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PROPERTIES OF LOCALLY CYCLIC GROUPS

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Locally cyclic group is a group every finite set of elements of which generates a cyclic subgroup. We give examples of periodic locally cyclic groups and locally cyclic torsion-free groups. Properties of locally cyclic groups are studied. A locally cyclic group cannot be mixed, that is, it cannot contain elements of finite and infinite order simultaneously. A locally cyclic group is Abelian. By their properties periodic locally cyclic groups and locally cyclic torsion-free groups are distinguished. The Sylow subgroups of a periodic locally cyclic group are cyclic or quasi-cyclic. A periodic locally cyclic group decomposes into a direct product of Sylow subgroups. By N. F. Sesekin and A. I. Starostin the following theorem is proved: a locally finite group, all Sylow p-subgroups of which are quasi-cyclic, is a complete periodic locally cyclic group. Here, in addition to this theorem, we consider the structure of a complete periodic locally cyclic group. A complete periodic locally cyclic group decomposes into a direct product of quasi-cyclic subgroups with distinct prime numbers. A complete periodic locally cyclic group is uniquely reconstructed by its lower layer. In this article an example is given of the fact that an arbitrary periodic locally cyclic group is not unique reconstructed by its lower layer. A torsion-free locally cyclic group is isomorphic to a subgroup of the additive group of rational numbers. A periodic locally cyclic group is layer-finite, that is a number of it's elements of each order is finite. A locally cyclic group can be either a layer-finite or a subgroup of additive groups of rational numbers. The results can be applied when encoding information in space communications.

Keywords: periodic group, locally cyclic group, quasi-cyclic group, complete group, layer finiteness.

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СВОЙСТВА ЛОКАЛЬНО-ЦИКЛИЧЕСКИХ ГРУПП

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Локально-циклическая группа – это группа, всякое конечное множество элементов которой порождает циклическую подгруппу. Приводятся примеры периодических локально-циклических групп и локально-циклических групп без кручения. Изучаются свойства локально-циклических групп. Локально-циклическая группа не может быть смешанной, т. е. она не может содержать одновременно элементы конечного и бесконечного порядка. Локальная-циклическая группа является абелевой. По своим свойствам различаются периодические локально-циклические группы и локально-циклические группы без кручения. Силовские подгруппы периодической локально-циклической группы являются циклическими или квазициклическими. Периодическая локально-циклическая группа разлагается в прямое произведение силовских подгрупп. Н. Ф. Сесекиным и А. И. Старостиным доказана теорема: локально-конечная группа, все силовские p-подгруппы которой квазицикличны, является полной периодической локально-циклической группой. Здесь в дополнение к этой теореме мы рассмотрим структуру полной периодической локально-циклической группы. Полная периодическая локально-циклическая группа разлагается в прямое произведение квазициклических p-подгрупп по различным простым числам p. Полная периодическая локально-циклическая группа единственным образом восстанавливается по своему нижнему слою. Приводится пример того, что произвольная периодическая локально-циклическая группа не единственным образом восстанавливается по своему нижнему слою. Локальная циклическая группа без кручения изоморфна некоторой подгруппе аддитивной группы рациональных чисел. Периодическая локально-циклическая группа слойно конечна, т. е. в ней конечно число элементов каждого порядка. Локально-

циклическая группа может быть либо слойно конечной, либо подгруппой аддитивной группы рациональных чисел. Результаты могут найти применение при кодировании информации, использующейся в сеансах космической связи.

Ключевые слова: периодическая группа, локально-циклическая группа, квазициклическая группа, полная группа, слойная конечность.

Introduction. The aim of this paper is to establish the basic properties of the locally cyclic group classes.

Definition. A group is said to be locally cyclic if every finite set of its elements generates a cyclic subgroup. Example of locally cyclic groups is an additive group of rational numbers, a quasi-cyclic group.

A quasi-cyclic group is a G group which is constructed as follows: supposing a cyclic group $\langle a_1 \rangle$ of prime order p is embedded in a cyclic group $\langle a_2 \rangle$ of order p^2, the last group, in its turn, is embedded in a cyclic group $\langle a_3 \rangle$ of order p^3 etc. The group is formed by combining an infinite chain of nested groups $\langle a_1 \rangle \subset \langle a_2 \rangle \subset ... \subset \langle a_n \rangle \subset ...$ In other words, the quasi-cyclic group can be defined as root groups of a unit of prime order p, order p^2, order p^3, etc.

The class of periodic locally cyclic groups is contained in the class of layer-finite groups whose properties are described in [1]. Layer-finite groups were investigated by S. N. Chernikov [2–4], R. Baer [5], Kh. Kh. Mukhamedzhan [6], Ya. D. Polovitsky [7]. Near layer-finite groups are described in the works of the author [8–11]. As S. N. Chernikov pointed out in the mathematical encyclopedia, layer-finite groups proved to be the most studied ones among the groups with finite classes of conjugate elements. This implies the finiteness of the conjugate elements classes in periodic locally cyclic groups. The arrangement of layer-finite groups with related classes is shown in the research paper [12].

Basic results. In this section we are going to consider the properties of locally cyclic groups.

A locally cyclic group cannot be mixed, that is, it cannot contain elements of finite and infinite order simultaneously:

Property 1. *Locally cyclic group can be either periodical or a torsion-free group.*

Evidence. It is obviously seen from the definition that a locally cyclic group cannot be mixed, that is, contain elements of both finite and infinite orders. Indeed, the subgroup, generated by an element of finite order and an element of infinite order, according to the definition of a locally cyclic group, must be cyclic. Thus there is a contradiction with the theorem on the structure of subgroups of an infinite cyclic group. It should be noted that each non-identity subgroup of an infinite cyclic group is infinite cyclic one. The property is proved.

Property 2. *Locally cyclic group is Abelian.*

Evidence. The validity of the statement immediately goes from the definition of a locally cyclic group.

Periodic locally cyclic groups and locally cyclic torsion-free groups differ substantially according to their properties.

At first, it is necessary to distinguish the properties of periodic locally cyclic groups.

Property 3. *Sylow p-subgroups of periodic locally cyclic group G are cyclic or quasi-cyclic.*

Evidence. Supposing that $\Im$ is a set of all *p*-elements of a locally cyclic group G for some prime number *p*. If $\Im$ is a finite set, then according to the definition of a locally cyclic group its elements generate a cyclic group. In the case if $\Im$ is an infinite set, we will prove that they form a quasi-cyclic group. Indeed, let us consider that $\Im$ is an infinite set. Let us deal with a subset $\{a_1, a_2, ..., a_n, ...\}$ of the set $\Im$. Group $\langle a_1, a_2 \rangle$, generated by the elements a_1 and a_2 is a cyclic p-subgroup, therefore, either $\langle a_1 \rangle \supseteq \langle a_2 \rangle$ or $\langle a_1 \rangle \subseteq \langle a_2 \rangle$. The bigger one of these groups is defined as $\langle a_{i_1} \rangle$. The same idea can be referred to the cyclic subgroups $\langle a_{i_1} \rangle$, $\langle a_3 \rangle$. The bigger one of these groups is defined as $\langle a_{i_2} \rangle$.

If we continue the argumentation in the same way, we will get an increasing chain of cyclic *p*-subgroups:

$$\langle a_{i_1} \rangle \subseteq \langle a_{i_2} \rangle \subseteq \langle a_{i_3} \rangle \subseteq ...$$

Obviously, the union A of these subgroups is a quasi-cyclic *p*-group. If not all elements of the set $\Im$ are in A, then the group, generated by an element $b \in \Im \setminus A$ and an element a of the same order of A is not cyclic. A contradiction with the definition of a locally cyclic group means that all elements of the set $\Im$ are contained in A. The property is proved.

Property 4. A periodic locally cyclic group decompounds into a direct product of Sylow subgroups.

Obviously, ***the evidence*** of this property follows from the fact that the periodic locally cyclic group is Abelian according to the property 2. The example of periodic locally cyclic group, which is different from quasi-cyclic group, is a direct product $\langle a_1 \rangle \times \langle a_2 \rangle \times ...$ of cyclic groups $\langle a_1 \rangle$, $\langle a_2 \rangle$, … of different direct orders p_1, p_2, … respectively.

Property 5. *A periodic locally cyclic group decompounds into a direct product of cyclic or quasi-cyclic p-subgroups with distinct prime numbers.*

Evidence of this property goes from the properties 3 and 4.

Locally finite groups are studied in the work of N. F. Sesekin and A. I. Starostin [13], all their Sylow-subgroups are quasi-cyclic.

Property 6. *Locally finite group, whose Sylow p-subgroups are quasi-cyclic, is a complete periodic locally cyclic group.*

In addition to this theorem, we are going to consider the structure of a complete periodic locally cyclic group:

Property 7. *A complete periodic locally cyclic group decompounds into a direct product of quasi-cyclic p-subgroups with distinct prime numbers.*

Evidence. A complete Abelian group decompounds into a direct product of quasi-cyclic *p*-subgroups and an additive group of rational numbers according to the theorem 9.1.6 from [14]. Since the additive group of rational numbers is not a periodic group we obtain a property statement (validity) using property 5. The property is proved.

Property 8. *A complete periodic locally cyclic group is uniquely recovered from its lower layer.*

The evidence *of this property goes from the property 7.*

At the same time, not every periodic locally cyclic group is uniquely recovered from its lower layer. For example, in two different periodic locally cyclic groups $C_{p^2} \times C_{q^\infty}$ *and* $C_{p^\infty} \times C_{q^2}$ *the lower layers are the same.*

Let us give one more characteristic of periodic locally cyclic groups developed by N. F. Sesekin and A. I. Starostin:

Property 9. *Only periodic locally cyclic groups are Abelian locally finite groups, all their Sylow p-subgroups are locally cyclic [13].*

Here it is necessary to give a description of a group that can be decompounded into a semidirect product of periodic locally cyclic groups:

Property 10. *If the group* G *is decompounded into a semidirect product of two periodic locally cyclic groups* $G = A\lambda B$*, whose orders of elements are relatively prime, then the* G *– locally finite group, whose Sylow p-subgroups are locally cyclic, and* A *contain the commutant of the group* G *[13].*

Let us now describe locally torsion-free cyclic groups.

Locally cyclic groups are groups of the first rank. Taking this into account that the following property is proved in the theorem 7.2.1 from [14]:

Property 11. *A local torsion-free cyclic group is isomorphic a subgroup of the additive group of rational numbers.*

Property 12. *A complete locally cyclic torsion-free group is isomorphic in relation to the additive group of rational numbers.*

Evidence. A complete locally cyclic torsion-free group in compliance with the property 2 and the theorem 9.1.6 from [14] decompounds into a direct product of quasi-cyclic *p*-subgroups and an additive group of rational numbers. Since quasi-cyclic subgroups of the direct product are not torsion-free groups, we can see the validity of the property. The property is proved.

Property 13. *A periodic locally cyclic group is layer-finite.*

Evidence. Let us consider that G is a periodic locally cyclic group. If $a_1, a_2, ..., a_n$ are elements of the order m of the group G, then the subgroup H, generated by these elements, is cyclic according to the definition of a locally cyclic group. Obviously, there is not more than $m-1$ elements of the order m in a cyclic group. Thus, we can say that $n \le m-1$ and this also means that the *m*-layer of the group *G* is finite and is *G* is a layer-finite group. The property is proved.

Property 14. *A locally cyclic group can be either leaf-finite or a subgroup of additive groups of rational numbers.*

The evidence of this property follows from the properties 1, 11, 13.

Definitions. Let us define the terms that were used in the work. A group *G* has a *finite rank r*. If *r* is the smallest number with the property defining that every finite generated subgroup of a group *G* is a part of a subgroup which contains r' components ($r' \le r$).

The lower layer of a group is the set of all its elements of prime order. A group is said to be layer-finite if the set of its elements of any given order is finite. The definitions of other terms used are standard and can be found in monographs [14; 15].

The conclusion. The main properties of locally cyclic groups are researched in this work. Periodic locally cyclic groups and locally cyclic torsion-free groups are described separately. Various examples of such groups are given.

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