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USE OF CONSERVATION LAWS TO SOLVE THE PROBLEM OF LOAD WAVE IN AN ELASTOPLASTIC ROD

*S. I. Senashov, I. L. Savostyanova, E. V. Filyushina

Siberian State University of Science and Technology
31, Krasnoyarsky Rabochy Av., Krasnoyarsk, 660037, Russian Federation
*E-mail: sen@sibsau.ru

The process of propagation of plastic deformations in a semi-infinite elastic plastic rod caused by dynamic loading applied to the end of the rod, which is not decreasing in time, is considered. The equations are written in the Lagrangian coordinate system. It is assumed that during deformation there is no lateral bulging of the rod and that the influence of transverse deformations of the rod on the process of propagation of longitudinal waves is negligible. At the initial moment, the rod is in a deformed and dormant state. Small deformations of the rod are considered. The density of the rod during deformation does not change. The only non-zero component of the stress tensor will be the component along the ox axis, non-zero components of the strain tensor will be the components along the Ox, Oy axes. As a result, a system of two quasilinear first-order homogeneous equations is constructed. The equations are hyperbolic. They are built for performance and ratio on them. Next, the equations are written in terms of Riemann invariants. For the equations constructed, the conservation laws are found in the case when the current remaining depends only on the functions sought. As a result, a system of linear equations with coefficients depending only on the required functions is obtained.

The construction of conservation laws is reduced to the solution of the boundary value problem for the known Euler–Poisson–Darboux equations. This problem is solved with the help of Riemann functions. The conservation laws allowed us to find the coordinates of the points of intersection of characteristics, and thus to solve the problem. In conclusion, the article considers the case when one of the characteristics crosses the line on which the initial conditions are given. In this case, as is known, the Cauchy problem cannot be solved. This leads to a procedure which, with the help of conservation laws, makes it possible to find out the solvability of the Cauchy problem. It is reduced to the solution of a simple integral equation by the method of successive approximations.

Keywords: conservation laws, wave loading, elastic-plastic rod, Cauchy problem.

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ИСПОЛЬЗОВАНИЕ ЗАКОНОВ СОХРАНЕНИЯ ДЛЯ РЕШЕНИЯ ЗАДАЧИ О ВОЛНЕ НАГРУЗКИ В УПРУГО-ПЛАСТИЧЕСКОМ СТЕРЖНЕ

*С. И. Сенашов, И. Л. Савостьянова, Е. В. Филюшина

Сибирский государственный университет науки и технологий имени академика М. Ф. Решетнева
Российская Федерация, 660037, г. Красноярск, просп. им. газ. «Красноярский рабочий», 31
*E-mail: sen@sibsau.ru

Рассмотрен процесс распространения пластических деформаций в полубесконечном упруго-пластическом стержне, вызванных приложенной к концу стержня динамической нагрузкой, не убывающей во времени. Уравнения записаны в лагранжевой системе координат. Предполагается, что в процессе деформации не происходит бокового выпучивания стержня и что влияние поперечных деформаций стержня на процесс распространения продольных волн пренебрежимо мало. В начальный момент стержень находится в деформированном состоянии и состоянии покоя. Рассмотрены малые деформации стержня. Плотность стержня в процессе деформирования не изменяется. Единственной отличной от нуля составляющей тензора напряжений будет компонента вдоль оси ох, отличными от нуля составляющими тензора деформаций будут компоненты вдоль осей Ох, Оу. В результате построена система двух квазилинейных однородных уравнений первого порядка. Уравнения являются гиперболическими. Для них построены характеристики и соотношения на них. Далее уравнения записаны в терминах инвариантов Римана. Для построенных уравнений найдены законы сохранения в случае, когда сохраняющийся ток зависит только от искомых функций. В результате получена система линейных уравнений с коэффициентами, зависящими только от искомых функций. Построение законов сохранения сведено к решению краевой задачи для известных уравнений Эйлера–Пуассона–Дарбу. Эта задача решена с помощью функций Римана. Законы сохранения позволили найти координаты точек пересечения характеристик, а значит, и решить поставленную задачу. В заключение рассмотрен случай, когда одна из характеристик пересекает линию, на которой заданы начальные условия. В этом случае, как известно, задача Коши *решена быть не может. Это приводит к процедуре, которая с помощью законов сохранения позволяет выяснить вопрос о разрешимости задачи Коши. Она сводится к решению несложного интегрального уравнения методом последовательных приближений.*

Ключевые слова: законы сохранения, волна нагрузки, упруго-пластический стержень, задача Коши.

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Introduction. Conservation laws, in relation to the differential equations, were published in Emma Neter's article [1] more than 100 years ago. She established the general principle connecting symmetry groups and conservation laws for the differential equations deduced from the variation principle. For more than 70 years all results in this field were based on this article. More general concepts allowing to calculate conservation laws for any systems of the differential equations appeared in A.M. Vinogradov's works [2; 3]. For a rather long time conservation laws occurred in literature as purely mathematical result, far from applications. In the works [4–6] it was shown how conservation laws can be used for the solution of Cauchy and Riemann problems and also accurate solutions of these tasks were made.

Later the method of conservation laws was applied to solution of free-boundary problem: elastic plasticity tasks [7–10]. For the first time, a special case of the task of the wave distribution, which is solved by means of conservation laws, is constructed in work [11; 12]. In this work more general case is considered and also condition under which there is a solution of Cauchy problem is formulated.

Derivations of the main equations

1. We will consider the process of plastic deformations propagation in a semi-infinite elastic plastic rod caused by dynamic loading time, $p(t)$ applied to the end of the rod, which is not decreasing in (i. e. $dp/dt \geq 0$).

We shall find a solution in the Langrangian coordinate system: we will take a rod axis for the axis x, we will choose the origin of coordinates $x = 0$ on the left end of the rod. Suppose that during deformation there is no lateral bulging of the rod and that the influence of transverse deformations of the rod on the process of propagation of longitudinal waves is negligible. Let us consider small deformations of the rod and assume that the rod density in the course of deformation does not change. The only component of tension tensor, other than zero, will be $\sigma_{xx} = \sigma$ other than zero components of a tensor of deformations will be $\varepsilon_{xx} = \varepsilon$ and $\varepsilon_{yy} = \upsilon\varepsilon$.

In this case, motion equation exclusive of massive external forces is as follows [13]:

$$\rho\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \tag{1}$$

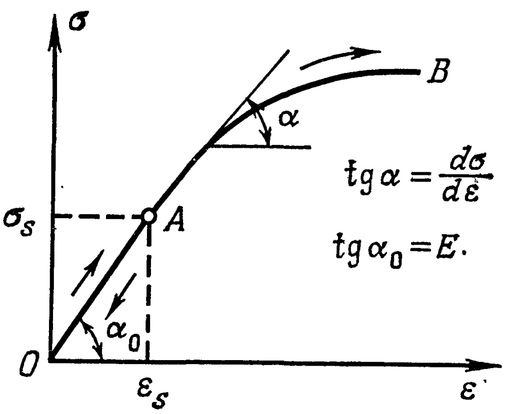
where $\sigma = \sigma_{xx}$ – component of the stress tensor; v – particles velocity along the axis Ox, ρ – density.

Since density is constant, without generality loss we further assume that $\rho = 1$.

Accepting the defining relation of the deformation plasticity theory (for the uniaxial stress) as follows

$$\sigma = \sigma(\varepsilon). \tag{2}$$

Supposing $\sigma(\varepsilon)$ is a steadily increasing along ε function (fig. 1) and for all ε derivative $d\sigma/d\varepsilon$ is a steadily decreasing function (i. e. $d^2\sigma/d^2\varepsilon < 0$).



Fig, 1. The process of propagation of plastic deformations in a semi-infinite elastoplastic rod

Рис. 1. Процесс распространения пластических деформаций в полубесконечном упруго-пластическом стержне

For tensions $\sigma \leq \sigma_s$ (σ_s is tensile yield) dependence $\sigma(\varepsilon)$, according to Hook's law, is linear:

$$\sigma = E\varepsilon, \tag{3}$$

where E is elasticity modulus (Young's modulus). Wherein the values of Young's modulus E have been sorted out as to under $\sigma = \sigma_s$ dependence (3) is continuous.

From the equation of through flow in case of minor deformations we obtain the following formula

$$\frac{d\varepsilon}{dt} = \frac{dv}{dx}. \tag{4}$$

Taking into account the dependence $\sigma = \sigma(\varepsilon)$ under load and introducing notation

$$a^2(\sigma) = \frac{\partial \sigma}{\partial \varepsilon}, \tag{5}$$

where $d\sigma/d\varepsilon$ is the rate of change to the curve $\sigma(\varepsilon)$; α is a constant, $0 < \alpha < 1$; $a^2(\sigma) = \sigma^{2\beta}$ velocity of longitudinal waves propagation in the rod, we have

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon}{d\sigma}\frac{\partial \sigma}{\partial t} = \frac{1}{a^2(\sigma)}\frac{\partial \sigma}{\partial t}. \tag{6}$$

Placing the relation (4) in (6), we obtain the system of two equations of partial derivatives of the first order [13]:

$$\frac{dv}{dt}=\frac{d\sigma}{dx},\quad \frac{\partial v}{\partial x}=\frac{1}{a^2(\sigma)}\frac{\partial\sigma}{\partial t},\qquad (7)$$

For two functions $v(x,t),\sigma(x,t)$.

In this equation $a(\varepsilon)$ is the velocity of longitudinal waves propagation in the rod.

Since the velocity of longitudinal waves propagation generally is the tension function, then, equation system (7) is the system of quasilinear equations with partial derivatives of the first order of hyperbolic type. For it we will determine characteristics and relations under characteristics.

Characteristics of equation system (7) are determined by integrating of differential equations' characteristics:

$$dx=\mp a(\sigma)dt.\qquad (8)$$

These equations generally cannot be integrated in plain (x,t) before the problem has been solved since a is the tension function $\sigma(x,t)$.

Along characteristics $dx=\mp\alpha(\sigma)dt$ the following relations are made

$$dv\mp\frac{1}{a(\sigma)}d\sigma=0.\qquad (9)$$

These relations are called differential equations of characteristics in hodograph plane (σ,v). After integrating we obtain

$$v=\mp\int_0^{\sigma}\frac{d\sigma_1}{a(\sigma_1)}+C_{1,2}\ \text{при}\ dx=\mp a(\sigma)dt.\qquad (10)$$

We will now consider the simplest case of load waves propagation in homogeneous half-infinite rod, which at the initial moment was in nonperturbed state.

We will consider the equation solution (7) under the given initial conditions (Cauchy conditions):

$$v(x,0)=v(x),\qquad (11)$$

and boundary condition

$$\sigma(0,t)=-p(t),\quad (p(t)>0),\ dx=\mp a(\sigma)dt.\qquad (12)$$

where, to ensure the load process there must be $p'(t)>0$.

Conditions (11)–(12) mean that at the initial moment the rod is in the deformed and dormant state. Meeting the initial conditions correlates with Cauchy problem solution in the domain (fig. 2), limited by axis *x* and positive characteristics *tsQ*.

2. For simplicity we will consider the following assertions for function (2)

$$\sigma=E\varepsilon,\ \text{under}\ \sigma<\sigma_z,$$
$$\sigma(\varepsilon)=\frac{1}{\alpha}\varepsilon^{\alpha},\ \text{under}\ 0<\alpha<1\quad dx=\mp a(\sigma)dt.\qquad (13)$$

General case is considered similarly.

For the continuity of function $\sigma(\varepsilon)$ at point ε_s we suppose $E=\frac{1}{\alpha}\varepsilon_s^{\alpha-1}$.

In this case the plane xOt splits into two domains: elastic bounded by axes *x* and direct t_sP and plastic domain, placed above the line t_sP. It should be noted that the equation of this line is as follows: $x=a_0(t-t_s)$, where t_s time point, when σ achieves the yield stress σ_s (fig. 2). In the plastic domain we have a linear problem which can be easily solved applying traditional methods. Hence, we will seek for the Cauchy problem solution for equations (7) only in the plastic domain.

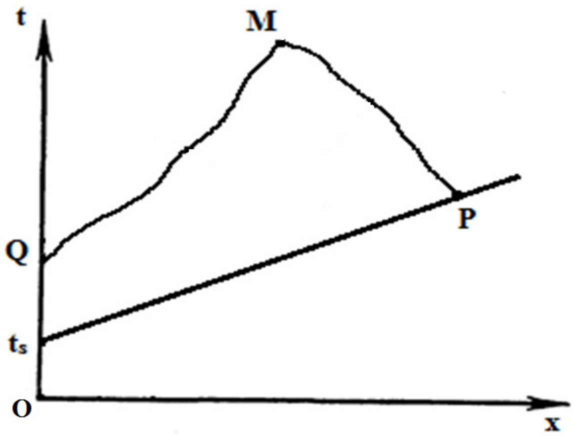


Fig. 2. Characteristics of equations (14)

Рис. 2. Характеристики уравнений (14)

Problem definition. To find the value of function $v(x,t),\sigma(x,t)$ at the point $M(x_m,t_m)$ if the values of the required function along t_sP and t_sQ are known. Here points $Q(0,t_q),P(x_p,t_p)$ are considered as intersection points of the correspondent characteristics with the axis Ot and the line t_sP, drawn from the point M. According to (13) the equations (7)–(10) will be as follows

$$\frac{dv}{dt}=\frac{d\sigma}{dx},\ \frac{\partial\sigma}{\partial t}=\sigma^{2\beta}\frac{\partial v}{\partial x},\ \beta=\frac{\alpha-1}{\alpha}.\qquad (14)$$

Characteristics of the present equation system according to (8) are as follows

$$dx=\mp\sigma^{\beta}dt.$$

Relations on characteristics (9), after integration will be

$$v\mp\frac{\sigma^{-\beta+1}}{-\beta+1}=C_{1,2},$$

where C_1, C_2 are random constants.

We will introduce Riemann's invariants under the formula $\xi=v+\frac{\sigma^{-\beta+1}}{-\beta+1},\eta=v-\frac{\sigma^{-\beta+1}}{-\beta+1}$, then the system (14) will be as follows

$$\frac{\partial\xi}{\partial t}-\sigma^{\beta}\frac{\partial\xi}{\partial x}=0,\quad \frac{\partial\eta}{\partial t}+\sigma^{\beta}\frac{\partial\eta}{\partial x}=0.\qquad (15)$$

Employment of conservation laws for equations describing the wave load in the elastic plastic rod. Conservation laws for the equation system (15) is founded as follows [5]

$$\partial_t A + \partial_x B = 0,$$

$$\frac{\partial}{\partial t} A(\xi,\eta) + \frac{\partial}{\partial x} B(\xi,\eta) =$$

$$= \left(\sigma^\beta \frac{\partial A}{\partial \xi} + \frac{\partial B}{\partial \xi}\right)\frac{\partial \xi}{\partial x} + \left(-\sigma^\beta \frac{\partial A}{\partial \eta} + \frac{\partial B}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = 0.$$

From here we obtain the equation to determine A and B

$$\sigma^\beta \frac{\partial A}{\partial \xi} + \frac{\partial B}{\partial \xi} = 0, \quad -\sigma^\beta \frac{\partial A}{\partial \eta} + \frac{\partial B}{\partial \eta} = 0. \tag{16}$$

Excluding from (16) function B we obtain the equation to determine function A:

$$\frac{8}{\beta(-\beta+1)}\frac{\partial^2 A}{\partial\xi\partial\eta} - \left(\frac{\partial A}{\partial \xi} - \frac{\partial A}{\partial \eta}\right)\frac{1}{\xi-\eta} = 0.$$

We will introduce in this equation the notation $\frac{8}{\beta(-\beta+1)} = \omega^{-1}$. As a result we obtain the Euler–Poisson–Darboux equation [14]:

$$\frac{\partial^2 A}{\partial\xi\partial\eta} - \frac{\omega}{\xi-\eta}\left(\frac{\partial A}{\partial \xi} - \frac{\partial A}{\partial \eta}\right) = 0. \tag{17}$$

To determine function B we get a similar equation

$$\frac{\partial^2 B}{\partial\xi\partial\eta} + \frac{\omega}{\xi-\eta}\left(\frac{\partial B}{\partial \xi} - \frac{\partial B}{\partial \eta}\right) = 0.$$

Applying (15) we will write the integral about closed path t_sQMP

$$\oint A dx - B dt = 0. \tag{18}$$

We will split this integral into four integrals taken about closed paths t_sP, PM, MQ, Qt_s.

About closed paths t_sP and Qt_s integrals can be calculated after determining A, B inclusive initial and boundary conditions (11), (12).

We will determine A and B so that along the characteristics PM and MQ integrals transform to zero. We have

$$\int_{PM} A dx - B dt = \int_{PM} \left(-\sigma^\beta A - B\right) dt.$$

We will calculate the obtained integral in parts

$$\int_{PM} \left(-\sigma^\beta A - B\right) dt = t\left(-\sigma^\beta A - B\right)\Big|_Q^M - \int_{PM} t d\left(-\sigma^\beta A - B\right).$$

Similarly along MQ we obtain

$$\int_{MQ} A dx - B dt = t\left(\sigma^\beta A - B\right)\Big|_M^Q - \int_{MQ} t d\left(\sigma^\beta A - B\right).$$

Finally obtain

$$d\left(\sigma^\beta A + B\right)\Big|_{\xi=\xi_0=\text{const}} = 0, \quad d\left(\sigma^\beta A - B\right)\Big|_{\eta=\eta_0=\text{const}} = 0.$$

From the first relation we have

$$\beta\sigma^{\beta-1}\left(\frac{1}{2}\sigma^\beta\right)A + \sigma^\beta A_\eta + B_\eta = -\frac{\beta}{2}A\sigma^{\beta-1} + 2A_\eta = 0$$

along $\xi = \xi_0$.

Since $\sigma^{\beta-1} = \frac{2}{(\xi-\eta)(-\beta+1)}$, we obtain differential equation for A along $\xi = \xi_0$

$$-\frac{\beta}{-\beta+1}A + A_\eta\left(\xi_0 - \eta\right) = 0.$$

By its integrating we obtain

$$\frac{\beta}{2(\beta-1)}\ln\left(\eta - \xi_0\right) = \ln A + \ln C_3,$$

or

$$A = C_3\left|\eta - \xi_0\right|^{\frac{\beta}{2(\beta-1)}}, \; B = -\sigma^\beta A - 1, \text{ along } \xi = \xi_0. \tag{19}$$

Similarly along $\eta = \eta_0$ we have

$$\beta\sigma^{\beta-1}\left(\frac{1}{2}\sigma^\beta\right)A + \sigma^\beta A_\xi - B_\xi =$$

$$= \beta\sigma^{\beta-1}\frac{1}{2}A\sigma^\beta + 2\sigma^\beta A_\xi = 0.$$

Therefore along $\eta = \eta_0$

$$A = C_4\left|\eta_0 - \xi\right|^{\frac{\beta}{2(\beta-1)}}, \; B = \sigma^\beta A. \tag{20}$$

Matching conditions (19) and (20) at the point $\xi = \xi_0$, $\eta = \eta_0$ gives $C_3 = C_4$.

Thus, for the final problem solution we have to solve the equation (17) with the restricted conditions (19) and (20).

To solve this problem Riemann function is used. It looks as follows:

$$w\left(\xi_0,\eta_0;\xi,\eta\right) =$$

$$= \left(\frac{\xi_0-\eta_0}{\xi_0-\eta}\right)^\omega \left(\frac{\xi_0-\eta_0}{\xi-\eta_0}\right)^\omega F\left(\omega,\omega;1,t\right), \tag{21}$$

where $1 - t = \frac{(\xi-\eta)(\xi_0-\eta_0)}{(\xi_0-\eta)(\xi-\eta_0)}$; F is hypergeometric polynomial of the second raw.

Suppose N is a random point from the t_sPMQ domain (fig. 3). We will connect the point N with MP characteristics $NK - \xi_N$, and with QM characteristics $NL - \eta_N$. As a result the value of function A at the point N will equal

$$A\left(N\right) = A\left(M\right)w\left(M\right) + \int_{KM} w\left(-\frac{A\omega}{(\xi-\eta)} + A_\xi\right)d\xi +$$

$$+ \int_{ML} w\left(\frac{A\omega}{(\xi-\eta)} + A_\eta\right)d\eta,$$

where function w is determined by the formula (21).
+ formula will define the function B values in point N

$$B(N)=B(M)w(M)+\int_{KM} w\left(-\frac{B\omega}{(\xi-\eta)}+B_\xi\right)d\xi+$$
$$+\int_{ML} w\left(\frac{B\omega}{(\xi-\eta)}+B_\eta\right)d\eta.$$

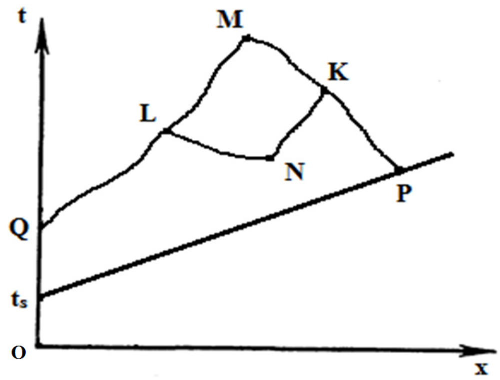


Fig. 3. The solution of the Cauchy problem

Рис. 3. Решение задачи Коши

Now from (18), taking into account the obtained relations we have

$$\oint_{t_sPMQ} Adx-Bdt=$$
$$=\oint_{t_sQ} Adx-Bdt+\oint_{Pt_s} Adx-Bdt+t_m-t_q=0.$$

Hence, we have

$$t_m-t_q=\oint_{t_sQ} Bdt-\oint_{Pt_s} Adx-Bdt.$$

Similar calculations help to find coordinates of x_m

$$x_m-x_q=\oint_{t_sQ} Bdt-\oint_{Pt_s} Adx-Bdt.$$

Later, according to values v,σ at points Q and P knowing the relations along characteristics PM and QM, we will find values $v(x_m,t_m)$ and $\sigma(x_m,t_m)$.

In conclusion, we will consider the question of Cauchy problem solvability which always arise when solving the systems of non-linear differential equations. As it is known, Cauchy problem is solvable if each characteristics crosses the lines Qt_s, t_sP only once [15]. Apparently, this question can as well be solved knowing the conservation laws. Suppose that characteristics QM cross the line t_sP at the point M (fig. 4). Then we get the conservation law

$$\oint_{t_sMQ} Adx-Bdt=0. \qquad (22)$$

Suppose as before

$$A=C_3\left|\eta-\xi_0\right|^{\frac{\beta}{2(\beta-1)}},\ B=-\sigma^\beta A-1,\ \text{along } \xi=\xi_0. \qquad (23)$$

Then, from (22) we obtain identical to the above

$$t_m-t_q=\oint_{t_sQ} Bdt-\oint_{Pt_s} Adx-Bdt. \qquad (24)$$

If the equation is (24) solvable, we can find the intersection point of the characteristic and the initial curve. In this case the Cauchy problem is unsolvable. Equation (24) can be easily solved applying the method of successive approximations.

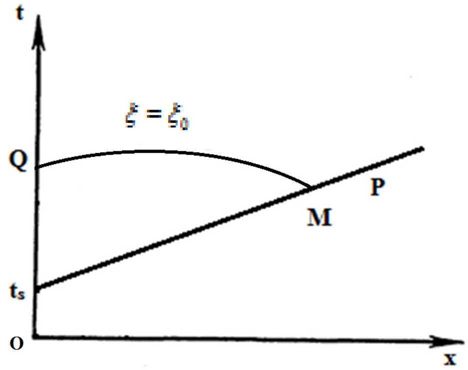


Fig. 4. Finding the intersection point of the characteristic and the curve on which the initial conditions are given

Рис. 4. Нахождение точки пересечения характеристики и кривой, на которой заданы начальные условия

Conclusion. Knowledge of conservation laws allowed to find coordinates of characteristics' points of intersection, and therefore to solve the problem discussed in the article. The case when one of characteristics crosses the line where initial conditions are set is considered. In this case, as we know, Cauchy problem cannot be solved. It leads to the procedure which, by means of conservation laws, allows to settle the issue of Cauchy problem solvability.

References

1. Neter E. [Invariant variational problems]. *Variatsionnye printsipy mekhaniki.* Moscow, Fizmatgiz Publ., 1959, P. 611–630 (In Russ.).

2. Vinogradov A. M., Krasil'shchik I. S., Lychagin V. V. *Simmetrii i zakony sokhraneniya* [Symmetries and conservation laws]. Moscow, Faktorial Publ., 1997, 464 p.

3. Vinogradov A. M. Local symmetries and conservation laws. *Acta appl. Math*. 1984, Vol. 2, No. 1, P. 21–78.

4. Senashov S. I. [On the laws of conservation of the equations of plasticity]. *Dokl. AN SSSR*. 1991, Vol. 320, No. 3, P. 606–608 (In Russ.).

5. Kiryakov P. P., Senashov S. I.,Yakhno A. N. *Prilozhenie simmetriy i zakonov sokhraneniya dlya resheniya differentsial'nykh uravneniy* [The application of symmetries and conservation laws for the solution of differential equations]. Novosibirsk, SO RAN Publ., 2001, 192 p.

6. Senashov S. I., Vinogradov A. M. Symmetries and Conservation Laws of 2-Dimensional Ideal Plasticity. *Proc. of Edinb. Math. Soc.* 31, 1988, P. 415–439.

7. Senashov C. I., Gomonova O. V., Yakhno A. N. *Matematicheskie voprosy dvumernykh uravneniy ideal'noy plastichnosti* [Mathematical problems of two-dimensional equations of ideal plasticity]. Krasnoyarsk, Sib. gos. aerokosmich. un-t, 2012, 139 p.

8. Senashov S. I., Cherepanova O. N., Kondrin A. V. [Ob uprugo plastchiskom kruchenii sterzhnya]. *Vestnik SibGAU*. 2013, No. 3(49), P. 100–103 (In Russ.).

9. Senashov S. I., Cherepanova O. N., Kondrin A. V. Elastoplastic Bending of Beam. *J. Siberian Federal Univ., Math. & Physics*. 2014, No. 7(2), P. 203–208.

10. Senashov S. I., Kondrin A. V., Cherepanova O. N. On Elastoplastic Torsion of a Rod with Multiply Connected Cross-Section. *J. Siberian Federal Univ., Math. & Physics.* 2015, Vol. 7(1), P. 343–351.

11. Senashov S. I., Filyushina E. V. [Analiticheskoe reshenie zadachi o volne nagruzki v uprugoplasticheskom sterzhne]. *Sb. nauchn. tr. "Dinamika sploshnoy sredy"*. Novosibirsk, 2012, Vol. 127, P. 94–97 (In Russ.).

12. Senashov S. I., Filyushina E. V. *Uprugoplasticheskie zadachi dlya ortotropnykh sred* [Elastic-plastic problems for orthotropic media]. Krasnoyarsk, SibGU Publ., 2017, 116 p.

13. Novatskiy V. *Volnovye zadachi teorii plastichnosti* [Wave problems of the theory of plasticity]. Moscow, Mir Publ., 1978, 307 p.

14. Stewart J. M. The Euler–Poisson–Darbous equations for relativists. *Gen. Relativ Gravit*. 2009, Vol. 2, P. 2045–2071.

15. Kachanov L. M. *Teoriya plastichnosti* [Theory of plasticity]. Moscow, Nauka Publ., 1969, 420 p.

Библиографические ссылки

1. Нетер Э. Инвариантные вариационные задачи // Вариационные принципы механики. М. : Физматгиз, 1959. С. 611–630.

2. Виноградов А. М., Красильщик И. С., Лычагин В. В. Симметрии и законы сохранения. М. : Факториал, 1997. 464 с.

3. Vinogradov A. M. Local symmetries and conservation laws // Acta appl. Math. 1984. Vol. 2, № 1. P. 21–78.

4. Сенашов С. И. О законах сохранения уравнений пластичности // Докл. АН СССР. 1991. Т. 320, № 3. С. 606–608.

5. Киряков П. П., Сенашов С. И., Яхно А. Н. Приложение симметрий и законов сохранения для решения дифференциальных уравнений. Новосибирск : СО РАН, 2001. 192 с.

6. Senashov S. I., Vinogradov A. M. Symmetries and conservation Laws of 2-dimensional equations of ideal plasticity // Proc. Edinburg Math. Soc. 1988. Vol. 31. P. 415–439.

7. Сенашов С. И., Гомонова О. В., Яхно А. Н. Математические вопросы двумерных уравнений идеальной пластичности / Сиб. гос. аэрокосмич. ун-т. Красноярск, 2012. 139 с.

8. Сенашов С. И., Черепанова О. Н., Кондрин А. В. Об упруго-пластическом кручении стержня // Вестник СибГАУ. 2013. № 3(49). С. 100–103.

9. Senashov S. I., Cherepanova O. N., Kondrin A. V. Elastoplastic Bending of Beam // J. Siberian Federal Univ., Math. & Physics. 2014. № 7(2). P. 203–208.

10. Senashov S. I., Kondrin A. V., Cherepanova O. N. On Elastoplastic Torsion of a Rod with Multiply Connected Cross-Section // J. Siberian Federal Univ., Math. & Physics. 2015. № 7(1). P. 343–351.

11. Сенашов С. И., Филюшина Е. В. Аналитическое решение задачи о волне нагрузки в упругопластическом стержне // Динамика сплошной среды : сб. науч. тр. / Ин-т гидродинамики СО РАН. Новосибирск, 2012. Вып. 127. С. 94–97.

12. Сенашов С. И., Филюшина Е. В. Упругопластические задачи для ортотропных сред / СибГУ им. М. Ф. Решетнева. Красноярск, 2017. 116 с.

13. Новацкий В. Волновые задачи теории пластичности. М. : Мир, 1978. 307 с.

14. Stewart J. M. The Euler-Poisson-Darbous equations for relativists // Gen. Relativ Gravit. 2009. Vol. 2. P. 2045–2071.

15. Качанов Л. М. Теория пластичности. М. : Наука, 1969. 420 с.