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NOTE ON COMPLEXITY OF QUANTUM TRANSMISSION PROCESSES

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*In 1989, Ohya propose a new concept, so-called Information Dynamics (ID), to investigate complex systems according to two kinds of view points. One is the dynamics of state change and another is measure of complexity. In ID, two complexities C^S and T^S are introduced. C^S is a measure for complexity of system itself, and T^S is a measure for dynamical change of states, which is called a transmitted complexity. An example of these complexities of ID is entropy for information transmission processes. The study of complexity is strongly related to the study of entropy theory for classical and quantum systems. The quantum entropy was introduced by von Neumann around 1932, which describes the amount of information of the quantum state itself. It was extended by Ohya for C^*-systems before CNT entropy. The quantum relative entropy was first defined by Umegaki for σ-finite von Neumann algebras, which was extended by Araki and Uhlmann for general von Neumann algebras and *-algebras, respectively. By introducing a new notion, the so-called compound state, in 1983 Ohya succeeded to formulate the mutual entropy in a complete quantum mechanical system (i.e., input state, output state and channel are all quantum mechanical) describing the amount of information correctly transmitted through the quantum channel. In this paper, we briefly review the entropic complexities for classical and quantum systems. We introduce some complexities by means of entropy functionals in order to treat the transmission processes consistently. We apply the general frames of quantum communication to the Gaussian communication processes. Finally, we discuss about a construction of compound states including quantum correlations.*

Key words: *quantum communication channel, von Neumann entropy, $\mathcal{S}$-mixing entropy, Ohya mutual entropy, C^*-system.*

1. Introduction

In [1], Ohya introduced Information Dynamics (ID) synthesizing dynamics of state change and complexity of state. Based on ID, one can study various problems of physics and other fields. Channel and two complexities are key concepts of ID.

Let us briefly review ID for quantum communication processes.

Let $\mathcal{H}_k$ $(k = 1, 2)$ be complex separable Hilbert spaces. We denote the set of all bounded linear operators on $\mathcal{H}_k$ by $\mathbf{B}(\mathcal{H}_k)$ $(k = 1, 2)$ and we express the set of all density operators on $\mathcal{H}_k$ by $\mathfrak{S}(\mathcal{H}_k)$ $(k = 1, 2)$. Let $(\mathbf{B}(\mathcal{H}_k), \mathfrak{S}(\mathcal{H}_k))$ $(k = 1, 2)$ be input $(k = 1)$ and output $(k = 2)$ quantum systems, respectively.

1.1. Quantum Channels

A mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$ is called a quantum channel Λ^*.

(1) Λ^* is called a linear channel if Λ^* satisfies the affine property such as $\Lambda^*(\sum_k \lambda_k \rho_k) = \sum_k \lambda_k \Lambda^*(\rho_k)$ for any $\rho_k \in \mathfrak{S}(\mathcal{H}_1)$ and any nonnegative number $\lambda_k \in [0,1]$ with $\sum_k \lambda_k = 1$.

For the quantum channel Λ^*, the dual map Λ of Λ^* is defined by

$$\operatorname{tr} \Lambda^*(\rho)B = \operatorname{tr} \rho\Lambda(B), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_1), \forall B \in \mathbf{B}(\mathcal{H}_2).$$

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(2) Λ^* is called a completely positive (CP) channel if Λ^* is linear channel and its dual map $\Lambda : \mathbf{B}(\mathcal{H}_2) \to \mathbf{B}(\mathcal{H}_1)$ of Λ^* holds

$$\left\langle x, \sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j\, x \right\rangle \geqslant 0 \quad (\forall x \in \mathcal{H}_1)$$

for any $n \in \mathbf{N}$, any $\{\overline{A}_i\} \subset \mathbf{B}(\mathcal{H}_2)$ and any $\{A_i\} \subset \mathbf{B}(\mathcal{H}_1)$.

One can describe almost all physical transform of states by using the CP channel [2–5].

1.2. Quantum Communication Channel

Here we explain the quantum communication channels as an example of the quantum channels.

In order to consider influence of the environment such as noise and loss, we suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ to be complex separable Hilbert spaces of noise and loss systems, respectively. Quantum channel of quantum communication process with noise and loss was discussed by [2, 6].

1.3. Noisy quantum channel and Generalized Beam Splitter

For an input state ρ in $\mathfrak{S}(\mathcal{H}_1)$ and a noise state $\xi \in \mathfrak{S}(\mathcal{K}_1)$, Ohya and NW defined in [6] a generalized beam splitting Π^* by

$$\Pi^*(\rho \otimes \xi) \equiv V\,(\rho \otimes \xi)\, V^*,$$

where V is a linear mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by for the n_1, m_1, j, $(n_1 + m_1 - j)$ photon number state vectors $|n_1\rangle \in \mathcal{H}_1$, $|m_1\rangle \in \mathcal{K}_1$, $|j\rangle \in \mathcal{H}_2$, $|n_1 + m_1 - j\rangle \in \mathcal{K}_2$

$$V(|n_1\rangle \otimes |m_1\rangle) = \sum_{j}^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1 + m_1 - j\rangle$$

and

$$\begin{aligned} C_j^{n_1,m_1} &= \sum_{r=L}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1! m_1! j! (n_1 + m_1 - j)!}}{r!(n_1 - j)!(j - r)!(m_1 - j + r)!} \\ &\quad \times \alpha^{m_1 - j + 2r} \left(-\bar{\beta}\right)^{n_1 + j - 2r} \end{aligned} \tag{1}$$

α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. K and L are constants given by $K = \min\{n_1, j\}$, $L = \max\{m_1 - j, 0\}$. For the coherent input state $\rho = |\theta\rangle \langle\theta| \otimes |\kappa\rangle \langle\kappa| \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$, the output state of Π^* is obtained by

$$\begin{aligned} \Pi^*\left(|\theta\rangle \langle\theta| \otimes |\kappa\rangle \langle\kappa|\right) &= |\alpha\theta + \beta\kappa\rangle \langle\alpha\theta + \beta\kappa| \\ &\quad \otimes \left|-\bar{\beta}\theta + \bar{\alpha}\kappa\right\rangle \left\langle -\bar{\beta}\theta + \bar{\alpha}\kappa\right| . \end{aligned}$$

By using Π^*, Ohya and NW introduced in [6] the noisy quantum channel Λ^* with a fixed noise state $\xi \in \mathfrak{S}(\mathcal{K}_1)$ defined by

$$\Lambda^*(\rho) \equiv \mathrm{tr}_{\mathcal{K}_2} \Pi^*(\rho \otimes \xi) = \mathrm{tr}_{\mathcal{K}_2} V\,(\rho \otimes \xi)\, V^*. \tag{2}$$

The generalized beam splitting Π^* with the vacuum noise state $\xi_0 = |0\rangle\langle 0|$ is called the beam splitter Π_0^* given by

$$\Pi_0^*\left(|\theta\rangle\langle\theta| \otimes \xi_0\right) = |\alpha\theta\rangle\langle\alpha\theta| \otimes \left|-\bar{\beta}\theta\right\rangle\left\langle -\bar{\beta}\theta\right|$$

for the coherent input state $\rho \otimes \xi_0 = |\theta\rangle\langle\theta| \otimes |0\rangle\langle 0| \in \mathfrak{S}\left(\mathcal{H}_1 \otimes \mathcal{K}_1\right)$. The beam splitter Π_0^* was described by means of the lifting $\mathcal{E}_0^*$ from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ in the sense of Accardi and Ohya [7] as follows

$$\mathcal{E}_0^*\left(|\theta\rangle\langle\theta|\right) = |\alpha\theta\rangle\langle\alpha\theta| \otimes |\beta\theta\rangle\langle\beta\theta|.$$

Based on the liftings, the beam splitting was studied by Accardi–Ohya and Fichtner–Freudenberg–Libsher [8]. Moreover, the noisy quantum channel Λ_0^* with the vacuum noise state $\xi_0 = |0\rangle\langle 0|$ is called the attenuation channel given by Ohya [2] as

$$\Lambda_0^*(\rho) \equiv \operatorname{tr}_{\mathcal{K}_2} \Pi_0^*(\rho \otimes \xi_0) = \operatorname{tr}_{\mathcal{K}_2} V_0(\rho \otimes |0\rangle\langle 0|)V_0^*, \tag{3}$$

which plays an important role for investigating the quantum communication processes.

2. Complexities

Two kind of complexities $C^{\mathcal{S}}(\rho)$, $T^{\mathcal{S}}(\rho;\Lambda^*)$ are used in ID. $C^{\mathcal{S}}(\rho)$ is a complexity of a state ρ measured from a subset $\mathcal{S}$ and $T^{\mathcal{S}}(\rho;\Lambda^*)$ is a transmitted complexity according to the state change from ρ to $\Lambda^*\rho$. These complexities should fulfill the following conditions: Let $\mathcal{S}$, $\overline{\mathcal{S}}$, $\mathcal{S}_t$ be subsets of $\mathfrak{S}(\mathcal{H}_1)$, $\mathfrak{S}(\mathcal{H}_2)$, $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, respectively.

(1) For any $\rho \in \mathcal{S}$, $C^{\mathcal{S}}(\rho)$ and $T^{\mathcal{S}}(\rho;\Lambda^*)$ are nonnegative ($C^{\mathcal{S}}(\rho) \geqslant 0$, $T^{\mathcal{S}}(\rho;\Lambda^*) \geqslant 0$).

(2) For a bijection j from $\operatorname{ex}\mathfrak{S}(\mathcal{H}_1)$ to $\operatorname{ex}\mathfrak{S}(\mathcal{H}_1)$,

$$C^{\mathcal{S}}(\rho) = C^{\mathcal{S}}(j(\rho))$$

is hold, where $\operatorname{ex}\mathfrak{S}(\mathcal{H}_1)$ is the set of extremal point of $\mathfrak{S}(\mathcal{H}_1)$.

(3) For $\rho \otimes \sigma \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\rho \in \mathfrak{S}(\mathcal{H}_1)$, $\sigma \in \mathfrak{S}(\mathcal{H}_2)$,

$$C^{\mathcal{S}_t}(\rho \otimes \sigma) = C^{\mathcal{S}}(\rho) + C^{\overline{\mathcal{S}}}(\sigma).$$

It means that the complexity of the state $\rho \otimes \sigma$ of totally independent systems are given by the sum of the complexities of the states ρ and σ.

(4) $C^{\mathcal{S}}(\rho)$ and $T^{\mathcal{S}}(\rho;\Lambda^*)$ satisfy the following inequality $0 \leqslant T^{\mathcal{S}}(\rho;\Lambda^*) \leqslant C^{\mathcal{S}}(\rho)$.

(5) If the channel Λ^* is given by the identity map id, then $T^{\mathcal{S}}(\rho;id) = C^{\mathcal{S}}(\rho)$ is hold.

One of the example of the above complexities are the Shannon entropy $S(p)$ for $C^{\mathcal{S}}(p)$ and classical mutual entropy $I(p;\Lambda^*)$ for $T^{\mathcal{S}}(p;\Lambda^*)$. Let us consider these complexities for quantum systems.

2.1. Example of Complexity $C^{\mathcal{S}}(\rho)$

2.1.1. von Neumann Entropy and $\mathcal{S}$-mixing entropy

One of the example of the complexity $C^{\mathcal{S}}(\rho)$ of ID in quantum system is the von Neumann entropy $S(\rho)$ [9] described by

$$C^{\mathcal{S}}(\rho) \Leftrightarrow S(\rho) = -\operatorname{tr}\rho\log\rho$$

for any density operators $\rho \in \mathfrak{S}(\mathcal{H}_1)$, which satisfies the above conditions (1), (2), (3).

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ be a C*-dynamical system and $\mathcal{S}$ be a weak* compact and convex subset of $\mathfrak{S}(\mathcal{A})$. For example, $\mathcal{S}$ is given by $\mathfrak{S}(\mathcal{A})$ (the set of all states on $\mathcal{A}$), $I(\alpha)$ (the set of all invariant states for α), $K(\alpha)$ (the set of all KMS states), and so on. Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on $ex\mathcal{S}$ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu, \tag{4}$$

where $\mathrm{ex}\,\mathcal{S}$ is the set of all extreme points of $\mathcal{S}$. The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex. We denote the set of all such measures by $M_\varphi(\mathcal{S})$, and define

$$\begin{aligned} D_\varphi(\mathcal{S}) = \Big\{ M_\varphi(\mathcal{S}); \quad & \exists \mu_k \subset \mathbb{R}^+ \text{ and } \{\varphi_k\} \subset \mathrm{ex}\, S \\ s.t. \quad \sum_k \mu_k = 1, \quad & \mu = \sum_k \mu_k \delta(\varphi_k) \Big\}, \end{aligned} \tag{5}$$

where $\delta(\varphi)$ is the Dirac measure concentrated on an initial state φ. For a measure $\mu \in D_\varphi(\mathcal{S})$, we put

$$H(\mu) = -\sum_k \mu_k \log \mu_k. \tag{6}$$

The C*-entropy of a state $\varphi \in \mathcal{S}$ with respect to $\mathcal{S}$ (**$\mathcal{S}$-mixing entropy**) is defined by

$$C^{\mathcal{S}}(\varphi) \Leftrightarrow S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); & \mu \in D_\varphi(\mathcal{S})\} \\ +\infty & \text{if } D_\varphi(\mathcal{S}) = \varnothing. \end{cases} \tag{7}$$

It describes the amount of information of the state φ measured from the subsystem $\mathcal{S}$. We denote $S^{\mathfrak{S}(\mathcal{A})}(\varphi)$ by $S(\varphi)$ if $\mathcal{S} = \mathfrak{S}(\mathcal{A})$. It is an extension of von Neumann's entropy.

This entropy (mixing $\mathcal{S}$-entropy) of a general state φ satisfies the following properties [10].

THEOREM 2.1. *When $\mathcal{A} = B(\mathcal{H})$ and $\alpha_t = Ad(U_t)$ (i.e., $\alpha_t(A) = U_t^* A U_t$ for any $A \in \mathcal{A}$) with a unitary operator U_t, for any state φ given by $\varphi(\,\cdot\,) = \mathrm{tr}\,\rho\,\cdot$ with a density operator ρ, the following facts hold:*

(1) *$S(\varphi) = -\mathrm{tr}\,\rho \log \rho$.*

(2) *If φ is an α-invariant faithful state and every eigenvalue of ρ is non-degenerate, then $S^{I(\alpha)}(\varphi) = S(\varphi)$, where $I(\alpha)$ is the set of all α -invariant faithful states.*

(3) *If $\varphi \in K(\alpha)$, then $S^{K(\alpha)}(\varphi) = 0$, where $K(\alpha)$ is the set of all KMS states.*

THEOREM 2.2. *For any $\varphi \in K(\alpha)$, we have*

(1) *$S^{K(\alpha)}(\varphi) \leqslant S^{I(\alpha)}(\varphi)$.*

(2) *$S^{K(\alpha)}(\varphi) \leqslant S(\varphi)$.*

2.2. Example of Transmitted Complexity $T^{\mathcal{S}}(\rho; \Lambda^*)$

The classical mutual entropy $I(p; \Lambda^*)$ defined by using the joint probability distribution between the input state and the output state is an example of the transmitted complexity $T^{\mathcal{S}}(p; \Lambda^*)$ of ID. In general, there does not exit the joint states in the quantum system [11]. We need to introduce the compound state in quantum system instead of the joint probability distribution in classical system.

2.3. Compound state

The quantum mutual entropy $I(\rho, \Lambda^*)$ should satisfy the following three conditions:

1) If the channel is given by the identity channel id, then $I(\rho; id) = S(\rho)$ (von Neumann entropy) is hold.
2) If the system is classical, then the quantum mutual equals to the classical mutual entropy.
3) The quantum mutual entropy should satisfy the Shannon's type inequalities:

$$0 \leqslant I(\rho, \Lambda^*) \leqslant S(\rho).$$

Ohya introduced two compound states σ_0 and σ_E. σ_0 is the trivial compound state given by

$$\sigma_0 = \rho \otimes \Lambda^* \rho.$$

σ_E is the compound state representing a certain correlation between the input state and the output state given by

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$$

associated with the Schatten–von Neumann (one dimensional spectral) decomposition [12] $\rho = \sum_n \lambda_n E_n$ of the input state ρ.

2.4. Ohya Mutual Entropy for density operator

An example of the transmitted complexity $T^{\mathcal{S}}(\rho; \Lambda^*)$ of ID in quantum system is the Ohya mutual entropy with respect to the initial state ρ and the quantum channel Λ^* defined by

$$T^{\mathcal{S}}(\rho; \Lambda^*) \Leftrightarrow I(\rho; \Lambda^*) \equiv \sup \left\{ \sum_n S(\sigma_E, \sigma_0), \rho = \sum_n \lambda_n E_n \right\},$$

where $S(\cdot, \cdot)$ is the Umegaki's relative entropy [13] denoted by

$$S(\rho, \sigma) \equiv \begin{cases} \operatorname{tr} \rho (\log \rho - \log \sigma) & (\text{when } \overline{\operatorname{ran} \rho} \subset \overline{\operatorname{ran} \sigma}) \\ \infty & (\text{otherwise}) \end{cases} \tag{8}$$

which was extended to more general quantum systems by Araki and Uhlmann [1, 3, 4, 14, 16]. The Ohya mutual entropy holds the above conditions (1), (4), (5) such as

$$0 \leqslant I(\rho, \Lambda^*) \leqslant S(\rho),$$
$$I(\rho, id) = S(\rho).$$

The capacity means the ability of the information transmission of the channel, which is used as a measure for construction of channels. The quantum capacity is formulated by taking the supremum of the Ohya mutual entropy with respect to a certain subset of the initial state space. The quantum capacity of quantum channel was studied in [17–20].

THEOREM 2.3. *Let Φ_E be a compound state w.r.t. the initial state ρ, the quantum CP channel Λ^* and aSchatten decomposition of $\rho = \sum_k \lambda_k E_k$ defined by*

$$\Phi_E = \sum_n (I \otimes V_n) \left[\sum_k \sqrt{\lambda_k} |x_k\rangle \otimes |x_k\rangle \right] \left[\sum_{k'} \sqrt{\lambda_{k'}} \langle x_{k'}| \otimes \langle x_{k'}| \right] (I \otimes V_n^*)$$

under the condition

$$\sum_n (I \otimes V_n^*) (I \otimes V_n) = I \otimes I$$

and Λ^* is given by $\Lambda^*(\rho) = \sum_n V_n \rho V_n^*$. By defining the compound state Φ_E, one can obtain the following theorem.

THEOREM 2.4. *For the compound state Φ_E given above, one can obtain two marginal states as follows*

$$\begin{aligned} \mathrm{tr}_{\mathcal{H}_2} \Phi_E &= S(\rho), \\ \mathrm{tr}_{\mathcal{H}_1} \Phi_E &= S\left(\Lambda^* \rho\right). \end{aligned}$$

The upper bound of the relative entropy $S(\Phi_E, \rho \otimes \Lambda^* \rho)$ is obtained as follows:

$$S(\Phi_E, \rho \otimes \Lambda^* \rho) \leqslant 2S\left(\rho\right).$$

Let Ψ_E be a compoundstate defined by

$$\Psi_{E,\mu} = \mu\, \sigma_E + (1-\mu)\, \Phi_E \quad (\mu \in [0{,}1]).$$

We have the following theorem.

THEOREM 2.5. *For the compound state $\Psi_{E,\mu}$ w.r.t. $\mu \in [0,1]$ given above, one can obtain two marginal states as follows*

$$\begin{aligned} tr_{\mathcal{H}_2} \Psi_{E,\mu} &= S(\rho), \\ tr_{\mathcal{H}_1} \Psi_{E,\mu} &= S\left(\Lambda^* \rho\right). \end{aligned}$$

The upper bound of the relative entropy $S(\Psi_{E,\mu}, \rho \otimes \Lambda^* \rho)$ is obtained as follows:

$$S(\Psi_{E,\mu}, \rho \otimes \Lambda^* \rho) \leqslant (2-\mu)\, S(\rho) \quad (\mu \in [0{,}1]).$$

2.5. Ohya Mutual Entropy for general C*-system

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha\,(G))$ be a unital C^*-system and $\mathcal{S}$ be a weak* compact convex subset of $\mathfrak{S}(\mathcal{A})$. For an initial state $\varphi \in \mathcal{S}$ and a channel $\Lambda^* : \mathfrak{S}\,(\mathcal{A}) \to \mathfrak{S}\,(\mathcal{B})$, two compound states [3, 10] are defined by

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega \; d\mu, \tag{9}$$

$$\Phi_0 = \varphi \otimes \Lambda^* \varphi. \tag{10}$$

The compound state $\Phi_\mu^{\mathcal{S}}$ expresses the correlation between the input state φ and the output state $\Lambda^*\varphi$. The mutual entropy with respect to $\mathcal{S}$ and μ is given by

$$I_\mu^{\mathcal{S}}(\varphi;\Lambda^*) = S\left(\Phi_\mu^{\mathcal{S}}, \Phi_0\right) \tag{11}$$

and the mutual entropy with respect to $\mathcal{S}$ is defined by Ohya [3, 10] as

$$T^{\mathcal{S}}(\varphi;\Lambda^*) \Leftrightarrow I^{\mathcal{S}}(\varphi;\Lambda^*) = \sup\left\{I_\mu^{\mathcal{S}}(\varphi;\Lambda^*)\ ;\ \mu \in M_\varphi(\mathcal{S})\right\}. \tag{12}$$

2.6. Other Mutual Entropy Type Measures

Recently, several mutual entropy type measures were proposed by Shor [21] and Bennet et al [22, 23], which defined by using the entropy exchange [24] given by

$$S_e(\rho,\Lambda^*) = -\mathrm{tr}W\log W, \tag{13}$$

where W is a matrix $W = (W_{ij})_{i,j}$ with the elements

$$W_{ij} \equiv \mathrm{tr}A_i^*\rho A_j \tag{14}$$

obtained by means of the input state ρ and the CP channel Λ^* described by a Stinespring–Sudarshan–Kraus form

$$\Lambda^*(\,\cdot\,) \equiv \sum\nolimits_j A_j^* \cdot A_j. \tag{15}$$

Based on the entropy exchange, the coherent entropy $I_C(\rho;\Lambda^*)$ [15] and the Lindblad-Nielson entropy $I_L(\rho;\Lambda^*)$ [23] were defined by

$$I_C(\rho;\Lambda^*) \equiv S(\Lambda^*\rho) - S_e(\rho,\Lambda^*), \tag{16}$$
$$I_L(\rho;\Lambda^*) \equiv S(\rho) + S(\Lambda^*\rho) - S_e(\rho,\Lambda^*). \tag{17}$$

2.7. Comparison among these quantum mutual entropy type measures

In this section, we compare with these mutual types measures.

By comparing these mutual entropies for quantum information communication processes, we have the following theorem [25]:

Theorem 2.6. *Let $\{A_j\}$ be a projection valued measure with $\dim A_j = 1$. For arbitrary state ρ and the quantum channel $\Lambda^*(\,\cdot\,) \equiv \sum_j A_j \cdot A_j^*$, one has*

(1) $0 \leqslant I(\rho;\Lambda^*) \leqslant \min\{S(\rho), S(\Lambda^*\rho)\}$ *(Ohya mutual entropy),*
(2) $I_C(\rho;\Lambda^*) = 0$ *(coherent entropy),*
(3) $I_L(\rho;\Lambda^*) = S(\rho)$ *(Lindblad entropy).*

For the attenuation channel Λ_0^*, one can obtain the following theorems [25]:

Lemma 2.1. *For the attenuation channel Λ_0^* and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

there exists a unitary operator U such that

$$UWU^* = \lambda|0\rangle\langle 0| + (1-\lambda)\big|-\overline{\beta}\theta\big\rangle\big\langle-\overline{\beta}\theta\big|.$$

THEOREM 2.7. *For the attenuation channel Λ_0^* and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

the entropy exchange is obtained by

$$S_e\left(\rho, \Lambda_0^*\right) = -\mathrm{tr} W \log W = -\sum_{j=0}^{1} \mu_j \log \mu_j,$$

where

$$\mu_j = \frac{1}{2}\left\{1 + (-1)^j \sqrt{1 - 4\lambda\left(1-\lambda\right)\left(1 - \exp\left(-\left|\beta\right|^2\left|\theta\right|^2\right)\right)}\right\} \quad \left(j = 0, 1\right).$$

THEOREM 2.8. *For any state $\rho = \sum_n \lambda_n \left|n\right\rangle\left\langle n\right|$ and the attenuation channel Λ_0^* with $\left|\alpha\right|^2 = \left|\beta\right|^2 = \frac{1}{2}$, one has*

(1) $0 \leqslant I\left(\rho; \Lambda_0^*\right) \leqslant \min\left\{S(\rho), S\left(\Lambda_0^*\rho\right)\right\}$ *(Ohya mutual entropy),*
(2) $I_C\left(\rho; \Lambda_0^*\right) = 0$ *(coherent entropy),*
(3) $I_L\left(\rho; \Lambda_0^*\right) = S(\rho)$ *(Lindblad entropy).*

THEOREM 2.9. *For the attenuation channel Λ_0^* and the input state*

$$\rho = \lambda\left|0\right\rangle\left\langle 0\right| + \left(1-\lambda\right)\left|\theta\right\rangle\left\langle\theta\right|,$$

we have

(1) $0 \leqslant I\left(\rho; \Lambda_0^*\right) \leqslant \min\left\{S(\rho), S\left(\Lambda_0^*\rho\right)\right\}$ *(Ohya mutual entropy),*
(2) $-S(\rho) \leqslant I_C\left(\rho; \Lambda_0^*\right) \leqslant S(\rho)$ *(coherent entropy),*
(3) $0 \leqslant I_L\left(\rho; \Lambda_0^*\right) \leqslant 2S\left(\rho\right)$ *(Lindblad entropy).*

It shows that the coherent entropy holds $I_C\left(\rho; \Lambda_0^*\right) < 0$ for $\left|\alpha\right|^2 < \left|\beta\right|^2$ and the Lindblad entropy satisfies $I_L\left(\rho; \Lambda_0^*\right) \geqslant S(\rho)$ for $\left|\alpha\right|^2 > \left|\beta\right|^2$. From the above theorems, we can conclude that the transmitted complexity in quantum system is the Ohya mutual entropy and it is most fitting measure for studying the efficiency of information transmission in quantum communication processes. It means that Ohya mutual entropy can be considered as the transmitted complexity for quantum communication processes.

THEOREM 2.10. *For the attenuation channel Λ_0^* and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if $\lambda = \frac{1}{2}$ and $\beta = \sqrt{\frac{2}{3}}$, then there exists a compound state Φ satisfying

$$I_L(\rho; \Lambda_0^*) = S\left(\Phi, \rho \otimes \Lambda_0^*\rho\right).$$

THEOREM 2.11. *For the attenuation channel Λ_0^* and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if $\lambda = \frac{1}{2}$ *and* $\alpha = 1$*, then there exists a compound state* Φ *satisfying*

$$S\left(\Phi, \rho \otimes \Lambda_0^* \rho\right) = S\left(\rho\right).$$

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