Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki
[J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2020, vol. 24, no. 2, pp. 379–389
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu1734

Short Communications

MSC: 26A33, 32A10

α-Differentiable functions in complex plane

***R. Pashaei*[1], *A. Pishkoo*[2], *M. S. Asgari*[1], *D. Ebrahimi Bagha*[1]**

[1] Islamic Azad University, Central Tehran Branch, Tehran, Iran.
[2] Nuclear Science and Technology Research Institute, Tehran, Iran.

Abstract

In this paper, the conformable fractional derivative of order α is defined in complex plane. Regarding to multi-valued function $z^{1-\alpha}$, we obtain fractional Cauchy–Riemann equations which in case of $\alpha = 1$ give classical Cauchy–Riemann equations. The properties relating to complex conformable fractional derivative of certain functions in complex plane have been considered. Then, we discuss about two complex conformable differential equations and solutions with their Riemann surfaces. For some values of order of derivative, α, we compare their plots.

Keywords: conformable fractional derivative, Cauchy–Riemann equations, limit based fractional derivative.

Received: 9th August, 2019 / Revised: 19th February, 2020 / Accepted: 16th March, 2020 / First online: 25th May, 2020

Short Communication

Please cite this article in press as:
Pashaei R., Pishkoo A., Asgari M. S., Ebrahimi Bagha D. α-Differentiable functions in complex plane, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2020, vol. 24, no. 2, pp. 379–389. doi: 10.14498/vsgtu1734.

Authors' Details:

Ronak Pashaei; Ph.D. Student in Mathematics (Analysis); Dept. of Mathematics; Faculty of Basic Sciences

Amir Pishkoo https://orcid.org/0000-0001-6211-5493
Physics and Accelerators Research School; e-mail: apishkoo@gmail.com

Mohammad Sadegh Asgari https://orcid.org/0000-0002-0675-0262
Associate Professor of Pure Mathematics; Dept. of Mathematics; Faculty of Basic Sciences; e-mail: moh.asgari@iauctb.ac.ir

Davood Ebrahimi Bagha; Professor in Mathematics; Dept. of Mathematics; Faculty of Basic Sciences; e-mail: dav.ebrahimi_bagha@iauctb.ac.ir

1. Introduction

The *fractional calculus* is an area of intensive research and development that can be historically divided into old and new parts. In [1], the old part of fractional calculus is referred to during the period 1695–1970. The books [2,3] are valuable resources for enthusiasts in detailed historical background. About the starting point of fractional derivative, it should be said that L'Hopital asked the question "what does derivative of order 1/2 mean?" namely $\frac{d^{1/2}}{dx^{1/2}}f$ in 1695. Two years later, in a letter to J. Wallis, Leibniz discussed the infinite product of Wallis for π and used the notation $d^{1/2}y$ to denote the derivative of order 1/2 [4,5]. Many researchers have been trying to generalize the concept of an ordinary derivative and integral to a fractional derivative and integral. Discussing the inversion of the integral equation by Grünwald in 1867, and proposing the sum of orders in the product of fractional derivatives by Letnikov in 1868 opened new ways. In 1872, Letnikov clarified the generalization of Cauchy's integral formula and utilized fractional derivatives to address differential equations. Relative to theory and applications of fractional calculus these references should be suggested [6–8].

There are many different types of fractional derivatives which have been suggested by famous researchers such as Riemann, Liouville, Riesz, Caputo, etc. The fractional derivatives of non-integer orders are utilized in the applied sciences to describe the processes and systems. Most of the fractional derivatives of non-integer orders form integro-differential operators. We can name them as "*integral based*" fractional derivatives which have a set of non-standard properties [10–14].

On the other hand, in recent years a few types of operators have been proposed that are attempted to be classified entitled "*limit based*" fractional derivatives. In 2014, Khalil et al. from one side and a few months later Katugampola from the other side proposed two limit based fractional derivatives as conformable derivatives [11,12]. However, the main idea in these definitions has been originated from works of Tarasov within the framework of the model of continuous fractal media in [15] (for instance see Eq. (1.1)) and also works of Li and Ostoja–Starzewski in [16–19] which is called "fractal derivative":

$$\frac{\partial f(x)}{\partial x^\alpha} = \frac{|x|^{1-\alpha}}{\alpha}\frac{\partial f(x)}{\partial x}.$$

In this paper, we focus on just the limit based form of fractional derivative, namely conformable fractional derivative, in complex plane.

Complex functions provide an almost inexhaustible supply of harmonic functions which means that solutions to the two-dimensional Laplace equation. In modern mathematics, the fractional derivatives of non-integer order have been introduced by such famous mathematicians as Riemann, Liouville, Riesz, Erdelyi, Kober and other. Many of definitions for the fractional derivatives are of type an integral form such as Riemann–Liouville definition and Caputo definition. In general, the fractional derivatives of non-integer orders include a set of non-standard properties [11,13,20,21].

There exists inconsistencies in the existing fractional derivatives (integral based) as follows [11].

1. All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions: $D_a^\alpha(fg) = fD_a^\alpha g + gD_a^\alpha f$.
2. All fractional derivatives do not satisfy the known formula of the derivative

of the quotient of two functions:

$$D_a^\alpha\Big(\frac{f}{g}\Big) = \frac{gD_a^\alpha f - fD_a^\alpha g}{g^2}.$$

3. All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.
4. Fractional derivatives do not have a corresponding *Rolle's Theorem.*
5. Fractional derivatives do not have a corresponding *Mean Value Theorem.*
6. All fractional derivatives do not obey the *Chain Rule*:

$$D_a^\alpha(f\circ g)(t) = f^{(\alpha)}\big(g(t)\big)g^{(\alpha)}(t).$$

The conformable fractional derivative has been applied in a variety of methods introduced to solve fractional differential equations. These methods include variational iteration method, sub-equation method, functional variable method, differential transform method.

In [22] different types of fractional-order logistic models in the framework of Caputo type fractional operators generated by conformable derivatives are discussed. In [23], R. W. Ibrahim et al. establish new analytic solution collections of nonlinear conformable time-fractional water wave dynamical equation in a complex domain. A new fractional model for the falling body problem has been suggested in [24], and [25]. The conformable two dimensional wave equation is solved by using differential transform method [26]. W. Chung [27], in his paper, uses the conformable fractional derivative and integral to give the fractional Newtonian mechanics. His model has been applied for fractional harmonic oscillator problem, the fractional damped oscillator problem, and the forced oscillator problem in the one-dimensional fractional dynamics. Since the conformable derivative is theoretically very more comfortable to handle, in [28], the mathematical modeling method for the fractional Bergman's model which involves fractional conformable derivative in Liouville–Caputo sense, and the fractional operators of Attangana–Baleanu–Caputo fractional derivative, is introduced. In many problems, analytic and exact solutions of fractional differential equations are not available, and numerical solutions are possible. Problems involving conformable derivative may be solved via shifted Legendre polynomials [29]. Sometimes the problem is to solve fractional conformable differential equation with integral boundary condition [30].

In the last few decades physicists, applied scientists and engineers realized that fractional differential equations provide a natural framework for fractional modeling of different processes such as viscoelastic systems, signal processing, diffusion processes, control processing, etc. [20,31–36]. Several authors have introduced the fractional derivative of complex functions. M. D. Ortigueira defines a generalized Caputo derivative for complex functions with respect to a given direction of the complex plane [37]. Since the Caputo definition is very welcome in applied science and engineering, C. Li et. al [38] generalize the Caputo derivative in real line to that in complex plane and discuss its properties. In [39], S. Owa discusses how to extend the fractional derivative to analytical functions on the unit circle $U = \{z\in\mathbb{C} : |z| < 1\}$.

In 2014, R. Khalil et al. introduces limit based for fractional derivative which is called conformable fractional derivative [11] which is not really fractional. T. Abdeljawad [40] develops the definitions and the basic concepts in this new simple interesting fractional calculus. He proposes and discusses the conformable fractional versions of chain rule, exponential functions, integration by parts, Taylor power series expansions, Laplace transforms and linear differential systems.

2. Main Results

Many complex functions are not complex differentiable of integer order at a point or some points. It is possible to define a new complex derivative which can be α-complex differentiable, but not differentiable. Let, for instance, $\alpha = 1/2$, $f(z) = z^{\alpha}|_{\alpha=1/2} = \sqrt{z} = z^{1/2}$ which is not differentiable at $z = 0$ ($f'(z)|_{z=0}$ does not exist). Can we define a new derivative which is differentiable at $z = 0$?

Definition 2.1. *A complex function $f(z)$ is conformable fractional differentiable at a point $z \in \mathbb{C}$ if and only if the following limiting difference quotient exists:*

$$T_{\alpha}(f)(z) = \lim_{\varepsilon \to 0} \frac{f(z + \varepsilon z^{1-\alpha}) - f(z)}{\varepsilon}$$

for all z, and $\alpha \in (0, 1)$. If f is α-differentiable in an open set U, and $\lim_{z \to 0} f^{(\alpha)}(z)$ exists, then define $f^{(\alpha)}(0) = \lim_{z \to 0} f^{(\alpha)}(z)$.

For $x > 0$ and $0 < \alpha < 1$ the function $x^{1-\alpha} = e^{(1-\alpha)\log x}$ is perfectly well defined and ready to handle. However, in the above definition the complex power function $z^{1-\alpha} = e^{(1-\alpha)\log z}$ in general is multiple valued. For a multiple valued function we can not talk about its derivatives unless we restrict ourselves to a single valued branch of logarithm, like principal value branch.

Theorem 2.1. *If a function $f(z)$ is α-differentiable at z_0 and $\alpha \in (0, 1]$, then f is continuous at z_0.*

Let $z \in \mathbb{C}$, $r = |z|$, and $\theta = \arg z$ then for all $n \in \mathbb{N}$ De Moivre's formula is

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Let $w \in \mathbb{C}$, $w \neq 0$ with $w = \rho(\cos\varphi + i\sin\varphi)$ then for n-th roots of $w = z^n$ there exist the number of n by the following formula

$$\sqrt[n]{z} = \sqrt[n]{re^{i\theta}} = r^{1/n}(e^{i\theta})^{1/n} = r^{1/n}\Big[\cos\Big(\frac{2k\pi + \theta}{n}\Big) + i\sin\Big(\frac{2k\pi + \theta}{n}\Big)\Big].$$

When one takes conformable fractional derivative of function $f(z)$ the following things can happen.

Remark 1. *Before derivation the function $f(z) = z^{1/n}$ at $z = z_0$ has "n" roots while after derivation $T_{\alpha}f(z)$ at $z = z_0$ has "m" roots.*

Example 2.1. *Let $n = 2$, $\alpha = 1/3$, so $m = 6$*

$$T_{\alpha}z^{1/n}|_{n=2,\alpha=1/3} = T_{1/3}z^{1/2} = (1/2)z^{1/2-1/3} = (1/2)z^{1/6}.$$

Remark 2. *Before derivation the function $f(z)$ is holomorphic on $\mathbb{C}$ (entire function) while after conformable fractional derivation, the function $T_{\alpha}f(z)$ is not entire function (but holomorphic on $\mathbb{C} \setminus \{z_0\}$).*

EXAMPLE 2.2. *Let $f(z) = z$ (holomorphic on $\mathbb{C}$), $\alpha = 1/3$, so*

$$T_\alpha f(z)|_{\alpha=1/3} = T_{1/3} f(z) = z^{2/3} \text{ (holomorphic on } \mathbb{C} \setminus \{0\}).$$

REMARK 3. *Before derivation the function $f(z)$ is holomorphic on $\mathbb{C}\setminus\{z_0\}$ while after conformable fractional derivation, the function $T_\alpha f(z)$ is entire function (holomorphic on $\mathbb{C}$).*

EXAMPLE 2.3. *Let $f(z) = z^{1/2}$, $\alpha = 1/2$, so*

$$T_\alpha f(z)|_{\alpha=1/2} = T_{1/2} f(z) = 1/2.$$

THEOREM 2.2. *Let $\alpha \in (0,1]$, and $f(z)$, $g(z)$ be α-differentiable at a point z_0. Then*

1. $T_\alpha\big(c_1 f(z) + c_2 g(z)\big) = c_1 T_\alpha f(z) + c_2 T_\alpha g(z)$ *for all* c_1, $c_2 \in \mathbb{C}$.
2. $T_\alpha(z^c) = c z^{c-\alpha}$ *for all* $c \in \mathbb{C}$.
3. $T_\alpha(\mu) = 0$ *for all constant functions* $f(z) = \mu$.
4. $T_\alpha\big(f(z)g(z)\big) = f(z) T_\alpha g(z) + g(z) T_\alpha f(z)$.
5. $T_\alpha\Big(\dfrac{f(z)}{g(z)}\Big) = \dfrac{g(z)T_\alpha f(z) - f(z) T_\alpha g(z)}{g^2(z)}$.
6. *If, in addition, f is analytic, then* $T_\alpha f(z)\big|_{z=z_0} = z_0^{1-\alpha} \frac{d}{dz} f(z_0)$.

Complex conformable fractional derivative of certain complex functions are as follows:

– $T_\alpha(z^c) = c z^{c-\alpha}$ for all $c \in \mathbb{C}$;
– $T_\alpha(1) = 0$;
– $T_\alpha(e^{cz}) = c z^{1-\alpha} e^{cz}$, $c \in \mathbb{C}$;
– $T_\alpha(\sin cz) = c z^{1-\alpha} \cos cz$, $c \in \mathbb{C}$;
– $T_\alpha(\cos cz) = c z^{1-\alpha} \sin cz$, $c \in \mathbb{C}$;
– $T_\alpha(\alpha^{-1} z^\alpha) = 1$.

Unlike the real line where there are only two directions to access a limiting point, in complex plane, there are an infinite variety of directions to access the point z. $\varepsilon = \lambda + i\omega$ accesses 0 through points in the plane not along the real axis or any line. The definition 2.1 requires that all of these "directional derivatives" must agree such that this requirement imposes severe restrictions on complex conformable fractional derivatives.

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic, then its first derivative is $f'(z)$ or $\frac{d}{dz} f(z)$, and we have Cauchy–Riemann equations for $u(x,y)$ and $v(x,y)$. However, if instead of $f'(z)$ we have conformable fractional derivative of the function $f(z)$, namely $T_\alpha f(z) = z^{1-\alpha} \frac{df}{dz}(z)$, then we can verify its real and imaginary parts for the generalized Cauchy–Rieman equations. So we give the following theorem as the necessary condition.

THEOREM 2.3. *A complex function $f(z) = u(x,y) + iv(x,y)$ depending on $z = x + iy$ has α-conformable fractional derivative of $f(z)$ of order α if and only if its real and imaginary parts are continuously differentiable and satisfy the following "conformable Cauchy–Riemann" equations:*

$$\operatorname{Re}(z^{1-\alpha}) \frac{\partial u}{\partial x} - \operatorname{Im}(z^{1-\alpha}) \frac{\partial v}{\partial x} = \operatorname{Re}(z^{1-\alpha}) \frac{\partial v}{\partial y} + \operatorname{Im}(z^{1-\alpha}) \frac{\partial u}{\partial y};$$

$$\operatorname{Im}(z^{1-\alpha})\frac{\partial u}{\partial x} + \operatorname{Re}(z^{1-\alpha})\frac{\partial v}{\partial x} = \operatorname{Im}(z^{1-\alpha})\frac{\partial v}{\partial y} - \operatorname{Re}(z^{1-\alpha})\frac{\partial u}{\partial y}.$$

Proof. As in classical approach, we first choose $\varepsilon = \lambda$ (namely $\omega = 0$) for which we have

$$T_\alpha f(z)\big|_{\varepsilon=\lambda} = \left[\operatorname{Re}(z^{1-\alpha}) + i\operatorname{Im}(z^{1-\alpha})\right]\frac{df(z)}{dz}\Big|_{\varepsilon=\lambda} = \\ = \left[\operatorname{Re}(z^{1-\alpha}) + i\operatorname{Im}(z^{1-\alpha})\right]\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right].$$

Now, choosing $\varepsilon = i\omega$ (namely $\lambda = 0$) gives us

$$T_\alpha f(z)\big|_{\varepsilon=i\omega} = \left[\operatorname{Re}(z^{1-\alpha}) + i\operatorname{Im}(z^{1-\alpha})\right]\frac{df(z)}{dz}\Big|_{\varepsilon=i\omega} = \\ = \left[\operatorname{Re}(z^{1-\alpha}) + i\operatorname{Im}(z^{1-\alpha})\right]\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]. \square$$

REMARK 4. *For the value of $\alpha = 1$, it gives*

$$\operatorname{Re}(z^{1-\alpha}) = \left[(x^2+y^2)^{(1-\alpha)/2}\cos[(1-\alpha)\arg(x+iy)]\right], \quad \lim_{\alpha\to 1}\operatorname{Re}(z^{1-\alpha}) = 1;$$
$$\operatorname{Im}(z^{1-\alpha}) = \left[(x^2+y^2)^{(1-\alpha)/2}\sin[(1-\alpha)\arg(x+iy)]\right], \quad \lim_{\alpha\to 1}\operatorname{Im}(z^{1-\alpha}) = 0.$$

So we obtain classical Cauchy–Riemann equations from their fractional counterpart.

3. Complex Conformable Differential Equations

Let $T_\alpha(z^c) = cz^{c-\alpha}$ for all $c \in \mathbb{C}$. Multiplying both sides by the coefficient c^{-1} gives

$$T_\alpha(c^{-1}z^c) = z^{c-\alpha} \text{ for all } c \in \mathbb{C}.$$

EXAMPLE 3.1. *If $c = \alpha$ we obtain the simplest complex conformable differential equation*:

$$T_\alpha f(z) - 1 = 0,$$

where its solution is $f(z) = \alpha^{-1}z^\alpha$.

In Fig. 1 vertical axis shows the real part of $f(z) = 2\sqrt{z}$ while he imaginary part of $f(z) = 2\sqrt{z}$ has been represented by the coloration of the points.

EXAMPLE 3.2. *Now let is calculate $T_\alpha(e^{\alpha^{-1}z^\alpha})$ which gives us $e^{\alpha^{-1}z^\alpha}$. So we deduce another complex conformable differential equation*:

$$T_\alpha g(z) - g(z) = 0,$$

where its solution is $g(z) = e^{\alpha^{-1}z^\alpha}$.

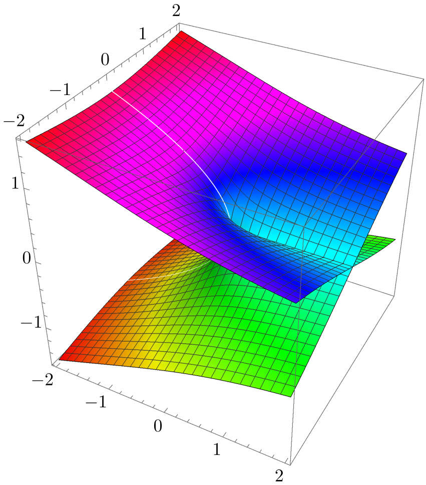

Figure 1. Riemann surface for the function $f(z) = 2\sqrt{z}$ for which $\alpha = 1/2$

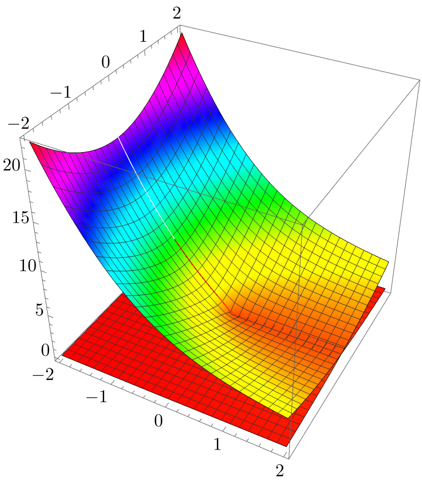

Figure 2. Riemann surface for the function $g(z) = e^{2\sqrt{z}}$ for which $\alpha = 1/2$

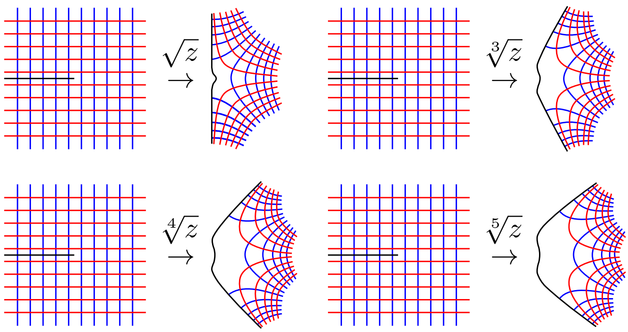

Figure 3. Mapping of lines by $z^{1/2}$, $z^{1/3}$, $z^{1/4}$, $z^{1/5}$ for $x \in [-1, 1]$

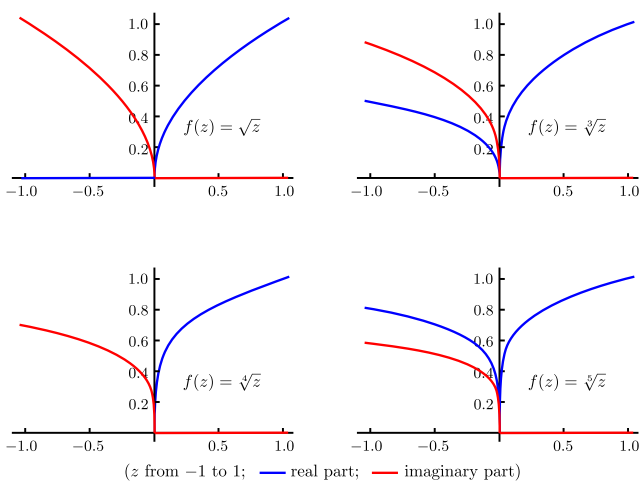

Figure 4. Real and imaginary parts of functions $z^{1/2}$, $z^{1/3}$, $z^{1/4}$, $z^{1/5}$

Similarly, in Fig. 2 vertical axis shows the real part of $g(z) = e^{2\sqrt{z}}$ while he imaginary part of $g(z) = e^{2\sqrt{z}}$ has been represented by the coloration of the points.

To solve boundary value problems for the Laplace equation, the study of analytic maps is very important. Comparing the plots with different values of α has been shown in Fig. 3 and Fig. 4.

Competing interests. We declare that we have no conflicts of interest in the authorship or publication of this paper.

Authors' contributions and responsibilities. Each author has participated in the article concept development and in the manuscript writing. The authors are absolutely responsible for submitting the final manuscript in print. Each author has approved the final version of manuscript. Mrs Ronak Pashaei is Ph.D. student in mathematics whose supervisors are Dr. Mohammad Sadegh Asgari and Dr. Amir Pishkoo, and her advisor is Dr. Davood Ebrahimi Bagha.

Acknowledgments. The authors are grateful to the reviewers and the editor for valuable remarks which contributed to the improvement of the paper.

References

1. Tenreiro Machado J., Kiryakova V., Mainardi F. A poster about the recent history of fractional calculus, *Fract. Calc. Appl. Anal.*, 2010, vol. 13, no. 3, pp. 329–334, http://eudml.org/doc/219594.
2. Samko S. G., Kilbas A. A., Marichev O. I. *Fractional Integrals and Derivatives: Theory and Applications*. Neq York, Gordon and Breach Science Publ., 1993, xxxvi+976 pp.
3. Oldham K., Spanier J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Mathematics in Science and Engineering, vol. 111. New York, London, Academic Press, 1974, xiii+234 pp.
4. de Oliveira E. C. *Solved Exercises in Fractional Calculus*, Studies in Systems, Decision and Control, vol. 240. Cham, Springer, 2019, xviii+321 p. doi: 10.1007/978-3-030-20524-9.
5. Tenreiro Machado J., Kiryakova V., Mainardi F. Recent history of fractional calculus, *Comm. Nonlinear Science Numerical Simulation*, 2011, vol. 16, no. 3, pp. 1140–1153. doi: 10.1016/j.cnsns.2010.05.027.
6. Kilbas A. A., Srivastava H. M., Trujillo J. J. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Amsterdam, Elsevier, 2006, xv+523 pp.
7. Podlubny J. *Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, vol. 198. San Diego, CA, Academic Press, 1999, xxiv+340 pp.
8. Miller K. S., Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York, John Wiley and Sons, 1993, xiii+366 pp.
9. Pashaei R., Asgari M., Pishkoo A. Conformable derivatives in Laplace equation and fractional Fourier series solution, *Int. Ann. Sci.*, 2019, vol. 9, no. 1, pp. 1–7. doi: 10.21467/ias.9.1.1-7.
10. Ortigueira M. D. *Fractional Calculus for Scientists and Engineers*, Lecture Notes in Electrical Engineering, vol. 84. Dordrecht, Springer, 2011, xiv+152 p. doi: 10.1007/978-94-007-0747-4.
11. Khalil R., Al Horani M., Yousef A., Sababheh M. A new definition of fractional derivative, *J. Comp. Appl. Math.*, 2014, vol. 264, pp. 65–70. doi: 10.1016/j.cam.2014.01.002.
12. Katugampola U. N. *A new fractional derivative with classical properties*, 2014, arXiv: 1410.6535 [math.CA].

13. Ortigueira M. D., Tenreiro Machado J. A. What is a fractional derivative?, *J. Comp. Phys.*, 2015, vol. 293, pp. 4–13. doi: 10.1016/j.jcp.2014.07.019.
14. Tarasov V. E. No nonlocality. No fractional derivative, *Commun. Nonlinear Sci. Numer. Simulat.*, 2018, vol. 62, pp. 157–163, arXiv: 1803.00750 [math.CA]. doi: 10.1016/j.cnsns.2018.02.019.
15. Tarasov V. E. Electromagnetic fields on fractals, *Modern Phys. Lett. A.*, 2006, vol. 21, no. 20, pp. 1587–1600, arXiv: 0711.1783. doi: 10.1142/S0217732306020974.
16. Li J., Ostoja-Starzewski M. Fractal solids, product measures and fractional wave equations, *Proc. R. Soc. A.*, vol. 465, no. 2108, pp. 2521–2536. doi: 10.1098/rspa.2009.0101.
17. Ostoja-Starzewski M., Li J. Fractal materials, beams, and fracture mechanics, *Z. angew. Math. Phys.*, 2009, vol. 60, no. 6, pp. 1194–1205. doi: 10.1007/s00033-009-8120-8.
18. Ostoja-Starzewski M. Electromagnetism on anisotropic fractal media, *Z. angew. Math. Phys.*, 2013, vol. 64, no. 2, pp. 381–390. doi: 10.1007/s00033-012-0230-z.
19. Ostoja-Starzewski M., Li J., Joumaa H., Demmie P. N. From fractal media to continuum mechanics, *Z. angew. Math. Mech.*, vol. 94, no. 5, pp. 373–401. doi: 10.1002/zamm.201200164.
20. Ortigueira M. D. *Fractional Calculus for Scientists and Engineers*, Lecture Notes in Electrical Engineering, vol. 84. Dordrecht, Springer, 2011, xiv+154 p. doi: 10.1007/978-94-007-0747-4.
21. Olver P. J. *Complex Analysis and Conformal Mapping*, Lecture Notes, 2018, 84 pp., http://www-users.math.umn.edu/~olver/ln_/cml.pdf.
22. Abdeljawada T., Al-Mdallal Q. M., Jarad F. Fractional logistic models in the frame of fractional operators generated by conformable derivatives, *Chaos, Solitons and Fractals*, 2019, vol. 119, pp. 94–101. doi: 10.1016/j.chaos.2018.12.015.
23. Ibrahim R. W., Meshram C., Hadid S. B., Momani S. Analytic solutions of the generalized water wave dynamical equations based on time-space symmetric differential operator, *J. Ocean Eng. Sci.*, 2020, vol. 5, no. 2, pp. 186–195. doi: 10.1016/j.joes.2019.11.001.
24. Ebaid A., Masaedeh B., El-Zahar E. A new fractional model for the falling body problem, *Chinese Phys. Lett.*, vol. 34, no. 2, 020201. doi: 10.1088/0256-307X/34/2/020201.
25. Alharbi F.M., Baleanu D., Ebaid A. Physical properties of the projectile motion using the conformable derivative, *Chinese J. Phys.*, 2019, vol. 58, pp. 18–28. doi: 10.1016/j.cjph.2018.12.010.
26. Kaabar M.K.A. *Novel methods for solving the conformable wave equation*, 2019, https://hal.archives-ouvertes.fr/hal-02267015.
27. Chung W. S. Fractional Newton mechanics with conformable fractional derivative, *J. Comp. Appl. Math.*, 2015, vol. 290, pp. 150–158. doi: 10.1016/j.cam.2015.04.049.
28. Morales-Delgado V. F., Gómez-Aguilar J. F., Taneco-Hernández M. A. Mathematical modeling approach to the fractional Bergman's model, *Discrete Contin. Dyn. Syst. Ser. S*, 2020, vol. 13, no. 3, pp. 805–821. doi: 10.3934/dcdss.2020046.
29. Çerdik Yaslan H., Mutlu F. Numerical solution of the conformable differential equations via shifted Legendre polynomials, *Int. J. Comp. Math.*, 2020, vol. 97, no. 5, pp. 1016–1028. doi: 10.1080/00207160.2019.1605059.
30. Meng S., Cui Y. The extremal solution to conformable fractional differential equations involving integral boundary condition, *Mathematics*, 2019, vol. 7, no. 2, 186. doi: 10.3390/math7020186.
31. Sibatov R. T. Anomalous grain boundary diffusion: Fractional calculus approach, *Adv. Math. Phys.*, vol. 2019, 8017363, 9 p. doi: 10.1155/2019/8017363
32. Alikhanov A. A. A new difference scheme for the time fractional diffusion equation, *J. Comp. Phys.*, 2015, vol. 280, pp. 424–438, arXiv: 1404.5221 [math.NA]. doi: 10.1016/j.jcp.2014.09.031.
33. Amanov D., Ashyralyev A. Initial-boundary value problem for fractional partial differential equations of higher order, *Abstract and Applied Analysis*, vol. 2012, 973102, 16 p. doi: 10.1155/2012/973102

34. *Application of Fractional Calculus in Physics*, ed. R. Hilfer. Singapore, World Scientific, 2000. doi: 10.1142/3779.
35. Muslih S. I., Baleanu D. Fractional multipoles in fractional space, *Nonlinear Anal. RWA*, 2007, vol. 8, no. 1, pp. 198–203. doi: 10.1016/j.nonrwa.2005.07.001.
36. Tavazoei M. S., Haeri M. Stabilization of unstable fixed points of chaotic fractional order systems by a state fractional PI controller, *Eur. J. Control*, 2008, vol. 14, no. 3, pp. 247–257. doi: 10.3166/ejc.14.247-257.
37. Ortigueira M. D. A coherent approach to non-integer order derivatives, *Signal Processing*, 2006, vol. 86, no. 10, pp. 2505–2515. doi: 10.1016/j.sigpro.2006.02.002.
38. Li C., Dao X, Guoa P. Fractional derivatives in complex planes, *Nonlinear Analysis: Theory, Methods, Applications*, 2009, vol. 71, no. 5–6, pp. 1857–1869. doi: 0.1016/j.na.2009.01.021.
39. Owa S. Some properties of fractional calculus operators for certain analytic functions, *RIMS Kôkyûroku*, 2009, vol. 1626, pp. 86–92, http://hdl.handle.net/2433/140314.
40. Abdeljawad T. On conformable fractional calculus, *J. Comp. Appl. Math.*, 2015, vol. 279, pp. 57–66. doi: 10.1016/j.cam.2014.10.016.

Вестн. Сам. гос. техн. ун-та. Сер. Физ.-мат. науки. 2020. Т. 24, № 2. С. 379–389
ISSN: 2310-7081 (online), 1991-8615 (print) https://doi.org/10.14498/vsgtu1734

УДК 517.548

α-Дифференцируемые функции в комплексной плоскости

***R. Pashaei*[1], *A. Pishkoo*[2], *M. S. Asgari*[1], *D. Ebrahimi Bagha*[1]**

[1] Islamic Azad University, Central Tehran Branch, Tehran, Iran.
[2] Nuclear Science and Technology Research Institute, Tehran, Iran.

Аннотация

В комплексной плоскости вводится взвешенная дробная производная порядка α. Относительно многозначной функции $z^{1-\alpha}$ получены дробные уравнения Коши–Римана, которые при $\alpha = 1$ совпадают с классическими уравнениями Коши–Римана. Для некоторых функций в комплексной плоскости рассмотрены свойства, относящиеся к комплексной взвешенной дробной производной. Обсуждаются два комплексных дифференциальных уравнения специальной формы. Для некоторых значений α приводятся римановы поверхности их решений и сравниваются их графики.

Ключевые слова: взвешенная дробная производная, уравнения Коши–Римана, предельная дробная производная.

Получение: 9 августа 2019 г. / Исправление: 19 февраля 2020 г. / Принятие: 16 марта 2020 г. / Публикация онлайн: 25 мая 2020 г.

Конкурирующие интересы. Мы заявляем, что у нас нет конфликта интересов в отношении авторства и публикации этой статьи.

Авторская ответственность. Мы несем полную ответственность за предоставление окончательной рукописи в печать. Каждый из нас одобрил окончательную версию рукописи. Ronak Pashaei — соискатель степени Ph.D. в математике. Доктор Mohammad Sadegh Asgari и доктор Amir Pishkoo являются её руководителями. Доктор Davood Ebrahimi Bagha является её советником.

Благодарности. Авторы благодарны рецензентам и редактору за ценные замечания, которые способствовали улучшению рукописи статьи.

Краткое сообщение

Образец для цитирования
Pashaei R., Pishkoo A., Asgari M. S., Ebrahimi Bagha D. α-Differentiable functions in complex plane, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.], 2020, vol. 24, no. 2, pp. 379–389. doi: 10.14498/vsgtu1734.

Сведения об авторах

Ronak Pashaei; Ph.D. Student in Mathematics (Analysis); Dept. of Mathematics; Faculty of Basic Sciences

Amir Pishkoo https://orcid.org/0000-0001-6211-5493
Physics and Accelerators Research School; e-mail: apishkoo@gmail.com

Mohammad Sadegh Asgari https://orcid.org/0000-0002-0675-0262
Associate Professor of Pure Mathematics; Dept. of Mathematics; Faculty of Basic Sciences; e-mail: moh.asgari@iauctb.ac.ir

Davood Ebrahimi Bagha; Professor in Mathematics; Dept. of Mathematics; Faculty of Basic Sciences; e-mail: dav.ebrahimi_bagha@iauctb.ac.ir