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ABOUT TORSION OF PARALLELEPIPED AROUND THREE AXIS

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The theory of limit state deals with statically determinate condition of solids. In this case the system is closed due to extreme conditions, such properties of matter such as viscosity, elasticity, etc. cannot influence the limit state. In other words, when reaching the limit state the nature of the relationship between stress and strain has no effect on the ultimate state. The study of such systems has been consistently pursued by D. D. Ivlev and his coauthors. To the equilibrium equations they attached two or an equation relating the components of the stress tensor. This led to the closure of the system of equilibrium equations. In the theory of plasticity equations, which are closed with a single yield stress are studied well. The most well-known system describing the ultimate state of deformable bodies are well-studied equations describing the torsion of the plastic bodies, the two-dimensional stationary problem of the theory of plasticity. The article discusses some other systems of equations which are closed only by one equation of flow, which corresponds to the classical theory of plasticity. It is assumed that the components of the velocity vector depend only on two spatial coordinates. In addition, for the component of velocity vector conditions of deformations compatibility are performed identically. The constructed systems can be used to describe the twisting of the parallelepiped around the three orthogonal axes. For the constructed system of equations point group symmetries, conservation laws have been found. It is shown that the system allows 8 -dimensional Lie algebra. On the basis of the symmetry group some classes of invariant solutions of rank 1 have been constructed. They depend on arbitrary functions of one variable. It is shown that these solutions can be used to describe plastic torsion of a parallelepiped around three orthogonal axes. It is shown that the system admits infinite series of conservation laws. The concluding paragraph describes the construction of elastic solutions to the problem. It is shown that it boils down to finding three harmonic functions.

Keyword: plasticity theory, limit state, the exact solutions.

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О КРУЧЕНИИ ПАРАЛЛЕЛЕПИПЕДА ВОКРУГ ТРЕХ ОСЕЙ

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Теория предельного состояния имеет дело со статически определимым состоянием твердых тел. В этом случае система замкнута за счет предельных условий, и такие свойства материи, как вязкость, упругость и т. п., на предельное состояние влиять не могут. Другими словами, при достижении предельного состояния характер связи между напряжениями и деформациями не оказывает влияния на предельное состояние. Исследование таких систем последовательно проводил Д. Д. Ивлева и его соавторы. К уравнениям равновесия они присоединяли два или уравнения, связывающие компоненты тензора напряжений. Это приводило к замкнутости системы уравнений равновесия. В теории пластичности хорошо изучены уравнения, которые замыкаются одним пределом текучести. К наиболее известным системам, описывающим предельное состояние деформируемых тел, относятся хорошо исследованные уравнения, описывающие кручение пластических тел, двумерные задачи стационарной теории пластичности. Рассмотрены некоторые другие системы уравнений, которые замыкаются только одним уравнениям текучести, что соответствует классической теории пластичности. Предполагается, что компоненты вектора скоростей зависят только от двух пространственных координат. При этом для компонент вектора скорости деформаций выполняются тождественно условия совместности деформаций. Построенные системы могут быть использованы для описания кручения параллелепипеда вокруг трех ортогональных осей. Для построенной системы уравнений найдены точечные группы симметрий, законы сохранения. Показано, что система допускает восьмимерную алгебру Ли. На основе группы симметрий построены некоторые классы инвариантных решений ранга 1. Они зависят от произвольных функций одной переменной. Показано, что эти решения можно использовать для описания пластического кручения параллелепипеда вокруг трех ортогональных осей. Показано, что система допускает бесконечную серию законов *сохранения. Описано построение упругого решения поставленной задачи. Показано, что оно сводится к нахождению трех гармонических функций.*

Ключевые слова: теория пластичности, предельное состояние, точные решения.

Introduction. Some tasks of the deformable solid body mechanics are studied rather well. These are so-called statically definable tasks. These tasks deal with torsion of prismatic bars and with plane strain state. They belong to the wide range of tasks – limit state of deformable bodies. The theory of the limit state is one of the fundamental sections of the deformable solid body mechanics [1]. The theory of the limit state deals with statically definable state of solid bodies. In this case the system is closed at the expense of limit conditions and such properties of matter as viscosity, elasticity, etc. cannot influence the limit state. In other words once the limit state has been achieved, the nature of relation between stress and strain does not influence the limit state. Some of such systems are considered in [1–3].

In the first part one system of plasticity equations which describes the limit state is considered. This system can be used for the description of plastic current around three orthogonal axes.

Problem setting. Suppose $x = x_1$, $y = x_2$, $z = x_3$ – orthogonal axes, u, v, w – components of velocity deformation vector, e_{ij} – components of velocity deformation tensor, σ_{ij} – components of stress tensor. Components of stress tensor conforms with the equilibrium equations

$$\partial_i \sigma_{ij} = 0. \tag{1}$$

On the repeating indexes summing is supposed. Deviator of stress tensor and deformation velocity tensor are coaxial

$$\sigma_{ij} - \delta_{ij} p = \lambda e_{ij}, \tag{2}$$

where δ_{ij} – Kronecker delta; λ – unspecified nonnegative function, $3p = \sigma_{ij}$.

Equation system (1)–(2) closes by Mises yield condition

$$\left(\sigma_{11} - p\right)^2 + \left(\sigma_{22} - p\right)^2 + \left(\sigma_{33} - p\right)^2 + \\ + 2\left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right) = 2k_S^2. \tag{3}$$

It is known [1] that in case of prismatic bar torsion around oz axis, the field of deformation velocities is as folows

$$u = -yz,\ v = xz,\ w = w(x,y). \tag{4}$$

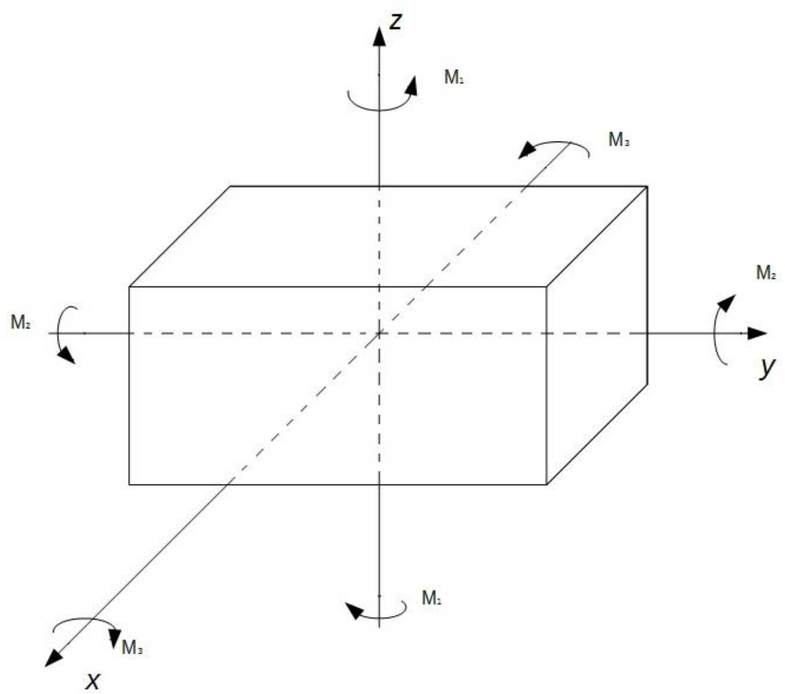
Generalizing ratios (1) we will demand

$$u = u(y,z),\ v = v(x,z),\ w = w(x,y). \tag{5}$$

We will construct the system of equations corresponding to the field of deformation velocities. As a result we receive the following system which will be researched in the study presented

$$\partial_y \tau^1 + \partial_z \tau^2 = \partial_x p,\ \ \partial_x \tau^1 + \partial_z \tau^3 = \partial_y p, \\ \partial_x \tau^2 + \partial_y \tau^3 = \partial_z p,\ \left(\tau^1\right)^2 + \left(\tau^2\right)^2 + \left(\tau^3\right)^2 = k_S^2. \tag{6}$$

The system of equations (6) can be used, in particular, for the description of rectangular parallelogram in the plastic state torsion around three axes (see a figure.).



The torsion of parallelepiped around the three axes

Кручение параллелепипеда вокруг трех осей

We will assume that the parallelogram twists around axes of ox, oy, oz in equal and opposite pairs of forces with the moments M_1, M_2, M_3. At the same time there are some limit moments M_1, M_2, M_3 when the parallelepiped passes into the plastic state and begins to twist. From the system (6) it is visible that such task is statically definable and can serve for limit moment values finding via formulas

$$M_1^* = \iint \left(y\tau^2 - z\tau^1 \right) dydz,$$

$$M_2^* = \iint \left(z\tau^1 - x\tau^3 \right) dxdz, \quad (7)$$

$$M_3^* = \iint \left(x\tau^3 - y\tau^2 \right) dydz.$$

Except the moments (7) the body is affected by hydrostatic pressure

$$P\big|_{\sum} = P_0,$$

$\sum$ – lateral surface of a parallelepiped.

We will now study some properties of the system (7).

1. Characteristic surfaces of system (6).

The system (6) contains the finite ratio, connecting values τ^1, τ^2, τ_3. After having differentiated it on x, y, z we receive the system

$$\partial_y \tau^1 + \partial_z \tau^2 = \partial_x p, \ \ \partial_x \tau^1 + \partial_z \tau^3 = \partial_y p,$$

$$\partial_x \tau^2 + \partial_y \tau^3 = \partial_z p,$$

$$\tau^1 \partial_x \tau^1 + \tau^2 \partial_x \tau^2 + \tau^3 \partial_x \tau^3 = 0, \quad (8)$$

$$\tau^1 \partial_y \tau^1 + \tau^2 \partial_y \tau^2 + \tau^3 \partial_y \tau^3 = 0,$$

$$\tau^1 \partial_z \tau^1 + \tau^2 \partial_z \tau^2 + \tau^3 \partial_z \tau^3 = 0.$$

Let's represent the equation of the characteristic surface of the equation system (9) as

$$\psi = \psi(x, y, z), \quad (9)$$

Characteristic surfaces of the system (8) are found from the determinant

$$\begin{vmatrix} \partial_x \psi & \partial_y \psi & \partial_z \psi & 0 \\ \partial_y \psi & \partial_x \psi & 0 & \partial_z \psi \\ \partial_z \psi & 0 & \partial_x \psi & \partial_y \psi \\ 0 & \tau^1 & \tau^2 & \tau^3 \end{vmatrix} = 0. \quad (10)$$

Note. It is easy to see that all three latter equations of the system (8) give identical lines in the determinant (10).

Expending the determinant (10) on the last line we receive

$$\tau^1 \partial_z \psi \left((\partial_z \psi)^2 - (\partial_x \psi)^2 - (\partial_y \psi)^2 \right) +$$

$$+ \tau^2 \partial_y \psi \left((\partial_y \psi)^2 - (\partial_x \psi)^2 - (\partial_z \psi)^2 \right) +$$

$$+ \tau^3 \partial_x \psi \left((\partial_x \psi)^2 - (\partial_y \psi)^2 - (\partial_z \psi)^2 \right) = 0.$$

This equation can be written as

$$\tau^1 n_3 \left(2n_3^2 - 1 \right) + \tau^2 n_2 \left(2n_2^2 - 1 \right) + \tau^3 n_1 \left(2n_1^2 - 1 \right) = 0, \quad (11)$$

where $n_1 = \dfrac{\partial_x \psi}{\sqrt{(\nabla \psi)^2}}, \ \ n_2 = \dfrac{\partial_y \psi}{\sqrt{(\nabla \psi)^2}}, \ \ n_3 = \dfrac{\partial_z \psi}{\sqrt{(\nabla \psi)^2}}.$

One of solutions of the equation (11) which does not depend on values τ^1, τ^2, τ^3 is

$$2n_i^2 = 1, \qquad i = 1, 2, 3.$$

Therefore, an angle between the normal to a characterristic surface $\psi(x, y, z) = 0$ and vector n equals $\pm \pi/4$. Set of elements of the characteristic surface forms the solution cone $\pm \pi/4$ around the direction which is defined by the third root of the equation (11) and depends on tension.

Point symmetries of the equation system (6). Point symmetries are widely used in the studies of differential equations. Necessary data on symmetries and their application to the equations of plasticity and elastic plasticity can be found in [4–8]. Since the system (6) contains finite ratio, we should work with its consequences, which looks like (9), where for convenience the following designations are entered

$$\partial_x \tau^1 = q_1^1, \ \partial_x \tau^2 = q_1^2, \ \partial_x \tau^3 = q_1^3, \ \partial_x p = q_1^4 \ \text{etc.},$$

$$q_2^1 + q_3^2 = q_1^4, \ \ q_1^1 + q_3^3 = q_2^4, \ \ q_1^2 + q_2^3 = q_3^4,$$

$$\tau^1 q_1^1 + \tau^2 q_1^2 + \tau^3 q_1^3 = 0,$$

$$\tau^1 q_2^1 + \tau^2 q_2^2 + \tau^3 q_2^3 = 0, \quad (12)$$

$$\tau^1 q_3^1 + \tau^2 q_3^2 + \tau^3 q_3^3 = 0,$$

$$\left(\tau^1\right)^2 + \left(\tau^2\right)^2 + \left(\tau^3\right)^2 = k_S^2.$$

We will search for point symmetries relative to which the diversity determined by the system of equations (12) is invariant.

According to Lie-Ovsyannikov's technique, we will search for the admissible operator of point symmetry in view of

$$X = \xi^j \frac{\partial}{\partial x_j} + \eta^i \frac{\partial}{\partial \tau^i}, \ \ j = 1, 2, 3; \ \ i = 1, 2, 3, 4. \quad (13)$$

We continue the operator (13) on the first derivatives by formulas

$$\tilde{X} = X + \varsigma_k^i \frac{\partial}{\partial q_k^i}, \quad (14)$$

where $\varsigma_k^i = D_k\left(\eta^i\right) - q_\beta^i D_k\left(\zeta^\beta\right), \ D_k = \dfrac{\partial}{\partial x_k} + q_k^i \dfrac{\partial}{\partial \tau^i}.$

With the operator (14) we affect the system of equations (13) and transfer to the diversity set by this system. As a result we receive polynoms of the second level in relation to "internal" – endogenous variables q_k^2, q_k^3. "External" – exogenetic variables q_k^1, q_k^4 are determined from the system (12) via endogenous variables. In the received polynoms of the second level we equate coefficients to zero in case of the first and second levels of endogenous variables. It allows to receive the redefined system of linear differential equations with respect to coefficients q_k^2, q_k^3. Solving this system, obtained is the following result.

Theorem. The system of equations (6) allows Lie algebra L_8, generated by operators

$$X_i = \frac{\partial}{\partial x_i},\ X_4 = x_i \frac{\partial}{\partial x_i},\ X_5 = \frac{\partial}{\partial p},$$

$$X_{12} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \tau^2 \frac{\partial}{\partial \tau^1} - \tau^1 \frac{\partial}{\partial \tau^2},$$

$$X_{13} = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + \tau^1 \frac{\partial}{\partial \tau^3} - \tau^3 \frac{\partial}{\partial \tau^1},$$

$$X_{23} = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + \tau^3 \frac{\partial}{\partial \tau^2} - \tau^2 \frac{\partial}{\partial \tau^3}.$$

Availability of the operators X_i, i = 1, 2, 3, 4 means that the system (6) allows shifts and stretching on axes x, y, z $x_i^{'} = x_i + a_i$, $i = 1, 2, 3$, $x_i^{'} = x_i \exp a_4$, shift for hydrostatical pressure $p' = p + a_5$, as well as rotation around three coordinate axes.

2. Invariant solutions of equation system (6).

2.1. Let's create invariant solution relative to subalgebra generated by the operator

$$X_3 = \frac{\partial}{\partial z}.$$

This type of solution should be searched in the following view

$$\tau^i = \tau^i(x, y),\ p = p(x, y). \quad (15)$$

We add (15) in system (6) and we obtain

$$\partial_y \tau^1 = \partial_x p,\ \partial_x \tau^1 = \partial_y p,\ \partial_x \tau^2 + \partial_y \tau^3 = 0,$$
$$\left(\tau^1\right)^2 + \left(\tau^2\right)^2 + \left(\tau^3\right)^2 = k_S^2. \quad (16)$$

From (16) easily obtain

$$\tau^1 = f(x+y) + g(x-y),\ p = f(x+y) - g(x-y),$$
$$\partial_y \tau^1 + \partial_z \tau^2 = \partial_x p. \quad (17)$$

Now functions τ^2, τ^3 are determines from the equation systems

$$\partial_x \tau^2 + \partial_y \tau^3 = 0,\ \left(\tau^2\right)^2 + \left(\tau^3\right)^2 = k_S^2 - \left(\tau^1\right)^2. \quad (18)$$

The equation system (18) describes the bar torsion in the conditions when the yield stress (limit of fluctuation) depends on variables x, y. These tasks are considered in [4] and in the literature quoted.

2.2. Let's construct the invariant decision relative to subalgebra generated by the operator X_{12}. This operator in cylindrical coordinate system $r\theta z$ looks like $X_{12} = \dfrac{\partial}{\partial \theta}$. In this case the system (6) will be written as follows

$$\partial_\theta \tau_{r\theta} + \partial_z \tau_{rz} = r\partial_r p,\ r\partial_r \tau_{r\theta} + r\partial_z \tau_{\theta z} + 2\tau_{r\theta} = \partial_\theta p,$$
$$r\partial_r \tau_{rz} + \partial_\theta \tau_{\theta z} + \tau_{rz} = \partial_z p, \quad (19)$$
$$\tau_{r\theta}^2 + \tau_{rz}^2 + \tau_{\theta z}^2 = k_S^2.$$

Invariant solution in this case is determined from the following system

$$\partial_z \tau_{rz} = r\partial_r p,\ r\partial_r \tau_{r\theta} + r\partial_z \tau_{\theta z} + 2\tau_{r\theta} = 0,$$
$$r\partial_r \tau_{rz} + \tau_{rz} = \partial_z p, \quad (20)$$
$$\tau_{r\theta}^2 + \tau_{rz}^2 + \tau_{\theta z}^2 = k_S^2.$$

In this case τ_{rz} is determined from the linear differential equation

$$r\partial_r \tau_{rz} + r^2 \partial_r^2 \tau_{rz} - \tau_{rz} + r^2 \partial_z^2 \tau_{rz} = 0.$$

Other functions are determined from the system (21).

3. Conservation laws of equation system (6).

Conservation laws are applied to solutions of elastic – plasticity equations. Necessary determination and examples conservation laws usability can be found in [9–15].

Let's find conservation laws of equation system (6) in the following view

$$\partial_x A\left(\tau^1, \tau^2, \tau^3, p\right) + \partial_y B\left(\tau^1, \tau^2, \tau^3, p\right) +$$
$$+ \partial_z C\left(\tau^1, \tau^2, \tau^3, p\right) = 0.$$

The equate is done on the account of equation system (6). From this follow the ratio

$$X_{12} A - \tau^2 \partial_p B + \tau^1 \partial_p C = 0,$$
$$X_{13} B - \tau^3 \partial_p A + \tau^1 \partial_p C = 0,\ X_{12} B - \tau^2 \partial_p A = 0,$$
$$X_{12} C - \tau^1 \partial_p A = 0,$$
$$X_{13} C + \tau^1 \partial_p B = 0,\ X_{13} A - \tau^3 \partial_p B = 0,$$

where $X_{12} = -\tau^2 \partial_{\tau^1} + \tau^1 \partial_{\tau^2}$, $X_{13} = -\tau^3 \partial_{\tau^1} + \tau^1 \partial_{\tau^3}$.

Let's show that these equations are compatible. Suppose $\partial_p A = \partial_p B = \partial_p C = 0$, than – one of the solutions will be the infinite series

$$A(S), B(S), C(S),$$

where $S = \left(\tau^1\right)^2 + \left(\tau^2\right)^2 + \left(\tau^3\right)^2$, $A(S), B(S), C(S)$ – random differentiable functions.

Remark. Are there other laws? Not stated, but according to the author other conservation laws do not exist.

4. It is clear, for the system (6) tension state is the most relevant. Supposing it is known. Than to find three components of the velocity vector we have three equations

$$\tau^1 = \lambda e_{12},\ \tau^2 = \lambda e_{13},\ \tau^3 = \lambda e_{23}, \quad (21)$$

where

$$2e_{12} = \partial_y u + \partial_x v,\ 2e_{13} = \partial_z u + \partial_x w,$$
$$2e_{23} = \partial_z v + \partial_y w. \quad (22)$$

Let's show that the equations (21) can be solved in terms of deformation velocity tensor components. It is known that except the equations (21) deformation velocity tensor components shall satisfy equations of compatibility as well. Owing to ratios (22) and (5) only six of them remain.

$$\partial_{xy}^2 e_{12} = 0,\ \partial_{xz}^2 e_{13} = 0,\ \partial_{yz}^2 e_{23} = 0,$$
$$\partial_x \left(\partial_x e_{23} - \partial_z e_{12} - \partial_y e_{13}\right) = 0,$$
$$\partial_y \left(\partial_y e_{13} - \partial_z e_{12} - \partial_x e_{23}\right) = 0, \quad (23)$$
$$\partial_z \left(\partial_z e_{12} - \partial_y e_{13} - \partial_x e_{23}\right) = 0.$$

Theorem. The compatibility equations of deformation speeds are done identically.

In this case from (21) we have

$$\left(\tau^1\right)^2\left(e_{12}^2+e_{13}^2+e_{23}^2\right)=k_S^2e_{12}^2,$$
$$\left(\tau^2\right)^2\left(e_{12}^2+e_{13}^2+e_{23}^2\right)=k_S^2e_{23}^2, \qquad (24)$$
$$\left(\tau^3\right)^2\left(e_{12}^2+e_{13}^2+e_{23}^2\right)=k_S^2e_{13}^2.$$

Equation system (24) is a system of linear homogeneous equations relevant to variables $e_{12}^2, e_{13}^2, e_{23}^2$. Its determinant is

$$\begin{vmatrix} \left(\tau^1\right)^2-k_S^2 & \left(\tau^1\right)^2 & \left(\tau^1\right)^2 \\ \left(\tau^2\right)^2 & \left(\tau^2\right)^2-k_S^2 & \left(\tau^2\right)^2 \\ \left(\tau^3\right)^2 & \left(\tau^3\right)^2 & \left(\tau^3\right)^2-k_S^2 \end{vmatrix}.$$

This determinant equals zero as the amount of all lines is equal to zero. It means that the system (24) has only two independent equations for three components of a deformation speed tensor. For example, value e_{23}^2 can be picked up randomly, thus for the given tension state, defined from the system (6), velocity field is defined with the functional arbitrariness.

5. In this part we will consider three-dimensional equations of elasticity in static. The system of equilibrium equations is described using equations (1), relation between components of stress tensor and deformation tensor is as follows

$$\varepsilon_{11}=\frac{\left(\sigma_{11}-v\left(\sigma_{22}+\sigma_{33}\right)\right)}{E},$$
$$\varepsilon_{22}=\frac{\left(\sigma_{22}-v\left(\sigma_{11}+\sigma_{33}\right)\right)}{E}, \qquad (25)$$
$$\varepsilon_{33}=\frac{\left(\sigma_{33}-v\left(\sigma_{22}+\sigma_{11}\right)\right)}{E},$$
$$\varepsilon_{12}=\frac{\sigma_{12}}{2\mu},\ \ \varepsilon_{13}=\frac{\sigma_{13}}{2\mu},\ \ \varepsilon_{23}=\frac{\sigma_{23}}{2\mu},$$

where ε_{ij} deformation tensor components, E,μ,v elastic constants.

Supposing vector deformation components are as follows

$$w_1=w_1\left(y,z\right),\ \ w_2=w_2\left(x,z\right),\ \ w_3=w_3\left(x,z\right). \qquad (26)$$

Inserting (26) into (25) we obtain

$$\partial_y w_1+\partial_x w_2=\frac{\sigma_{12}}{2\mu},\ \ \partial_z w_1+\partial_x w_3=\frac{\sigma_{13}}{2\mu}, \qquad (27)$$
$$\partial_z w_2+\partial_y w_3=\frac{\sigma_{23}}{2\mu}.$$

In this case equation (1) with regard to (26), (27) is as follows

$$\partial_{yy}w_1+\partial_{zz}w_1=0,\ \ \partial_{xx}w_2+\partial_{zz}w_2=0, \qquad (28)$$
$$\partial_{xx}w_2+\partial_{yy}w_3=0.$$

It is shown that components of deformation vector are harmonic functions. The solutions obtained here can be used for the description of elastic status of the parallelepiped twisted around three orthogonal axes. The moments are defined from formulas (7). As $\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}=0$, dilatation (volume change) is equal to zero. The solution (27) describes vortex movement characterized by the vector $\overline{\omega}$

$$\overline{\omega}=\begin{pmatrix} \overline{i} & \overline{j} & \overline{k} \\ \partial_x & \partial_y & \partial_z \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Herewith movement paths will be vortex lines which are defined from the equation

$$\frac{dx}{w_1}=\frac{dy}{w_2}=\frac{dz}{w_3}.$$

Conclusion. In the present work for the first time considered is the system which can be used for the analysis of stress state appearing under torsion of the parallelepiped around three orthogonal axes. At that it can be in either plastic or elastic state.

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