\frac{7}{2}\right\},Z\right]\frac{2^7\cdot 3^2\cdot 5\cdot 11\sigma^5}{14!}C_4Z^{13}\Bigg\}. \quad (17)$$

Complex velocity W can be expressed in two ways. Firstly, it can be obtained by integrating the first equation in system (9), but it is a second-order equation. Hence, we obtain a solution depending on six integration constants, whereas system (11) is a fifth-order equation system. Thus our problem is complicated by additional analysis of overdetermination. A correct expression for velocity W can be obtained by the integration of the third equation in system (9),

$$\Omega W = C_5Z + \left({}_0^3F\left[\left\{\frac{2}{3},\frac{5}{6},\frac{7}{6}\right\},Z\right]-1\right)\frac{i}{2}\sqrt[3]{\frac{\sigma}{36}}C_2Z-$$
$$-{}_1^4F\left[\left\{-\frac{1}{6}\right\},\left\{\frac{1}{2},\frac{2}{3},\frac{5}{6},\frac{5}{6}\right\},Z\right]\frac{2i\Gamma\left(-\frac{1}{6}\right)}{4!\Gamma\left(\frac{5}{6}\right)}C_1Z^2+$$
$$+{}_1^4F\left[\left\{-\frac{1}{6}\right\},\left\{\frac{5}{6},\frac{7}{6},\frac{4}{3},\frac{3}{2}\right\},Z\right]\frac{2\cdot 5\Gamma\left(-\frac{1}{6}\right)}{6!\Gamma\left(\frac{5}{6}\right)}C_4+$$
$$+{}_1^4F\left[\left\{\frac{1}{6}\right\},\left\{\frac{5}{6},\frac{7}{6},\frac{7}{6},\frac{4}{3}\right\},Z\right]\frac{2\cdot 5i\Gamma\left(\frac{1}{6}\right)}{6!\Gamma\left(\frac{7}{6}\right)}\sqrt[3]{\frac{\sigma^2}{6}}C_3Z^2+$$
$$+{}_1^4F\left[\left\{\frac{1}{3}\right\},\left\{\frac{7}{6},\frac{4}{3},\frac{4}{3},\frac{3}{2}\right\},Z\right]\frac{2^4\cdot 3\cdot 5\cdot 7i\sigma\Gamma\left(\frac{1}{3}\right)}{9!\Gamma\left(\frac{4}{3}\right)}C_4Z^3-$$
$$-{}_1^4F\left[\left\{\frac{1}{3}\right\},\left\{\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{11}{6}\right\},Z\right]\frac{2^2\cdot 5\sigma\Gamma\left(\frac{1}{3}\right)}{6!\Gamma\left(\frac{4}{3}\right)}C_1Z^3-$$
$$-{}_1^4F\left[\left\{\frac{1}{2}\right\},\left\{\frac{3}{2},\frac{5}{3},\frac{11}{6},\frac{13}{6}\right\},Z\right]\frac{\sigma}{4!}\sqrt[3]{\frac{\sigma}{36}}C_2Z^4-$$
$$-{}_1^4F\left[\left\{\frac{2}{3}\right\},\left\{\frac{5}{3},\frac{11}{6},\frac{13}{6},\frac{7}{3}\right\},Z\right]\frac{2^4\cdot 3^2\cdot 7\sigma\Gamma\left(\frac{2}{3}\right)}{9!\Gamma\left(\frac{5}{3}\right)}\sqrt[3]{\frac{\sigma}{6}}C_3Z^5-$$
$$-{}_1^4F\left[\left\{\frac{5}{6}\right\},\left\{\frac{3}{2},\frac{5}{3},\frac{11}{6},\frac{11}{6}\right\},Z\right]\frac{2^3\cdot 3^2\cdot 7i\sigma^2\Gamma\left(\frac{5}{6}\right)}{9!\Gamma\left(\frac{11}{6}\right)}C_1Z^6-$$
$$-{}_1^4F\left[\left\{\frac{5}{6}\right\},\left\{\frac{11}{6},\frac{13}{6},\frac{7}{3},\frac{5}{2}\right\},Z\right]\frac{2^3\cdot 5^2\cdot 11\cdot 13\cdot 83\sigma^2\Gamma\left(\frac{5}{6}\right)}{13!\Gamma\left(\frac{11}{6}\right)}C_4Z^6-$$
$$-{}_1^4F\left[\left\{\frac{7}{6}\right\},\left\{\frac{11}{6},\frac{13}{6},\frac{13}{6},\frac{7}{3}\right\},Z\right]\frac{2\cdot 3^2i\sigma^2\Gamma\left(\frac{7}{6}\right)}{9!\Gamma\left(\frac{13}{6}\right)}\sqrt[3]{\frac{\sigma^2}{6}}C_3Z^8-$$
$$-{}_1^4F\left[\left\{\frac{4}{3}\right\},\left\{\frac{13}{6},\frac{7}{3},\frac{7}{3},\frac{5}{2}\right\},Z\right]\frac{2^3\cdot 5^2\cdot 11\cdot 13i\sigma^3\Gamma\left(\frac{4}{3}\right)}{13!\Gamma\left(\frac{7}{3}\right)}C_4Z^9+$$
$$+{}_1^4F\left[\left\{\frac{4}{3}\right\},\left\{\frac{7}{3},\frac{5}{2},\frac{8}{3},\frac{17}{6}\right\},Z\right]\frac{2^3\cdot 3\cdot 5\cdot 7\sigma^3\Gamma\left(\frac{4}{3}\right)}{11!\Gamma\left(\frac{7}{3}\right)}C_1Z^9+$$
$$+{}_1^4F\left[\left\{\frac{3}{2}\right\},\left\{\frac{5}{2},\frac{8}{3},\frac{17}{6},\frac{19}{6}\right\},Z\right]\frac{2^7\cdot 3^5\sigma^3}{13!}\sqrt[3]{\frac{\sigma}{36}}C_2Z^{10}+$$

$$+{}_1^4F\left[\left\{\frac{5}{3}\right\},\left\{\frac{8}{3},\frac{17}{6},\frac{19}{6},\frac{10}{3}\right\},Z\right]\frac{2^3\cdot 3^4\cdot 5\sigma^3\Gamma\left(\frac{5}{3}\right)}{13!\Gamma\left(\frac{8}{3}\right)}\sqrt[3]{\frac{\sigma^2}{6}}C_3Z^{11}+$$
$$+{}_1^4F\left[\left\{\frac{11}{6}\right\},\left\{\frac{17}{6},\frac{19}{6},\frac{10}{3},\frac{7}{2}\right\},Z\right]\frac{2^6\cdot 5\cdot 11\sigma^4\Gamma\left(\frac{11}{6}\right)}{14!\Gamma\left(\frac{17}{6}\right)}C_4Z^{12}-$$
$$-{}_1^4F\left[\left\{\frac{7}{3}\right\},\left\{\frac{10}{3},\frac{7}{2},\frac{11}{3},\frac{23}{6}\right\},Z\right]\frac{2^6\cdot 3\cdot 7^2\cdot 13\sigma^5\Gamma\left(\frac{7}{3}\right)}{17!\Gamma\left(\frac{10}{3}\right)}C_1Z^{15}-$$
$$-{}_1^4F\left[\left\{\frac{5}{2}\right\},\left\{\frac{7}{2},\frac{11}{3},\frac{23}{6},\frac{25}{6}\right\},Z\right]\frac{2^8\cdot 3^7\cdot 7\sigma^5}{19!}\sqrt[3]{\frac{\sigma}{36}}C_2Z^{16}-$$
$$-{}_1^4F\left[\left\{\frac{8}{3}\right\},\left\{\frac{11}{3},\frac{23}{6},\frac{25}{6},\frac{13}{3}\right\},Z\right]\frac{2^7\cdot 3^6\cdot 5\sigma^5\Gamma\left(\frac{8}{3}\right)C_3Z^{17}}{19!\Gamma\left(\frac{11}{3}\right)}\sqrt[3]{\frac{\sigma^2}{6}}-$$
$$-{}_1^4F\left[\left\{\frac{8}{3}\right\},\left\{\frac{23}{6},\frac{25}{6},\frac{13}{3},\frac{9}{2}\right\},Z\right]\frac{2^8\cdot 3^2\cdot 5^2\cdot 11\cdot 17\sigma^6\Gamma\left(\frac{17}{6}\right)}{21!\Gamma\left(\frac{23}{6}\right)}\sqrt[3]{\frac{\sigma^2}{6}}C_4Z^{18}. \quad (18)$$

Here $\Gamma(\lambda) = \int_0^{+\infty} r^{\lambda-1}\exp(-r)\,dr$ is the Euler gamma function [20]. Solutions (15), (17), (18) are written as a linear combination of generalized hypergeometric functions in the form of infinite power series. Consequently, examination of solution convergence is an urgent problem. Using the properties of convergence of these functions, we can state that the convergence radius for each solution equals infinity [20]. It means that the solution of our problem always converges.

Let us now formulate the boundary conditions to determine the integration constants of equation system (7). Without any claim to solution generality and without studying the shapes of the fluid, we give a simple example of such solutions. Let the simplest and most physically meaningful solution be a flow between two infinite disks rotating with free velocities. This is possible, since the solution can always be shifted along the Oz-axis, an appropriate distance between the disks being chosen. The disks must be nonuniformly heated to a proper temperature (the temperature must be related to the radius by a well-defined law). Assume that the lower boundary is isothermal (the temperature of the lower disk is zero); the adhesion condition holds on both boundaries. Thus, the following equalities are valid:

$$\text{when}\quad z=0,\quad U=V=0,\quad T_1=T_2=0;$$

$$\text{when}\quad z=h,\quad U=A\cos\alpha,\quad V=A\sin\alpha,\quad T_1=B,\quad T_2=C,\quad P_1=\tau_1,\quad P_2=\tau_2.$$

The conditions for determining the integration constants, written on the boundary, do not determine the integration constants uniquely. The system of linear algebraic equations for determining the integration constants is underdetermined, since there are few boundary conditions. To close the system of equations, we specify the fluid flow rate conditions as

$$\int_0^h U\,dz = Q_1,\quad \int_0^h V\,dz = Q_2.$$

When the flow rate is specified, it is generally assumed that $Q_1 = Q_2 = 0$ [19]. It is a physically justified convention [18, 19]. However, it is possible to obtain a solution by specifying a nonzero flow rate. Next, not limiting the generality of the reasoning, we will study the solutions simulating fluid motions with the zero flow rate.

For the above-introduced complex velocities, pressure and temperature (15), (17), (18), in view of transformations (10), the following boundary conditions are valid:

$$\text{when} \quad z = 0: \quad W = 0, \quad \Theta_1 = 0; \tag{19}$$

$$\text{when} \quad z = h: \quad W = A\exp(i\alpha), \quad \Theta_1 = B + 2A\Omega\sin\alpha + i\,(C - 2A\Omega\cos\alpha),$$

$$\int_0^h W dz = 0.$$

With the boundary conditions (19), it follows from equality (15) that $C_1 = 0$, and from solution (18) that $C_4 = 0$. Thus, the expressions for the determination of hydrodynamic fields become much simpler. Remember that, in the computation of the integration constants, the variable $z = h\sigma Z$ is to be replaced.

4. Analysis of the solutions. Using conditions (19), it is easy, but rather cumbersome, to compute the complex C_2, C_3, and C_5 which are related to the dispersion relation σ (kinematic viscosity) by the fractional linear law. Analyzing conditions (19), we find that the boundary value problem under study is meaningful at any positive value of σ. Nevertheless, the structure and behavior of the solutions depend considerably on the value of σ.

It is well known that, to analyze the polynomial solutions of system (7), it would suffice to localize the roots of the linear combination of the generalized hypergeometric functions [20]. Any hypergeometric function is known to be monotonically increasing with a zero value at the origin. Therefore, the appearance of other zero values is possible only when there is a combination of functions with at least one negative weight coefficient [20]. The appearance of negative coefficients depends on σ and the function domain, which also depends on the dispersion relation $Z \in \left[0, \frac{1}{\sigma}\right]$. Temperature is characterized by three types of solutions. When $\sigma \geqslant \frac{3}{2}$, the function describing temperature is a strictly monotonically increasing or decreasing function (fig. 1). If the evaluation $\frac{9}{10} < \sigma < \frac{3}{2}$ is valid for the dispersion relation, there is exactly one zero value of temperature inside the fluid layer (fig. 2). With the further decrease, temperature gets a finite number of zeros (fig. 3). Consequently, the fluid layer is divided into the regions with positive and negative temperature, which are characterized by alternating signs. The analysis of pressure is similar to that of temperature.

Velocities are characterized by only two qualitative behaviors of the functions (figs. 2 and 3). Thus, at any σ, there appear fluid counterflows. It was shown in [21–27] that the appearance of counterflows is a purely nonlinear effect, i. e., the viscous forces are comparable to the convective acceleration, and even without Coriolis forces. However, when σ tends to zero, counterflow zones are formed, whose thickness decreases with σ. Other values of the dispersion relation are characteristic of velocity, and they are easy to compute. However, the values obtained for the temperature and pressure fields are a good reference point for refining the characteristic dispersion numbers of velocities.

When analyzing the solutions (fig. 3), we can see their “fluctuation”. In [28] localized convective flows in terms were studied in the axisymmetric formulation. Solution “damping” was observed there. We use a wider class of solutions. There may be not only “damping” (when σ is close to one), but also “resonance” initiation. Yet, the solution does not become singular, and the Boussinesq approximation is not violated in the simulation of the convective flow. This difference is attributed to the account of the quadratic summands for temperature. If temperature is considered to be a linear function with respect to the horizontal coordinates, the solutions obtained in this study coincide with those discussed in [22].

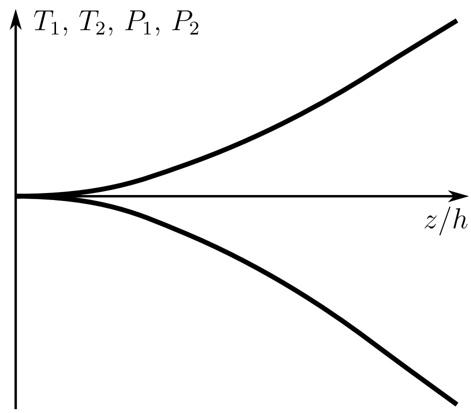

Figure 1. A qualitative form of the solutions T_1, T_2, P_1, P_2 of system (7) when $\sigma \geqslant \frac{3}{2}$ ($1-\tau_1 > 0$, $\tau_2 > 0$, $A > 0$, $B > 0$, $C > 0$; $2-\tau_1 < 0$, $\tau_2 < 0$, $A < 0$, $B < 0$, $C < 0$)

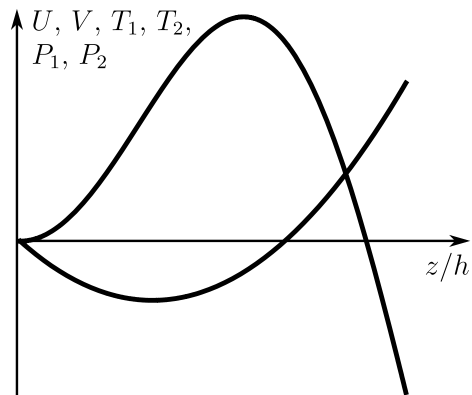

Figure 2. A qualitative form of the solutions T_1, T_2, P_1, P_2, U, V of system (7) when $\frac{9}{10} < \sigma < \frac{3}{2}$ ($1-\tau_1 > 0$, $\tau_2 > 0$, $A > 0$, $B > 0$, $C > 0$; $2-\tau_1 < 0$, $\tau_2 < 0$, $A < 0$, $B < 0$, $C < 0$)

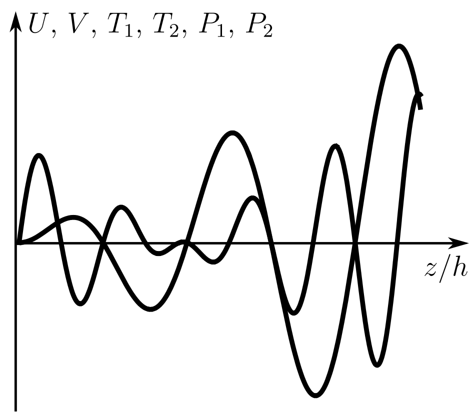

Figure 3. A qualitative form of the solutions T_1, T_2, P_1, P_2, U, V of system (7) when $\sigma \leqslant \frac{9}{10}$ ($1-\tau_1 > 0$, $\tau_2 > 0$, $B > 0$, $C > 0$; $2-\tau_1 < 0$, $\tau_2 < 0$, $B < 0$, $C < 0$)

5. Solution of equation with the Prandtl number different from one. An important particular case of equation (13) was studied above; namely, when the values of kinematic viscosity and thermal diffusivity of the fluid coincide. Let us now write the solution of equation (13) in the general case:

$$\Theta_1 = \mathsf{Ai}\Big(\xi + \frac{4\Omega z}{\sqrt{\nu\chi}}\sqrt[3]{\frac{\nu^2\chi^2}{16\Omega^2}}\Big)\left(C_1\mathsf{Ai}\left(\xi\right) + C_2\mathsf{Bi}\left(\xi\right)\right) + \\ + \mathsf{Bi}\Big(\xi + \frac{4\Omega z}{\sqrt{\nu\chi}}\sqrt[3]{\frac{\nu^2\chi^2}{16\Omega^2}}\Big)\left(C_3\mathsf{Ai}\left(\xi\right) + C_4\mathsf{Bi}\left(\xi\right)\right). \quad (20)$$

Here

$$\xi = \sqrt[3]{\nu^2\chi^2}\frac{i\left(\nu - \chi\right) + 2\sqrt{\nu\chi}\Omega z}{\nu\chi\sqrt[3]{16\Omega^2}}$$

is an auxiliary variable introduced in order to make the solution notation concise; $\mathsf{Ai}\left(\xi\right)$ and $\mathsf{Bi}\left(\xi\right)$ are the Airy function and the associated function [20]. Solution (20) is complex-valued, since it is written for arbitrary differences between ν and χ. This is done for the convenience of the solution analysis. Solution (20) being known, pressure S_1 and velocity W are computed similarly to the previous case.

When the dissipative coefficients ν and χ of the nonisothermal fluid are equal, solution (20) is reduced to formula (15). Studying the convergence of series (20), we find that the solution expressed in terms of this series converges on the solution domain, i. e. layer thickness.

The consideration of equation (13) with an arbitrary relation between ν and χ offers solutions to system (7), (8) in qualitative terms, see figs. 1 to 3. In the simulation of the hydrodynamic fields by the linear combination of the Airy functions the behavior of the solutions is predictable from the very beginning, since these special functions satisfy the simplest differential equation having a point where the fluctuation of the solution is replaced by its exponential growth. It was shown above that the solution does not go off to infinity.

6. Conclusion. The problem of simulating rotating fluid masses has a long history. To describe fluid motion (relative equilibrium), ordinary differential equations are used that enable one to obtain solutions simulating fluid flows by the theory of motion stability with the disturbance of the background (principal) flow. This paper discusses the exact solution to an overdetermined boundary value problem, which describes stationary flow at any scales where the Boussinesq approximation is valid. In this case, stationary dynamic fluid equilibria are characterized by the formation of counterflows, the sign of velocity being able to alternate several times, depending on the boundary conditions and the geometric dimensions of the fluid layer. A similar situation is true for temperature and pressure, namely, the formation of zones with the negative and positive values of temperature and pressure with respect to the reference value.

Competing interests. I declare that I have no competing interests.

Author's Responsibilities. I take full responsibility for submitting the final manuscript in print. I approved the final version of the manuscript.

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Динамические равновесия неизотермической жидкости

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Аннотация

В рамках точности приближения Буссинеска рассмотрены стационарные динамические равновесия вращающейся массы неизотермической жидкости. Показано, что в этом случае в жидкости наблюдается конечное число противотечений и усиление скоростей по сравнению с заданными на границе значениями, а также формирование зон положительного и отрицательного давления и температуры.

Ключевые слова: динамическое равновесие, вращающийся поток жидкости, точное решение, аппроксимация Буссинеска, противотечение, увеличенные скорости.

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