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CONSTRUCTION OF ELASTO-PLASTIC BOUNDARIES USING CONSERVATION LAWS

S. I. Senashov*, E. V. Filyushina, O. V. Gomonova

Reshetnev Siberian State Aerospace University
31, Krasnoyarsky Rabochy Av., Krasnoyarsk, 660037, Russian Federation
*E-mail: sen@sibsau.ru

The solution of elasto-plastic problems is one of the most complicated and actual problems of solid mechanics. Traditionally, these problems are solved by the methods of complex analysis, calculus of variations or semi-inverse methods. Unfortunately, all these methods can be applied to a limited number of problems only.

In this paper, a technique of conservation laws is used. This technique allows constructing analytical formulas to determine the elasto-plastic boundary for a wide class of problems. As a result, the elasto-plastic boundaries were constructed for twisted straight rods with cross sections limited by piecewise smooth contour, for flexible consoles with constant cross-sections, as well as for anti-plane problems. Computer programs for construction of elasto-plastic boundaries for twisted straight rods were written using obtained technique.

In this work, the elasto-plastic boundary arising during the torsion of a straight beam of arbitrary cross section, which is limited by a piecewise smooth contour is constructed; and the elasto-plastic boundaries for the problems of a consol bending and anti-plane deformation are found. The plan of the paper is the following. In the first section the basic equations of elasticity and boundary problems are considered; in the second section the basic equations of the theory of ideal plasticity of von Mises are given; in the third section the conditions on the boundaries of the elastic and plastic domains are formulated. The fourth section is devoted to torsion of elastic prismatic rods; the fifth one describes elastic bending of bars; in the sixth section the plane problem of theory of elasticity is given. The seventh section covers an anti-plane problem of elasticity theory; in the eighth section, conservation laws for the equations of elasticity are constructed; in the ninth one, conservation laws of two-dimensional equations of plasticity are discussed. In the tenth section an elasto-plastic boundary of a twisted straight rod is found; in the eleventh one an elasto-plastic boundary in the bended console is given; and finally, in the twelfth section a method for the construction of elasto-plastic boundaries for large areas is described.

Keywords: conservation laws, elasto-plastic boundary, exact solutions, elasticity, plasticity, elasto-plasticity.

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ПОСТРОЕНИЕ УПРУГО-ПЛАСТИЧЕСКИХ ГРАНИЦ С ПОМОЩЬЮ ЗАКОНОВ СОХРАНЕНИЯ

С. И. Сенашов*, Е. В. Филюшина, О. В. Гомонова

Сибирский государственный аэрокосмический университет имени академика М. Ф. Решетнева
Российская Федерация, 660037, г. Красноярск, просп. им. газ. «Красноярский рабочий», 31
*E-mail: sen@sibsau.ru

Решение упруго-пластических задач – одна из сложнейших и актуальных проблем механики деформируемого твердого тела. Традиционно эти задачи решаются или методами ТФКП, вариационного исчисления, или полуобратными методами. К сожалению, все эти методы могут быть применены лишь к ограниченному числу задач. В работе используется техника законов сохранения. Это позволяет построить аналитические формулы для нахождения упруго-пластической границы для широкого класса задач. В результате удалось построить упруго-пластические границы: для скручиваемых прямолинейных стержней, сечение которых ограничено кусочно-гладким контуром; для изгибаемых консолей постоянного сечения, а также для антиплоских задач. Разработанная методика позволила написать компьютерные программы для построения упруго-пластических границ для скручиваемых прямолинейных стержней. В предлагаемой работе построена упруго-пластическая граница, возникающая при кручении прямолинейного бруса произвольного сечения, которое ограничено кусочно-гладким контуром, а также упруго-пластическая граница в задачах об изгибе консоли и антиплоской деформации. В первом разделе статьи рассмотрены основные уравнения упругости и краевые задачи, во втором – даны основные уравнения теории идеальной пластичности Мизеса, в третьем – сформулированы условия на границах упругих и пластических областей, в четвертом – рассмотрено кручение призматических упругих стержней, в пятом – описан упругий изгиб брусьев, в шестом – рассмотрена плоская задача теории упругости, *в седьмом – описана антиплоская задача теории упругости, в восьмом – построены законы сохранения для уравнений упругости, в девятом – рассмотрены законы сохранения двумерных уравнений пластичности, в десятом – найдена упруго-пластическая граница в скручиваемом прямолинейном стержне, в одиннадцатом – найдена упруго-пластическая граница в изгибаемой консоли, в двенадцатом – предложена методика для построения упруго-пластических границ для областей больших размеров.*

Ключевые слова: законы сохранения, упруго-пластическая граница, точное решение, упругость, пластичность, упруго-пластичность.

Introduction. Solution of elasto-plastic problems is one of the most complicated and actual problems of solid mechanics. It is determined by the fact that elasto-plastic boundary is not known in advance and should be defined during the solution of a problem. The elasto-plastic problems were considered by many well-known mechanicians. One can find a good review in works of B. D. Annin and G. P. Cherepanov [1], L. A. Galin [2; 3]. For the moment, a common approach for solving such problems has not been worked out yet. There are only a few single solutions for different special cases. As classical results one should consider an exact solution for the problem of elasto-plastic torsion of a rod with oval cross-section constructed by V. V. Sokolovsky, as well as solution of L. A. Galin for the problem of straining of a plane with circular hole.

An interesting theoretical result was obtained by B. D. Annin [1]. He proved the unique existence for the problem of elasto-plastic torsion of the rod with oval cross-section.

For solving of the elasto-plastic problem the methods of complex analysis, calculus of variations or semi-inverse methods were applied. In this paper, for construction of the elasto-plastic boundary the conservation laws were used. The conservation laws were applied in works [4–6] for the solving of the problem of 2-dimensional ideal plasticity; they allowed to obtain analytical solutions of Cauchy and Riemann problems. In following works of one of the co-authors of the present article the conservation laws were used for solving of some elasto-plastic problems [7; 8]. The obtained method allowed to write an algorithm (computer programs) for construction of elasto-plastic boundaries of twisted straight rods. For these programs the certificates of State registration are got [9; 10].

In present paper the elasto-plastic boundary for the problem of torsion of a straight beam of arbitrary cross-section, which is limited by a piecewise smooth contour is constructed; and the elasto-plastic boundaries for the problems of a consol bending and anti-plane deformation are found. For convenience, the article is divided into sections. In the first section the basic equations of elasticity and boundary problems are considered; in the second section the basic equations of the theory of ideal plasticity of von Mises are given; in the third section the conditions on the boundaries of the elastic and plastic domains are formulated. The fourth section is devoted to torsion of elastic prismatic rods; the fifth one describes elastic bending of bars; in the sixth section the plane problem of theory of elasticity is given. The seventh section covers an anti-plane problem of elasticity theory; in the eighth section, conservation laws for the equations of elasticity are constructed; in the ninth one, conservation laws of two-dimensional equations of plasticity are discussed. In the tenth section an elasto-plastic boundary of a twisted straight rod is found; in the eleventh one an elasto-plastic boundary in the bended console is given; and finally, in the twelfth section a method for the construction of elasto-plastic boundaries for large areas is described.

1. The Basic Equations of Elasticity and Boundary Problems. Let's consider steady-state equations of linear isotropic elasticity.

The equilibrium equations look like:

$$\begin{gathered}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X=0,\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+Y=0,\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0,\end{gathered}\tag{1}$$

here $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are components of a stress tensor, X, Y, Z are components of an external force affected to a unit of volume. The components of a stress tensor related to components of a strain tensor by means of Hook's law:

$$\begin{gathered}\sigma_x=\lambda\varepsilon+2\mu\varepsilon_x,\ \sigma_y=\lambda\varepsilon+2\mu\varepsilon_y,\ \sigma_z=\lambda\varepsilon+2\mu\varepsilon_z,\\ \tau_{xy}=2\mu\varepsilon_{xy},\ \tau_{xz}=2\mu\varepsilon_{xz},\ \tau_{yz}=2\mu\varepsilon_{yz}.\end{gathered}\tag{2}$$

Here $\varepsilon=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$, $\varepsilon_x=\frac{\partial u}{\partial x}$, $\varepsilon_y=\frac{\partial v}{\partial y}$, $\varepsilon_z=\frac{\partial w}{\partial z}$, $2\varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$, $2\varepsilon_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$, $2\varepsilon_{yz}=\frac{\partial u}{\partial z}+$ $+\frac{\partial w}{\partial y}$, and λ, μ are constants of Lamé, u, v, w are components of a vector of deformations, $\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$ are components of a strain tensor.

Taking into account (2), equations of theory of elasticity can be written using displacements:

$$\begin{gathered}(\lambda+\mu)\frac{\partial\varepsilon}{\partial x}+\mu\Delta u=0,\\ (\lambda+\mu)\frac{\partial\varepsilon}{\partial y}+\mu\Delta v=0,\\ (\lambda+\mu)\frac{\partial\varepsilon}{\partial z}+\mu\Delta w=0,\end{gathered}\tag{3}$$

here Δ is the Laplace operator.

On account of (2) the components of a stress tensor are in accord with the compatibility equations along with the equilibrium equations (1). The compatibility equations here are written in case of absence of external forces:

$$(1+\nu)\Delta\sigma_x + \frac{\partial^2\theta}{\partial x^2} = 0,\ (1+\nu)\Delta\sigma_y + \frac{\partial^2\theta}{\partial y^2} = 0,$$

$$(1+\nu)\Delta\sigma_z + \frac{\partial^2\theta}{\partial z^2} = 0,\ (1+\nu)\Delta\tau_{xy} + \frac{\partial^2\theta}{\partial x\partial y} = 0, \qquad (4)$$

$$(1+\nu)\Delta\tau_{xz} + \frac{\partial^2\theta}{\partial x\partial z} = 0,\ (1+\nu)\Delta\tau_{yz} + \frac{\partial^2\theta}{\partial y\partial z} = 0,$$

$\theta = \sigma_x + \sigma_y + \sigma_z$, ν is a Poisson's ratio.

Problems for elasticity equations are usually pose either using displacements (in this case one have to solve equations (3)) or using stresses (in that case one solves equations (1)–(3)).

If a problem is written using stresses, one should add boundary conditions to equations (1), (3):

$$\overline{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n,$$

$$\overline{Y} = \tau_{xy} l + \sigma_y m + \tau_{yz} n, \qquad (5)$$

$$\overline{Z} = \tau_{xz} l + \tau_{yz} m + \sigma_z n,$$

here l, m, n are direction cosines of an external normal line to the boundary surface at point under study, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are components of a vector of superficial forces affected to a unit of area.

If the problem is written in displacements then on a boundary S these displacements are specified:

$$u/_s = \overline{u},\ v/_s = \overline{v},\ w/_s = \overline{w}, \qquad (6)$$

here $\overline{u}$, $\overline{v}$, $\overline{w}$ are certain functions on S.

Remark. There are others problems in the theory of elasticity, but they are not adduced in this article.

2. The Basic Equations of the Theory of Plasticity of Mises. For steady-state equations of the theory of plasticity of Mises one should add the plasticity law of Mises to the equilibrium equations (1). This law looks like:

$$\left(\sigma_x - \frac{1}{3}\theta\right)^2 + \left(\sigma_y - \frac{1}{3}\theta\right)^2 + \left(\sigma_z - \frac{1}{3}\theta\right)^2 + \\ + 2\tau_{xy}^2 + 2\tau_{xz}^2 + 2\tau_{yz}^2 = 2k^2, \qquad (7)$$

here k is a yield point under simple shear.

In the case of plane deformation the plasticity law (7) can be reduce to form:

$$\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2 = 4k^2. \qquad (8)$$

In plastic domain, the components of deviator of the strain tensor relate to the components of tensor of a strain rate with correlations

$$\sigma_x - \tfrac{1}{3}\theta = S_x = \Lambda e_x,\ \sigma_y - \tfrac{1}{3}\theta = S_y = \Lambda e_y,$$

$$\sigma_z - \tfrac{1}{3}\theta = S_z = \Lambda e_z,\ \tau_{xy} = \Lambda e_{xy}, \qquad (9)$$

$$\tau_{yz} = \Lambda e_{yz},\ \tau_{xz} = \Lambda e_{xz},$$

here Λ is a nonnegative function obtained from (7):

$$e_x = \frac{\partial u^1}{\partial x},\ e_y = \frac{\partial u^2}{\partial y},\ e_z = \frac{\partial u^3}{\partial z},$$

$$2e_{xy} = \frac{\partial u^1}{\partial y} + \frac{\partial u^2}{\partial x},\ 2e_{xz} = \frac{\partial u^1}{\partial z} + \frac{\partial u^3}{\partial x}, \qquad (10)$$

$$2e_{yz} = \frac{\partial u^2}{\partial z} + \frac{\partial u^3}{\partial y},$$

here u^1, u^2, u^3 are components of the vector of strain rate.

3. Conditions on the Boundaries of the Elastic and Plastic domains. Determination of a boundary separating elastic and plastic domains is one of the most difficult problems of the solid mechanics. The boundary is not known in advance and is defined during the elasto-plastic problem solving. In some cases a shape of the boundary can be guessed by general considerations.

Assume that an elastic state of a medium continuously changes over to a yield state. In this case close to elasto-plastic boundary and on each side of it the Hook's law is applies. This fact leads to the continuity of all components of the stress tensor and strain tensor, on the elasto-plastic boundary.

4. Torsion of Elastic Prismatic Rods. Let's consider an elastic prismatic rod with a cross-section of a variable form. It's lateral surface is free from efforts, face planes have forces equivalent to rotational moment *M*.

Let the coordinate origin is placed in an arbitrary point of the face plane and axis *z* is parallel to generatrix of the rod. The boundary conditions (3) will look like:

$$\sigma_x l + \tau_{xy} m = 0,$$

$$\tau_{xy} l + \sigma_y m = 0, \qquad (11)$$

$$\tau_{xz} l + \tau_{yz} m = 0,$$

and on the face planes of the rod ($z = 0,\ z = l$)

$$\iint_\Omega \tau_{xz} dxdy = 0,\ \iint_\Omega \tau_{yz} dxdy = 0,$$

$$\iint_\Omega \sigma_z dxdy = 0,\ \iint_\Omega x\sigma_z dxdy = 0, \qquad (12)$$

$$\iint_\Omega y\sigma_z dxdy = 0,$$

$$\iint_\Omega \left(x\tau_{yz} - y\tau_{xz}\right) dxdy = M, \qquad (13)$$

here Ω is the area of a cross-section. As is the convention in the theory of torsion:

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \qquad (14)$$

and the remaining components of the stress tensor are in accord with equilibrium equations (1) which are the following form taking into account (14):

$$\frac{\partial\tau_{xz}}{\partial x} = 0,\ \frac{\partial\tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0. \qquad (15)$$

Compatibility equations (4) will look like

$$\frac{\partial^2\sigma_z}{\partial x^2}=\frac{\partial^2\sigma_z}{\partial y^2}=\frac{\partial^2\sigma_z}{\partial z^2}=\frac{\partial^2\sigma_z}{\partial x\partial y}=0,$$

$$(1+\nu)\Delta\tau_{xz}+\frac{\partial^2\sigma_z}{\partial x\partial y}=0, \qquad (16)$$

$$(1+\nu)\Delta\tau_{yz}+\frac{\partial^2\sigma_z}{\partial y\partial z}=0.$$

From the equations (16) one can get

$$\sigma_z = Azy+Bzx+Dx+Ey+Fz+H, \qquad (17)$$

here A, B, D, E, F, H are arbitrary constants.

By substituting (17) into (13) on gets that $\sigma_z=0$ in all alternate cross-sections of the rod. Therefore equations (15), (16) are reduced to the following:

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0, \qquad (18)$$

$$\Delta\tau_{xz}=0,\ \ \Delta\tau_{yz}=0. \qquad (19)$$

Let's transform the equations (18), (19). For this purpose we'll derive the equation (18) on x and subtract from it the first equation (19):

$$\frac{\partial}{\partial y}\left(\frac{\partial\tau_{yz}}{\partial x}-\frac{\partial\tau_{xz}}{\partial y}\right)=0. \qquad (20)$$

Now we'll derive the equation (18) on y and subtract from it the second equation (19).

$$\frac{\partial}{\partial x}\left(\frac{\partial\tau_{xz}}{\partial y}-\frac{\partial\tau_{yz}}{\partial x}\right)=0. \qquad (21)$$

It follows from (19) and (20)

$$\frac{\partial\tau_{xz}}{\partial y}-\frac{\partial\tau_{yz}}{\partial x}=C, \qquad (22)$$

here C is an arbitrary constant.

A system (18), (19) may be replaced be equations (18) and (22).

As from (12) one can get

$$\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\ \ \tau_{yz}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right), \qquad (23)$$

then

$$\frac{\partial\tau_{xz}}{\partial y}-\frac{\partial\tau_{yz}}{\partial x}=\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right). \qquad (24)$$

It is known that $\frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$ is the third component of the vector $\operatorname{rot}(u,v,w)$. We'll obtain

$$\frac{\partial\tau_{xz}}{\partial y}-\frac{\partial\tau_{yz}}{\partial x}=-2\mu\frac{\partial\omega_z}{\partial z}, \qquad (25)$$

here $\frac{\partial\omega_z}{\partial z}$ is the angle of a torsion per unit length of fiber of the rod. This angle is called twist and is denoted θ. From (25) and (22) one gets

$$C=-2\mu\frac{\partial\omega_z}{\partial z}=-2\mu\theta. \qquad (26)$$

We obtained finally that the problem of torsion of elastic prismatic rod comes to integration the following equations

$$\frac{\partial\tau_{xz}}{\partial y}+\frac{\partial\tau_{yz}}{\partial x}=0,\ \ \frac{\partial\tau_{xz}}{\partial y}-\frac{\partial\tau_{yz}}{\partial x}=-2\mu\theta, \qquad (27)$$

with boundary conditions

$$\tau_{xz}l+\tau_{yz}m=0. \qquad (28)$$

As the equations (27) come to Poisson's equation then it is the base of numerous examples of solving the problem of torsion of elastic prismatic rods.

This fact leads to analogies which permit to reduce the problem (27), (28) to others mechanical problems which solution is described by the same equations. Here are some of them: membrane analogy, some fluid-flow analogies, electrodynamic analogy. One can introduce a term torsional hardness $C=M/\theta$.

It is considered, the bigger a torsional hardness the better a rod resists to the torsion.

It was shown that among all the prismatic rods with the same area of lateral face, the biggest torsional hardness belongs to a rod with a circular cross-section.

Moreover, it is proved that among all the prismatic rods with multiply connected cross-section of the defined area and the defined total area of holes, the rod with ring-shaped cross-section which is bounded by two concentric circles has the highest torsional hardness. These and other problems of the theory of torsion of elastic bodies one can find in [11].

Saint-Venant noted an interesting fact: the maximum tangential stress as a rule is achieved upon the lateral face of a rod in the points the closest to a center of gravity of a cross-section.

5. Elastic Bending of Bars. Let's consider a prismatic rod bending by two equal and opposite moments M in one of the principal plane (fig. 1).

The coordinate origin is in the centre of gravity of a cross-section, the plane xz is in the main plane of bending. One gets the following elementary solution of the equations (1) in case of absence of body forces:

$$\sigma_z=\frac{Ex}{R},\ \ \sigma_y=\sigma_x=\tau_{xy}=\tau_{xz}=\tau_{yz}=0, \qquad (29)$$

here R is a radius of curvature of the bended rod; E is Young's modulus of stretch and compression.

Let's consider a common case of bending of a console with the constant cross-section, which is under the action of a force P applied to an end and which is parallel to one of a main axes of the cross-section (fig. 2).

Let's suppose that in console case, stresses allocate in a distance z from the fixed end in the same way as (29):

$$\sigma_z=\frac{P(l-z)x}{l}. \qquad (30)$$

Let's suppose now that in every point of the cross-sections tangential stresses τ_{xz} and τ_{yz} affected and the other components of the stress tensor $\sigma_x,\sigma_y,\tau_{xy}$ are equal to zero.

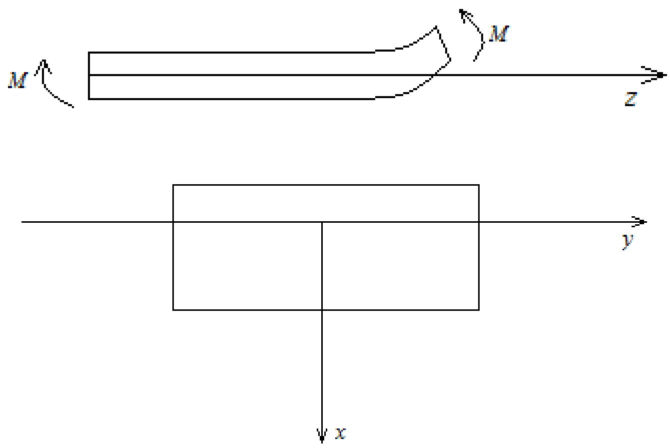

Fig. 1. Bending of a Bar

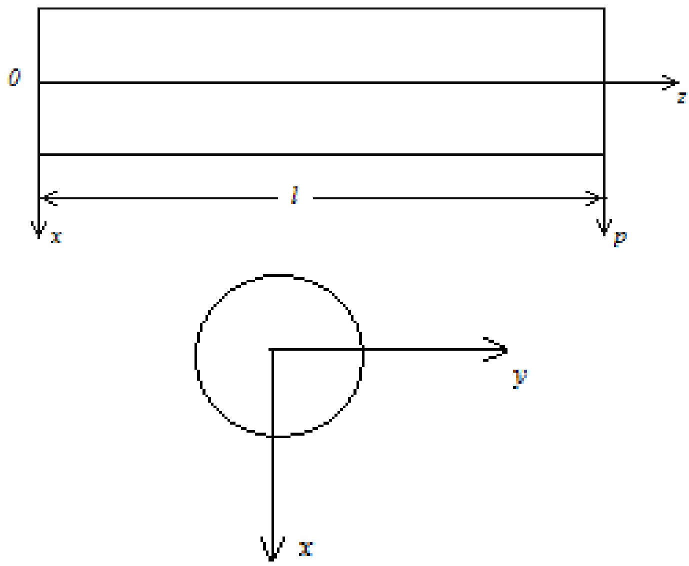

Fig. 2. Bending of a Console

By such suppositions, in case of absence of volume forces one gets from the equations (1)

$$\frac{\partial \tau_{xz}}{\partial z} = 0,\ \frac{\partial \tau_{yz}}{\partial z} = 0,\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{Px}{l}. \tag{31}$$

It is follows from (31) that tangent stresses do not depend on z, and they are the same for every cross-section.

The compatibility equations come to following:

$$\Delta\tau_{xz} = -\frac{P}{l(1+v)},\ \Delta\tau_{yz} = 0. \tag{32}$$

One gets in the same way as in the previous paragraph:

$$\begin{gathered} \frac{\partial}{\partial y}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right) = \frac{P}{l(1+v)}, \\ \frac{\partial}{\partial x}\left(\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x}\right) = 0. \end{gathered} \tag{33}$$

It is obtained from the formulas (33)

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -\frac{Py}{l(1+v)} + C,$$

here C is a constant. It is possible to show that $C = 0$ [11]. Then equations of the bending of console look like:

$$\begin{cases} \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = -\dfrac{Px}{l}, \\ \dfrac{\partial \tau_{xz}}{\partial y} - \dfrac{\partial \tau_{xz}}{\partial x} = -\dfrac{Py}{l(1+v)}. \end{cases} \tag{34}$$

One should add a boundary condition to these equations which is the following on the frontier of the contour

$$\tau_{xz} l + \tau_{yz} m = 0\ .$$

6. A Plane Problem of Elasticity Theory. In this section the equations of a plane problem of elasticity theory in displacements are given and some boundary problems are posed.

For a plane problem the following conditions are valid:

$$u = u(x,y),\ \ v = v(x,y),\ \ w = 0.$$

Then from (3) one gets

$$F_1 = (\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0,$$
$$F_2 = (\lambda+\mu)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0. \qquad (35)$$

The boundary conditions (5) look like

$$\sigma_x l + \tau_{xy} m = \left[(\lambda+\mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu \frac{\partial u}{\partial x}\right] l +$$
$$+\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) m = \overline{X},$$
$$\tau_{xy} l + \sigma_y m = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) l + \qquad (36)$$
$$+\left[(\lambda+\mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu \frac{\partial v}{\partial y}\right] m = \overline{Y}.$$

7. An Anti-plane Problem of Elasticity Theory. Let's consider equations (1), (2) when $u = v = 0$, $w = \omega(x,y)$. This case corresponds to so-called anti-plane elastic state.

Equations (1) come to

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0,\ \ X = Y = 0,$$

and compatibility conditions for deformations come to equation

$$\partial_y \tau_{xz} - \partial_x \tau_{yz} = 0.$$

Let the elastic body be affected by only its dead weigh, then if the axis *Oz* is up-directed one receives $z = -\rho g$, here ρ is a constant density.

Finally equations describing elastic state on condition that the deformation is anti-plane look as follow

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \rho g = 0,$$
$$\frac{\partial}{\partial y}\tau_{xz} - \frac{\partial}{\partial y}\tau_{yz} = 0. \qquad (37)$$

8. Conservation Laws for the Equations of Elasticity. Conservation laws are the fundamental laws of nature, they were determined in the beginning of the XXth century. A concept of the conservation laws appeared later after researches of E. Noether and her followings. The wide application of these laws to solving and investigations of some differential equations is relative to the last quarter of the XXth century. But significance and usefulness of this concept is not properly understood by majority of researchers even nowadays.

In this work the simplest definition of the conservation laws is given. For more details see [11] and cited literature there.

Let's $F_1 = 0,\ F_2 = 0$ is a system of two differential equations for two sought functions $u = u(x,y)$.

Definition. Conserved current for the system $F_1 = 0$, $F_2 = 0$, is a vector (A,B) which is

$$\frac{\partial}{\partial x}A + \frac{\partial}{\partial y}B = \Pi_1 F_1 + \Pi_2 F_2, \qquad (38)$$

here Π_i are some differentiation operators. It is assumed that both of them are not equal to zero simultaneously.

Let's find conservation laws for equations from the sections 4–7.

1. The equations describing the elastic torsion (27) in convenient denotation look as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = u_x + v_y = 0,$$
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y_x} = u_y - v_x = \alpha, \qquad (39)$$

here $u = \tau_{xz}; v = \tau_{yz}; \alpha = -2\mu\theta.$

Let

$$A = \alpha^1 u + \beta^1 v + \gamma^1,\ \ B = \alpha^2 u + \beta^2 v + \gamma^2, \qquad (40)$$

where $\alpha^i, \beta^i, \gamma^i$ are arbitrary functions of x, y.

From (38) with respect to (40) and (39) one obtains

$$\alpha^1_x u + \alpha^1 u_x + \beta^1_x v + \beta^1 v_x + \gamma^1_x + \alpha^2_y u + \alpha^2 u_y +$$
$$+\beta^2_y v + \beta^2 v_y + \gamma^2_y = \alpha^1(u_x + v_y) + \alpha^2(u_y - v_x - \alpha).$$

Here and further, subscript signifies a corresponding variable derivative.

One can get hence

$$\alpha^1 = \beta^2,\ \beta^1 = -\alpha^2,\ \gamma^1_x + \gamma^2_y = -\alpha^2\alpha,$$
$$\alpha^1_x + \alpha^2_y = 0,\ \beta^1_x + \beta^2_y = 0.$$

Or after simple conversion

$$\alpha^2 = -\beta^1,\ \beta^2 = \alpha^1,$$
$$\gamma^1_x + \gamma^2_y = -\alpha^2\alpha,\ \alpha^1_x - \beta^1_y = 0, \qquad (41)$$
$$\beta^1_x + \alpha^1_y = 0.$$

Therefore, a conserved current look as follows

$$A = \alpha^1 u + \beta^1 v + \gamma^1,$$
$$B = -\beta^1 u + \alpha^1 v + \gamma^2, \qquad (42)$$

here (α^1, β^1) is a solution of the Cauchy–Riemann system; γ^1, γ^2 are determined from the equation (41).

With respect to (42) the conservation law may be defined in the following form

$$\int_\Gamma (\alpha^1 u + \beta^1 v + \gamma^1) dy - (-\beta^1 u + \alpha^1 v + \gamma^2) dx = 0, \qquad (43)$$

where Γ is an arbitrary piecewise smooth closed contour.

Let (x_0, y_0) a point inside the domain bounded by Γ. One can choose

$$\alpha^1 = \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},$$
$$\beta^1 = -\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}. \tag{44}$$

Let Γ_1 is a circle $(x-x_0)^2+(y-y_0)^2=R^2$ (fig. 3).

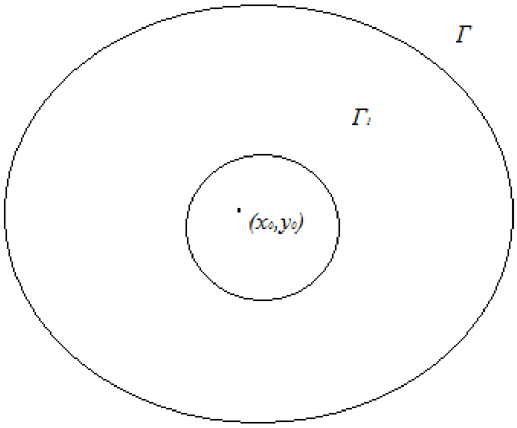

Fig. 3. A Circle Γ_1

It's not complicated to indicate that integral (44)

$$\int_\Gamma \left(\alpha^1 u+\beta^1 v+\gamma^1\right)dy-\left(-\beta^1 u+\alpha^1 v+\gamma^2\right)dx =$$
$$= -\int_{\Gamma_1} \left(\alpha^1 u+\beta^1 v+\gamma^1\right)dy-\left(-\beta^1 u+\alpha^1 v+\gamma^2\right)dx. \tag{45}$$

Let's calculate the circulation integral on Γ_1 using polar coordinates $x-x_0=R\cos\theta$, $y-y_0=R\sin\theta$:

$$\int_{\Gamma_1} \left(\alpha^1 u+\beta^1 v+\gamma^1\right)dy-\left(-\beta^1 u+\alpha^1 v+\gamma^2\right)dx =$$
$$= \int_0^{2\pi}\left[\left(\frac{\cos\theta}{R}u-\frac{\sin\theta}{R}v+\gamma^1\right)R\cos\theta+\right.$$
$$\left.+\left(\frac{\sin\theta}{R}u+\frac{\cos\theta}{R}v+\gamma^2\right)R\sin\theta\right]d\theta =$$
$$= \int_0^{2\pi}\left[u+\gamma^1 R\cos\theta+\gamma^2 R\sin\theta\right]d\theta =$$
$$= \int_0^{2\pi} u\,d\theta + R\int_0^{2\pi}\left(\gamma^1\cos\theta+\gamma^2\sin\theta\right)d\theta.$$

In the last expression R tends to zero ($R\to 0$), and using mean-value one gets

$$\int_{\Gamma_1} \left(\alpha^1 u+\beta^1 v+\gamma^1\right)dy-\left(-\beta^1 u+\alpha^1 v+\gamma^2\right)dx = 2\pi u\left(x_0,y_0\right).$$

Now from (45)

$$u(x_0,y_0)=$$
$$=\frac{1}{2\pi}\int_\Gamma \left(\alpha^1 u+\beta^1 v+\gamma^1\right)dy-\left(-\beta^1 u+\alpha^1 v+\gamma^2\right)dx. \tag{46}$$

Let this time

$$\alpha^1 = \frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} = a^2,$$
$$\beta^1 = \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} = \beta^2. \tag{47}$$

Let's calculate the circulation integral on Γ_1 in this case. In polar coordinates $x-x_0=R\cos\theta$, $y-y_0=R\sin\theta$:

$$\int_{\Gamma_1} = \int_0^{2\pi}\left\{\left[\frac{\sin\theta}{R}u+\frac{\cos\theta}{R}v+\gamma^1\right]R\cos\theta+\right.$$
$$\left.+\left[-\frac{\cos\theta}{R}u+\frac{\sin\theta}{R}v+\gamma^2\right]R\sin\theta\right\}d\theta =$$
$$= \int_0^{2\pi}\left[v+R\left(\gamma^1\cos\theta+\gamma^2\sin\theta\right)\right]d\theta.$$

On conditions that $R\to 0$ one obtains from the last formula and (46):

$$v(x_0,y_0)=\frac{1}{2\pi}\int_\Gamma\left(\alpha^2 u+\beta^2 v+\gamma^1\right)dy-$$
$$-\left(-\beta^2 u+\alpha^2 v+\gamma^2\right)dx. \tag{48}$$

Expressions (47) and (48) allow to calculate values u, v at any internal point of the domain enclosed by Γ if the values u, v on the contour are known. But $u=\tau_{xz}, v=\tau_{yz}$ therefore these two values are not known on Γ, it known only the expression $\tau_{xz}l+\tau_{yz}m=0$. Hence formulas (47), (48) don't allow to calculate the values of the stress tensor inside the domain and then don't allow to solve the problem of the rod torsion. But as one will see later these formulas allow to resolve the elasto-plastic problem which is more complicated.

Remark. The similar formulas can be obtained easily for the equations of anti-plane theory of elasticity, which is given in the section 7.

Equations describing the torsion of a bar have a form (34). Let us assume in these equations

$$u=\tau_{xz},\ v=\tau_{yz},$$

$$-P\,{}^{x}\!/_{l}=\omega_1,\ -P\,{}^{y}\!/_{l(1+v)}=\omega_2,$$

then

$$u_x+v_y=\omega_1,\ u_x-v_y=\omega_2. \tag{49}$$

The conserved current for this system will be found in a form

$$A=\alpha^1 u+\beta^1 v+\gamma^1,$$
$$B=\alpha^2 u+\beta^2 v+\gamma^2, \tag{50}$$

here $\alpha^i,\beta^i,\gamma^i$ are fuctions of x,y only.

Analogously to the previous clause, one can get

$$A_x+B_y=\alpha_x^1 u+\alpha^1 u_x+\beta_x^1 v+\beta^1 v_x+\gamma_x^1+$$
$$+\alpha_y^2 u+\alpha^2 u_y+\beta_y^2 v+\beta^2 v_y+\gamma_y^2=$$

$$= \alpha^1 \left(u_x + v_y - \omega_1\right) + \alpha^2 \left(u_y - v_x - \omega_2\right).$$

One can obtain from here

$$\alpha^1 = \beta^2,\ \beta^1 = -\alpha^2,\ \gamma^1_x + \gamma^2_y = -\alpha^1\omega_1 - \alpha^2\omega_2,$$
$$\alpha^1_x + \alpha^2_y = 0,\ \beta^1_x + \beta^2_y = 0.$$

After not complicated calculations

$$\alpha^2 = -\beta^1,\ \beta^2 = \alpha^1,$$
$$\gamma^1_x + \gamma^2_y = -\alpha^1\omega_1 - \alpha^2\omega_2, \tag{51}$$
$$\alpha^1_x - \beta^1_y = 0,\ \beta^1_x + \alpha^1_y = 0.$$

Hence, the conserved current looks like

$$\begin{aligned} A &= \alpha u + \beta v + \gamma^1, \\ B &= -\beta u + \alpha v + \gamma^2, \end{aligned} \tag{52}$$

here $(\alpha,\ \beta)$ is an arbitrary solution of Cauchy–Riemann equations; γ^1, γ^2 are determined from the equation (52).

For the current (52) a conservation law can be written as follows

$$\int_\Gamma \left(\alpha u + \beta v + \gamma^1\right)dy - \left(-\beta u + \alpha v + \gamma^2\right)dx = 0.$$

Acting much as the previous clause one can obtain finally

$$u(x_0 . y_0) = \frac{1}{2\pi}\int_\Gamma \left(\alpha^1 u + \beta^1 v + \gamma^1\right)dy - \left(-\beta^1 u + \alpha^1 v + \gamma^2\right)dx,$$

$$v(x_0 . y_0) = \frac{1}{2\pi}\int_\Gamma \left(\alpha^2 u + \beta^2 v + \gamma^1\right)dy - \left(-\beta^2 u + \alpha^2 v + \gamma^2\right)dx.$$

Remarks of the previous item are also correct for this problem.

Conservation laws of the plane theory of elasticity.

Let's find some conservation laws for equations describing 2-dimensional resilience (35) as it is in the work [12]. The conserved current is looked for in a form

$$A = \alpha_1 \left(\lambda + 2\mu\right)u_x + \beta_1\mu u_y + \gamma_1\mu v_x + \left(\lambda + 2\mu\right)\delta_1 v_y,$$

$$B = \alpha_2 u_x + \beta_2 u_y + \gamma_2 v_x + \delta_2 v_y,$$

here $\alpha_i, \beta_i, \gamma_i, \delta_i$, some functions of x, y.

From the relation

$$\frac{\partial}{\partial x}A + \frac{\partial}{\partial y}B = \alpha_1 F_1 + \gamma_1 F_2. \tag{53}$$

From (53) one can obtain

$$\alpha_2 = -\beta_1 + \left(\lambda + \mu\right)\gamma_1,$$
$$\beta_2 = \alpha_1\mu,\ \gamma_2 = -\delta_1 + \left(\lambda + \mu\right)\alpha_1, \tag{54}$$
$$\delta_2 = \left(\lambda + 2\mu\right)\gamma_1,$$

$$\left(\lambda + 2\mu\right)\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_2}{\partial y} = 0,$$
$$\frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_2}{\partial y} = 0,\ \mu\frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_2}{\partial y} = 0, \tag{55}$$
$$\frac{\partial \delta_1}{\partial x} + \frac{\partial \delta_2}{\partial y} = 0.$$

Making substitution of (54) into (55) one can get

$$\left(\lambda + 2\mu\right)\frac{\partial^2 \alpha_1}{\partial x^2} + \frac{\partial^2 \gamma_1}{\partial x \partial y}\left(\lambda + \mu\right) + \mu\frac{\partial^2 \alpha_1}{\partial y^2} = 0,$$

$$\mu\frac{\partial^2 \gamma_1}{\partial x^2} + \left(\lambda + \mu\right)\frac{\partial^2 \alpha_1}{\partial x \partial y} + \left(\lambda + 2\mu\right)\frac{\partial^2 \gamma_1}{\partial y^2} = 0,$$

or

$$\left(\lambda + 2\mu\right)\frac{\partial^2 \beta_1}{\partial x^2} + \frac{\partial^2 \delta_1}{\partial x \partial y}\left(\lambda + \mu\right) + \mu\frac{\partial^2 \beta_1}{\partial y^2} = 0,$$

$$\mu\frac{\partial^2 \delta_1}{\partial x^2} + \left(\lambda + \mu\right)\frac{\partial^2 \beta_1}{\partial x \partial y} + \left(\lambda + 2\mu\right)\frac{\partial^2 \delta_1}{\partial y^2} = 0.$$

It means that $\left(\alpha_1, \gamma_1\right)$ and $\left(\beta_1, \delta_1\right)$ are arbitrary solutions of the equations (35) which are coupled by correlations (55). This fact permits to construct an infinite system of conservation laws on a base of the exact solutions of equations of elasticity.

9. Conservation Laws of Two-dimensional Equations of Plasticity. Let us consider the following equations of two-dimensional plane theory of plasticity which can be obtained with ease from the equation of the section 2:

$$\frac{\partial \sigma}{\partial x} - 2k\left(\frac{\partial \theta}{\partial x}\cos 2\theta + \frac{\partial \theta}{\partial y}\sin 2\theta\right) = 0, \tag{56}$$

$$\frac{\partial \sigma}{\partial y} - 2k\left(\frac{\partial \theta}{\partial x}\sin 2\theta - \frac{\partial \theta}{\partial y}\cos 2\theta\right) = 0. \tag{57}$$

Here σ is a hydrostatic pressure, $\theta = \left(1, x\right) - \frac{\pi}{4}$, $\left(1, x\right)$ is an angle between the main direction of the stress tensor and the axis *Ox*.

Let's find conservation laws of a system (86), (57) in the form $C = C\left(\sigma, \theta\right)$ for which equality

$$\frac{\partial C}{\partial x} + \frac{\partial D}{\partial y} = 0$$

or owing to Green's formula

$$\oint_\Gamma D dx - C dy = 0 \tag{58}$$

is correct on account of system, i. e. relation

$$\frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial x} + \frac{\partial C}{\partial \theta \partial x} + \frac{\partial D}{\partial \sigma}\frac{\partial \sigma}{\partial y} + \frac{\partial D}{\partial \theta}\frac{\partial \theta}{\partial y} = 0 \tag{59}$$

has to be performed for all its solutions in a domain bounded by the smooth contour Γ.

Let's determine a system of plasticity in a normal matrix form [5]:

$$\begin{pmatrix} \dfrac{\partial \sigma}{\partial x} \\ \dfrac{\partial \theta}{\partial x} \end{pmatrix} + \begin{pmatrix} -\dfrac{\cos 2\theta}{\sin 2\theta} & -\dfrac{2k}{\sin 2\theta} \\ -\dfrac{1}{2k \sin 2\theta} & -\dfrac{\cos 2\theta}{\sin 2\theta} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \sigma}{\partial y} \\ \dfrac{\partial \theta}{\partial y} \end{pmatrix} = 0. \tag{60}$$

Multiplying this system by the vector $\left(\frac{\partial C}{\partial\sigma},\frac{\partial C}{\partial\theta}\right)$ one can get the following equation

$$\frac{\partial C}{\partial\sigma}\frac{\partial\sigma}{\partial x}+\frac{\partial C}{\partial\theta}\frac{\partial\theta}{\partial x}-\left(\frac{\partial C}{\partial\sigma}\frac{\cos 2\theta}{\sin 2\theta}+\frac{\partial C}{\partial\theta}\frac{1}{2k\sin 2\theta}\right)\frac{\partial\sigma}{\partial y}- -\left(\frac{\partial C}{\partial\sigma}\frac{2k}{\sin 2\theta}+\frac{\partial C}{\partial\theta}\frac{\cos 2\theta}{\sin 2\theta}\right)\frac{\partial\theta}{\partial y}=0. \tag{61}$$

Comparing the equations (60) and (61) it is possible to obtain two exppressions for the functions C and D:

$$\begin{aligned}\frac{\partial D}{\partial\sigma}&=-\frac{\partial C}{\partial\sigma}\frac{\cos 2\theta}{\sin 2\theta}-\frac{\partial C}{\partial\theta}\frac{1}{2k\sin 2\theta},\\ \frac{\partial D}{\partial\theta}&=-\frac{\partial C}{\partial\sigma}\frac{2k}{\sin 2\theta}-\frac{\partial C}{\partial\theta}\frac{\cos 2\theta}{\sin 2\theta}.\end{aligned} \tag{62}$$

Let's express the components $\frac{\partial C}{\partial\theta},\frac{\partial D}{\partial\theta}$ of the linear system (62) in an explicit form:

$$\begin{aligned}&\frac{\partial C}{\partial\theta}+2k\left(\frac{\partial D}{\partial\sigma}\sin 2\theta+\frac{\partial C}{\partial\sigma}\cos 2\theta\right)=0,\\ &\frac{\partial D}{\partial\theta}-2k\left(\frac{\partial D}{\partial\sigma}\cos 2\theta-\frac{\partial C}{\partial\sigma}\sin 2\theta\right)=0.\end{aligned} \tag{63}$$

It is possible to remark that by substitution $C=-y(\sigma,\theta)$, $D=x(\sigma,\theta)$, the system (63) coincides with a linearized plasticity system

$$\begin{aligned}&y_\theta-2k\left(-y_\sigma\cos 2\theta+x_\sigma\sin 2\theta\right)=0,\\ &x_\theta-2k\left(y_\sigma\sin 2\theta+x_\sigma\cos 2\theta\right)=0.\end{aligned} \tag{64}$$

This fact permits to use all the proprieties of this system during the conservation laws construction.

Thus, the linearization of the plasticity system is achieved without the requirement of the non being zero to Jacobian.

Further using the substitution

$$\xi=\sigma+2k\theta,\quad \eta=\sigma-2k\theta,$$

the system (63) comes to equations:

$$\begin{aligned}&\frac{\partial D}{\partial\xi}-\frac{\partial C}{\partial\xi}\operatorname{tg}\theta=0,\\ &\frac{\partial D}{\partial\eta}+\frac{\partial C}{\partial\eta}\operatorname{ctg}\theta=0.\end{aligned} \tag{65}$$

If insert new independent functions φ,ψ

$$\begin{aligned}\varphi&=D-\operatorname{tg}\theta C,\\ \psi&=D+\operatorname{ctg}\theta C,\end{aligned} \tag{66}$$

it is possible to obtain the system

$$\begin{aligned}&\frac{\partial\varphi}{\partial\xi}-\frac{1}{2}\operatorname{tg}\theta\left(\psi-\varphi\right)=0,\\ &\frac{\partial\varphi}{\partial\eta}+\frac{1}{2}\operatorname{ctg}\theta\left(\psi-\varphi\right)=0.\end{aligned} \tag{67}$$

Finally, by setting $\rho=\varphi\cos\theta$ one can come to equation:

$$\rho_{\xi\eta}-\frac{\rho}{4}=0. \tag{68}$$

And this is the well-known telegraph equation.

Thereby the construction of conservation laws for the plasticity equations comes to solving of the linear systems for which a lot of methods of resolution of equations and boundary problems are developed.

10. Elasto-plastic Boundary of a Twisted Straight Rod. Let's consider an elasto-plastic torsion of a straight rod which cross-section is bounded by a convex contour Γ.

If the twisting moment is rather significant, a plastic domain P forms in the rod. This domain arises on the external contour Γ. Suppose that the plastic domain is covered completely by the contour. In this case in the cross-section two domains appear, a plastic one P and an elastic one F. L is a boundary of these domains (fig. 4).

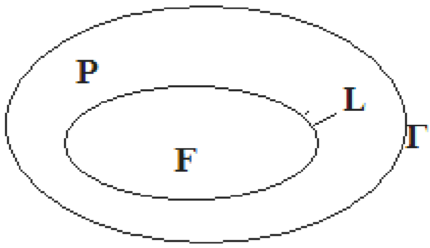

Fig. 4. Cross-section of the Twisted Rod

There are a lot of works devoted to solution of the problem of the stressed state of an elasto-plastic rod, but most of them are based on some assumptions concerning the form of boundary L which is not known in advance. A novel method of the determination of unknown boundary is proposed by B. D. Annin [1]. This method is based on contact transformations and it permits to define the boundary between elastic and plastic domains in the rods with oval cross-section. This problem one can find in [1] and in the bibliography cited there.

In the present work the stress state is defined in all internal points of the rod by means of conservation laws, and formulas for analytical calculations of these stresses are proposed in the case of a piecewise-smooth directed boundary of the cross-section. The conservation laws is used for a long time in a fruitful way for solving of various mathematical and mechanical problems. A summary of results and solved problems in different domains of mechanics can be found in [4; 5; 7].

Problem Definition

Let τ_{xz},τ_{yz} are single non-zero components of the stress tensor. In the elastic domain they satisfy the equilibrium equation

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0 \tag{69}$$

and the equations

$$\tau_{xz}=G\theta\left(\frac{\partial\psi}{\partial x}-y\right),\quad \tau_{yz}=G\theta\left(\frac{\partial\psi}{\partial y}+x\right). \tag{70}$$

Here function $\theta\psi(x,y)$ determines a deplanation warpingof the cross-section, θ is a constant, G is a Young modulus by shear.

Let's introduce the stress function φ as following

$$\tau_{xz} = \frac{\partial \varphi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad (71)$$

then to determine of φ in the elastic domain one can get the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = a, \qquad (72)$$

here $a = -2G\theta$ is non-zero constant.

In the plastic domain the components τ_{xz}, τ_{yz} along with the equilibrium equation satisfy the plasticity condition

$$\tau_{xz}^2 + \tau_{yz}^2 = 1. \qquad (73)$$

Here, to simplify the further calculations, the plasticity constant equals to one.

By introducing the stress function in this equation one can get

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = 1. \qquad (74)$$

Boundary Conditions. Let the lateral surface be free from stresses. It means that $\frac{\partial \varphi}{\partial l} = 0$ on the contour Γ. Here $\vec{l} = (l_1, l_2)$ is a tangent vector to the contour Γ. It follows that $\varphi = \text{const}$ among the contour. As Γ is a simply connected contour then $\varphi = 0$ on it.

Finally one gets the following problem.

It is necessary to resolve the following equation in the domain bounded by the curve L:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = a. \qquad (75)$$

In the domain bounded by curves L and Γ, i. e. in the plastic domain, the function φ satisfies the equation

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = 1. \qquad (76)$$

The following conditions apply on the contour Γ for the function φ

$$\varphi = 0, \qquad (77)$$

$$\frac{\partial \varphi}{\partial l} = 0 \text{ or } \frac{\partial \varphi}{\partial x} l_1 + \frac{\partial \varphi}{\partial y} l_2 = 0, \qquad (78)$$

on the frontier L the function φ is continued.

It is necessary to find φ in elastic and plastic domains and to determine a frontier L.

Let's introduce the denotation $\varphi_x = u$, $\varphi_y = v$. Then equations (75), (76) come to

$$F_1 = u_x + v_y - a = 0. \qquad (79)$$

$$u^2 + v^2 = 1. \qquad (80)$$

Owing to the denotation the following equality occurs:

$$F_2 = u_y - v_x = 0. \qquad (81)$$

Definition. A vector (A, B) is a conserved current for the system of the equations (79), (81) if there is the following correlation

$$\partial_x A + \partial_y B = \Delta_1 F_1 + \Delta_2 F_2 = 0. \qquad (82)$$

Here Δ_1, Δ_2 are some linear differential operators.

It means that for functions A and B the conservation is correct law for all solutions of the system (79), (81):

$$\partial_x A + \partial_y B = 0. \qquad (83)$$

The conservation law (83) owing to the equations (79), (77) look like

$$A_x + A_u u_x + A_v v_x + B_y + B_u u_y + B_v v_y = 0$$

or taking into account $u_x = a - v_y$ and $u_y = v_x$,

$$A_x + A_u a + A_v v_x + B_y + B_u v_x + B_v v_y = 0.$$

From the last expression follows that functions A and B satisfy the equations

$$A_x + A_u a + B_y = 0, \qquad (84)$$

$$B_v - A_u = 0, \quad A_v + B_u = 0. \qquad (85)$$

(84), (85) are Cauchy–Riemann equations.

Let's consider a domain D with the boundary Γ on condition that plastic domain P comprises completely the elastic domain F. Let Γ be a smooth directed contour, i. e. continuously differentiable without singular points.

It follows from the conservation law

$$\iint_D \left(\partial_x A + \partial_y B\right) dxdy = 0. \qquad (86)$$

From (86), using Green's formula one can obtain

$$\oint_\Gamma A dy - B dx = 0. \qquad (87)$$

Our objective is to find a domain F belonging with its boundary to the domain D where inequality $u^2 + v^2 < 1$ applies.

Let $A = \alpha u + \beta v$, $B = \alpha v - \beta u + \gamma$ then

$$A_x = \alpha_x u + \beta_x v + \beta v_x, \qquad (88)$$

$$B_y = \alpha_y v + \alpha v_y - \beta_y u - \beta u_y + \gamma_y. \qquad (89)$$

According to the conservation law (83) one can get the equation

$$\begin{aligned} A_x + B_y = \alpha_x u + \alpha u_x + \beta_x v + \beta v_x + \\ + \alpha_y v + \alpha v_y - \beta_y u - \beta u_y + \gamma_y = 0, \end{aligned} \qquad (90)$$

which contains conditions on functions α, β and γ.

$$\begin{cases} \alpha_x - \beta_y = 0, \\ \beta_x + \alpha_y = 0, \\ a\alpha + \gamma_y = 0. \end{cases} \qquad (91)$$

Let's consider two solutions of the system (90)

The first one is

$$\alpha_1 = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2},$$

$$\beta_1 = \frac{y - y_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}, \tag{92}$$

$$\gamma_{1y} = \frac{x - x_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2},$$

then

$$\gamma_1 = -a \cdot \operatorname{arctg} \frac{y - y_0}{x - x_0}. \tag{93}$$

Respectively, the second one is

$$\alpha_2 = \frac{y - y_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2},$$

$$\beta_2 = \frac{x - x_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}, \tag{94}$$

$$\gamma_{2y} = -a \frac{y - y_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2},$$

then $\gamma_2 = -\frac{a}{2} \cdot \ln\left(\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right)$.

Let's note the equation (87) for the functions A and B

$$\begin{aligned}
&\oint_\Gamma A dy - B dx = \oint_\Gamma \left(\alpha u + \beta v\right) dy - \left(\alpha v - \beta u + \gamma\right) dx = \\
&= \oint_\Gamma \left(-\alpha \frac{l_2}{l_1} + \beta\right) v dy - \left(\alpha \frac{l_1}{l_2} - \beta\right) u dx - \oint_\Gamma \gamma dx = \\
&= \oint_\Gamma \left(-\alpha \frac{l_2}{l_1} + \beta\right) \frac{\partial \varphi}{\partial y} dy - \left(\alpha \frac{l_1}{l_2} - \beta\right) \frac{\partial \varphi}{\partial x} dx - \oint_\Gamma \gamma dx = \\
&= \oint_\Gamma \left(-\alpha \frac{l_2}{l_1}\right) v dy - \left(\alpha \frac{l_1}{l_2}\right) dx - \\
&- \oint_\Gamma \gamma dx + \oint_\Gamma \beta \left(\frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial x} dx\right) = \\
&= \oint_\Gamma \alpha u dy - \left(\alpha v + \gamma\right) dx = 0.
\end{aligned} \tag{95}$$

Let's decompose the boundary Γ into parts, i. e. $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$; Γ_3 is a circle $(x - x_0)^2 + (y - y_0)^2 = R^2$ (fig. 5).

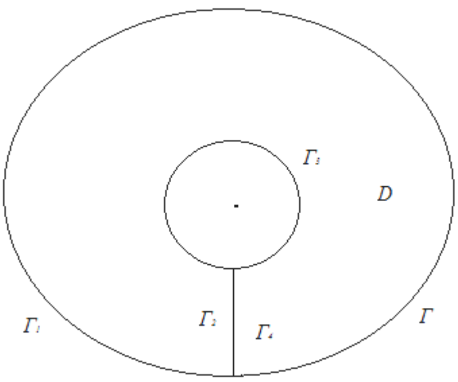

Fig. 5. Boundary Γ

In this case

$$\begin{aligned}
&\oint_\Gamma A dy - B dx = \oint_\Gamma \alpha u dy - \left(\alpha v + \gamma\right) dx = \\
&= \oint_{\Gamma_1} \alpha u dy - \left(\alpha v + \gamma\right) dx + \oint_{\Gamma_2} \alpha u dy - \\
&- \left(\alpha v + \gamma\right) dx + \oint_{\Gamma_3} \alpha u dy - \left(\alpha v + \gamma\right) dx + \\
&+ \oint_{\Gamma_4} \alpha u dy - \left(\alpha v + \gamma\right) dx = 0.
\end{aligned} \tag{96}$$

Obviously, $\oint_{\Gamma_2} \alpha u dy - \left(\alpha v + \gamma\right) dx + \oint_{\Gamma_4} \alpha u dy - (\alpha v + \gamma) dx = 0$. Taking into account this condition the equation (93) looks like

$$\oint_{\Gamma_3} \alpha u dy - \left(\alpha v + \gamma\right) dx = -\oint_{\Gamma_1} \alpha u dy - \left(\alpha v + \gamma\right) dx. \tag{97}$$

Let's calculate an integral $\oint_{\Gamma_1}$, Γ_1 is a circle of the radius R.

Let

$$\alpha = \alpha_1 = \frac{x - x_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2},$$

$$\beta = \beta_1 = -\frac{y - y_0}{\left(x - x_0\right)^2 + \left(y - y_0\right)^2}, \tag{98}$$

$$\gamma = \gamma_1 = -a \cdot \operatorname{arctg} \frac{y - y_0}{x - x_0}.$$

Introduce the polar coordinate system

$$\begin{cases} x - x_0 = R \cos \varphi \\ y - y_0 = R \sin \varphi \end{cases}, \tag{99}$$

then

$$\begin{cases} dx = R \sin \varphi d\varphi \\ dy = R \cos \varphi d\varphi \end{cases},$$

$$\alpha = \frac{\cos \varphi}{R}, \quad \beta = -\frac{\sin \varphi}{R}, \quad \gamma = -a\varphi. \tag{100}$$

As a result with $R \to 0$ one obtains

$$\oint_{\Gamma_1} \alpha u dy - \left(\alpha v + \gamma\right) dx = \pi u\left(x_0, y_0\right). \tag{101}$$

Analogically using $\alpha = \alpha_2$, $\beta = \beta_2$, $\gamma = \gamma_2$

$$\oint_{\Gamma_1} \alpha u dy - \left(\alpha v + \gamma\right) dx = \pi u\left(x_0, y_0\right). \tag{102}$$

Finally one can get from (82)

$$\oint_{\Gamma_3} \alpha_1 u dy - \left(\alpha_1 v + \gamma_1\right) dx = \pi u\left(x_0, y_0\right), \tag{103}$$

$$\oint_{\Gamma_3} \alpha_2 u dy - \left(\alpha_2 v + \gamma_2\right) dx = \pi u\left(x_0, y_0\right). \tag{104}$$

Let's determine the curve Γ_3 in parametric form:

$$x = f\left(t\right), \quad y = \varphi\left(t\right), \quad 0 \le t \le T, \tag{105}$$

$f'\left(t\right)$, $\varphi'\left(t\right)$ are derivatives of the functions $f\left(t\right)$ and $\varphi\left(t\right)$ respectively.

Hen functions $u(x_0, y_0)$, $v(x_0, y_0)$ from (98), (102) are found from the formulas

$$u(x_0,y_0)=\frac{1}{\pi}\int_0^T\left(\frac{(f(t)-x_0)\sqrt{(f'(t))^2+(\varphi'(t))^2}}{\sqrt{(f(t)-x_0)^2+(\varphi(t)-y_0)^2}}+\right.$$
$$\left.+af'(t)\operatorname{arctg}\frac{\varphi(t)-y_0}{f(t)-x_0}\right)dt;\tag{106}$$
$$v(x_0,y_0)=\frac{1}{\pi}\int_0^T\left(\frac{(\varphi(t)-y_0)\sqrt{(f'(t))^2+(\varphi'(t))^2}}{\sqrt{(f(t)-x_0)^2+(\varphi(t)-y_0)^2}}+\right.$$
$$\left.+\frac{a}{2}f'(t)\ln\left((f(t)-x_0)^2+(\varphi(t)-y_0)^2\right)\right)dt.$$

The solutions (102) and (103) were used respectively to obtain these relations.

Let's calculate now a value of the expression

$$u^2+v^2\tag{107}$$

in a point (x_0, y_0). The points where (107) is greater than or equal to one belong to the plastic domain, the points where the expression (107) is less than one belong to the elastic domain.

On the base of the formulas (103), (104) the programs were developed; they permit to construct plastic and elastic domains in a twisted rod with indicated accuracy.

The solutions obtained using the programs coincident rather well with the known solutions.

In this journal one can find some examples of calculation of elasto-plastic boundaries for some cross-section of the rolling section. These results belong to A. V. Kondrin and to the authors of the article. The article [13] gives examples of the calculation of elastic – plastic rods borders for rolling profile.

11. Elasto-plastic Boundary in the Bended Consol. Let's consider a consol with the permanent cross-section bounded by the contour Γ. The consol is under the concentrated force *P* on a free end in parallels to principal axes (fig. 6).

Let the component σ_z of the stress tensor is distributed along the consol as in the case of pure bending

$$\sigma_z=-\frac{p(i-z)x}{l}.$$

Let components of the stress tensor are

$$\sigma_x=\sigma_y=\sigma_{xy}=0.$$

Then the residual components of the stress tensor satisfy the equations

$$\frac{\partial\tau_{xz}}{\partial z}=\frac{\partial\tau_{yz}}{\partial z}=0,\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial x}=-\frac{px}{l}.\tag{108}$$

Usually the equations (108) replace by two compatibility equations

$$\Delta\tau_{xz}=-\frac{p}{l(1+v)},\quad \Delta\tau_{yz}=0,$$

here Δ is the Laplacian, v is a Poisson's ratio. This system is usually resolved by semi-inverse Saint-Venant method.

Let's rewrite the system (108) in terms of the vector of deformations (*u*; *v*; *w*). A boundary problem will be posed and resolved using conservation laws.

Using the formulas (2) which connect components of the stress tensor and strain tensor one can get

$$\begin{gathered}\sigma_x=\lambda\varepsilon+2\mu\frac{\partial u}{\partial x}=0,\ \sigma_y=\lambda\varepsilon+2\mu\frac{\partial v}{\partial y}=0,\\ \sigma_z=\lambda\varepsilon+2\mu\frac{\partial w}{\partial z}=-\frac{p(l-z)}{l}=\sigma,\\ \tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0,\ \tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\tau_1,\\ \tau_{yz}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=\tau_2,\end{gathered}\tag{109}$$

here λ, μ are Lamé coeffisients, $\varepsilon=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$, τ_1, τ_2 are sought functions of x, y.

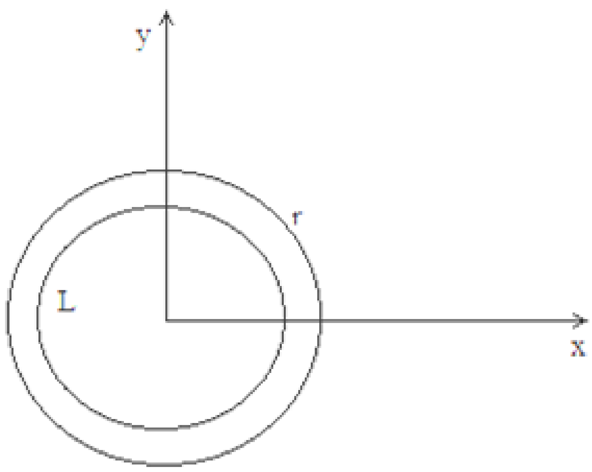

Fig. 6. Elasto-plastic Boundary in the Cross-section of the Consol

From the first, the second and the third equation one can get

$$\frac{\partial u}{\partial x}=A_1xz+B_1x,\ \ \frac{\partial v}{\partial y}=A_2xz+B_2x,$$

$$\frac{\partial w}{\partial z}=A_3xz+B_3x, \tag{110}$$

here constants A_i, B_i can be evaluated with ease per λ, μ, p, l.

From the equations (110) one can get

$$w=\frac{A_3xz^2}{2}+B_3xz+\omega(x,y),$$

$$u=-\frac{A_3z^3}{\sigma}-\frac{B_3z^3}{2}+\left(\tau_1/\mu-\omega_x\right)z+U(x,y),$$

$$v=\left(\tau_2/\mu-\omega_y\right)+V(x,y),$$

here ω, U, V are sought functions.

From the relation $\tau_{xy}=0$ one gets

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\left(\tau_1 y/\mu-\omega_{xy}\right)z+U_y+\left(\tau_2 x/\mu-\omega_{xy}\right)+V_x=0.$$

Relations

$$\frac{\partial u}{\partial x}=-\left(\partial\tau_{1x}/\mu-\omega_{xx}\right)z+U_x=A_1xz+B_1x,$$

$$\frac{\partial v}{\partial y}=\left(\partial\tau_{2y}/\mu-\omega_{yy}\right)z+U_y=A_2xz+B_2x$$

are substituted in the previous equation; the result is

$$\mu\left(\omega_{xx}+A_1x\right)+\mu\left(\omega_{yy}+A_2x\right)=-\frac{Px}{l}.$$

Suppose that the lateral surface of the beam if free from the stresses. It means that

$$\tau_{xz}n_2-\tau_{yz}n_1=0, \tag{111}$$

here (n_1, n_2) are an external normal line to the contour Γ. Suppose also that the plastic flow begins from the external side of the lateral surface of the beam. In this case the plasticity condition of Von Mises looks like

$$\tau_{xz}^2+\tau_{yz}^2=k^2, \tag{112}$$

here k is a constant of plasticity. Solving the system (111), (112) one can get

$$\tau_{xz}=\pm n_1k,\ \ \tau_{yz}=\pm n_2k.$$

Choosing the upper sign in these relations, one can pose the following problem.

It is necessary to resolve the equation

$$\omega_{xx}+\omega_{yy}=ax, \tag{113}$$

under the following conditions on Γ:

$$\omega_x=\frac{\left(n_1k-\dfrac{A_1x^2}{2}\right)}{\mu},\ \ \omega_y=-\frac{\left(n_2k-A_2xy\right)}{\mu}. \tag{114}$$

Remark. There are two domains in the cross-section of the beam, a plastic one and an elastic one. If the following condition satisfies in a point of the cross-section

$$\tau_{xz}^2+\tau_{yz}^2\langle 1,$$

Then the point falls into the elastic domain. The other points including the boundary of the contour Γ belong to the plastic domain.

The relation of a form

$$\partial_x A+\partial_y B=0, \tag{115}$$

is called a conservation law of (115) for the equation (113) if the equation (115) is correct for all solutions of the equation (113). Let the conserved current looks like

$$A=\alpha(x,y)\omega_x+\beta(x,y)\omega_y+\gamma(x,y),$$

$$B=\alpha^1(x,y)\omega_x+\beta^1(x,y)\omega_y+\gamma^1(x,y).$$

One can obtain from (113) and (115)

$$\begin{aligned}&\alpha\left(a-\omega_{yy}\right)+\alpha_x\omega_x+\beta\omega_{xy}+\beta_x\omega_y+\gamma_x+\\&+\alpha^1\omega_{xy}+\alpha^1_y\omega_x+\beta^1\omega_{yy}+\beta^1_y\omega_y+\gamma^1_y 0.\end{aligned} \tag{116}$$

The relation (116) is correct for all solutions of the equation (113) therefore it is follows from (116)

$$\alpha_x-\beta_y=0,\ \ \beta_x+\alpha_y=0,\ \ \alpha a+\gamma_x+\gamma^1_y=0. \tag{117}$$

The conservation law (115) can be written using the Green's formula:

$$\int-\left(\alpha\omega_x+\beta\omega_y+\gamma\right)dy+\left(-\beta\omega_x+\alpha\omega_y+\gamma^1\right)dx=0.$$

Let's consider two solutions of the equations (117).

The first one is the following

$$\alpha^1=\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},\ \ \beta^1=-\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2},$$

$$\gamma_x=0,\ \ \alpha^1xa=-\gamma^1_y,\ \ \gamma^1=-ax\cdot\operatorname{arctg}\left(\frac{y-y_0}{x-x_0}\right).$$

The second one looks like

$$\alpha^2=\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2},\ \ \beta^2=\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},$$

$$\gamma^2_y=0,\ \ \alpha^2xa=-\gamma_x.$$

Using the conservation law and applying it to the contour represented on the fig. 6 one can obtain on the analogy with the previous section

$$\int\left(\alpha\omega_x+\beta\omega_y+\gamma\right)dy-\left(-\beta\omega_x+\alpha\omega_y+\gamma^1\right)dx=$$

$$=-\int_{(x-x_0)^2+(y-y_0)^2=R^2}\left(\alpha\omega_x+\beta\omega_y+\gamma\right)dy-$$

$$-\left(-\beta\omega_x+\alpha\omega_y+\gamma^1\right)dx.$$

Let's calculate the second integral for the first and the second solutions of the equation (117). As a result the formulas for finding of $\omega_x(x_0,y_0)$, $\omega_y(x_0,y_0)$ are obtained.

These formulas permit to find a stress state in any point (x_0, y_0). It means that it is possible to determine for every point of the domain its belonging to either elastic or plastic zone.

The expounded method allow to construct a boundary between an elastic and a plastic domains with any prescribed accuracy using the computer. Preliminary calculations confirm this conclusion.

12. Elasto-plastic Boundaries for Large Areas. In this section only domains with smooth convex boundaries are regarded.

Let's consider a domain bounded by the contour Γ (fig. 7).

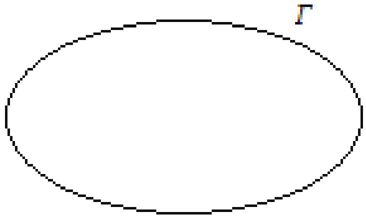

Fig. 7. Domain with the Contour Γ

Let boundary conditions (5) applies on Γ. They will look like

$$\sigma_x l + \tau_{xy} m = \overline{X}, \quad \tau_{xy} l + \sigma_y m = \overline{Y}, \tag{118}$$

Moreover, suppose that the loadings (118) bring the entire boundary and the nearby points to the plastic state. Then it is possible to introduce variables σ, θ by the following way

$$\sigma_x = \sigma - k\cos 2\theta, \ \sigma_y = \sigma + k\cos 2\theta, \ \tau_{xy} = k\sin 2\theta. \tag{119}$$

In this case conditions (118) look like

$$(\sigma - k\cos 2\theta)l + k\sin 2\theta m = \overline{X},$$
$$k\sin 2\theta l + (\sigma + k\cos 2\theta)m = \overline{Y}.$$

These conditions can be written as

$$\sigma = \overline{X}', \quad \theta = \overline{Y}'. \tag{120}$$

Thus, one gets the Cauchy problem on the boundary Γ in the plastic domain. Solving this problem using formulas of the section 9, one obtains two families of characteristic curves (fig. 8).

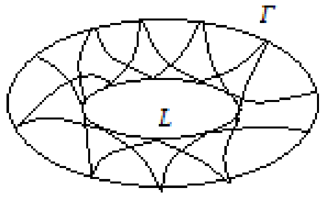

Fig. 8. Characteristic Curves

For these curves one can construct an envelope curve *L*. This line is the sought elasto-plastic boundary. It is enough to solve an elastic problem inside the domain to resolve completely elasto-plastic problem.

It is possible to set Cauchy problem for the system of plasticity (56), (57).

Let on the contour curve Γ denominated as *SP* there are starting data:

$$\sigma|_{SP} = \sigma_0', \quad \theta|_{SP} = \theta_0'. \tag{121}$$

Let describe a characteristic curve $PR : \eta_0 = \text{const}$ from the point *P* and a characteristic curve $RS : \xi_0 = \text{const}$ from the point *S*. Then a solution of Cauchy problem is determined in a curvilinear triangle ΔSPR (fig. 9).

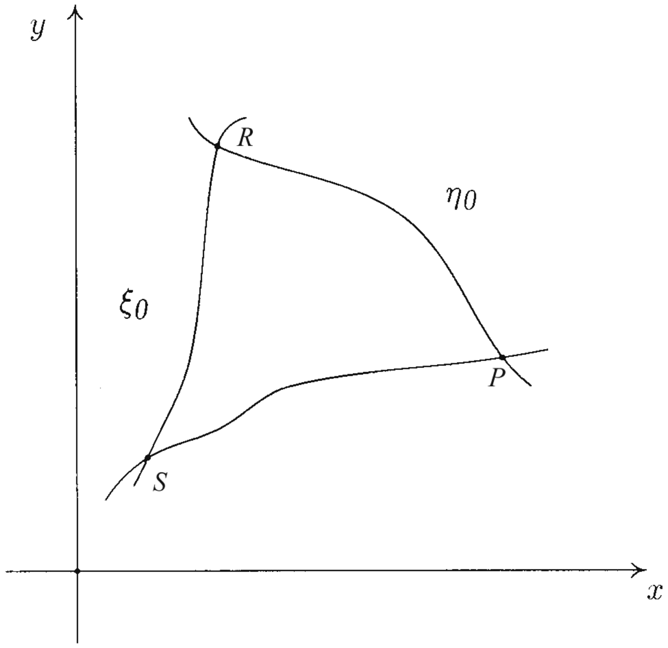

Fig. 9. Cauchy Problem

Thus it is necessary to find coordinates of a cross point R of the characteristic curves. If the coordinates of the point $R(x_R, y_R)$ and values ξ_0, η_0 of are known then it is possible to find values of functions σ, θ.

One can calculate an integral over the closed contour SPR. Using the correlation (58) in the Stokes theorem for a plane [11], one can conclude that this integral is equal to zero:

$$\int_{SPR} Ddx - Cdy = \int_{SP} + \int_{PR:\eta=\eta_0} + \int_{RS:\xi=\xi_0} = 0.$$

1. For the coordinate x_R one can get:

$$\int_{SPR} Ddx - Cdy = \int_{SPR} \left(D - C\frac{dy}{dx} \right) dx.$$

Integrals $\int\limits_{PR:\eta=\eta_0}$ and $\int\limits_{RS:\xi=\xi_0}$ are integrated by parts:

$$\int_{PR} \left(D - C\frac{dy}{dx} \right) dx = \int_{PR} \left(D + C\mathrm{ctg}\theta \right) dx =$$

$$= \int_{PR} \psi dx = x\psi\Big|_{x=x_P}^{x=x_R} - \int_{PR} x\frac{\partial\psi}{\partial\xi} d\xi,$$

$$\int_{RS} \left(D - C\frac{dy}{dx} \right) dx = \int_{RS} \left(D - C\mathrm{ctg}\theta \right) dx =$$

$$= \int_{RS} \varphi dx = x\varphi_{x=x_R}^{x=x_S} - \int_{RS} x\frac{\partial\varphi}{\partial\eta} d\eta.$$

Assuming $\varphi = 1$, $\psi = 0$ one can get the boundary conditions for the system (65) in the form:

$$\varphi\big|_{RS} = 1', \quad \psi\big|_{PR} = 0'. \tag{122}$$

Under such assumption the final expression for the coordinate x_R looks like:

$$\int_{SPR} Ddx - Cdy = \int_{SP} \left(Ddx - Cdy \right) + x\psi\Big|_{x=x_P}^{x=x_R} + x\varphi\Big|_{x=x_R}^{x=x_S} = 0,$$

$$x_R = \int_{SP} \left(Ddx - Cdy \right) + x_S'. \tag{123}$$

2. Similarly fort the coordinate y_R:

$$\int_{SPR} Ddx - Cdy = \int_{SPR} \left(D\frac{dx}{dy} - C \right) dy.$$

Integrals $\int\limits_{PR:\eta=\eta_0}$ and $\int\limits_{RS:\xi=\xi_0}$ are integrated by parts:

$$\int_{PR} \left(D\frac{dy}{dx} - C \right) dy = \int_{PR} \frac{D - C\frac{dy}{dx}}{\frac{dy}{dx}} dy =$$

$$= \int_{PR} \frac{D + C\mathrm{ctg}\theta}{-\mathrm{ctg}\theta} dy = \int_{PR} \frac{\psi}{-\mathrm{ctg}\theta} dy =$$

$$= -\frac{y\psi}{\mathrm{ctg}\theta}\Bigg|_{y=y_P}^{y=y_R} + \int_{PR} y\frac{\partial}{\partial\xi}\left(\frac{\psi}{\mathrm{ctg}\theta} \right) d\xi,$$

$$\int_{RS} \left(D\frac{dx}{dy} - C \right) dy = \int_{RS} \frac{D - C\mathrm{tg}\theta}{\mathrm{tg}\theta} dy =$$

$$= \int_{RS} \frac{\varphi}{\mathrm{tg}\theta} dy = \frac{y\varphi}{\mathrm{tg}\theta}\Bigg|_{x=x_R}^{x=x_S} - \int_{RS} y\frac{\partial}{\partial\eta}\left(\frac{\varphi}{\mathrm{tg}\theta} \right) d\eta.$$

Assuming $\varphi = \mathrm{tg}\theta$, $\psi = 0$ one can get the boundary conditions for the system (65) in the form:

$$\varphi\big|_{RS} = \mathrm{tg}\frac{\eta - \xi_0}{2}', \quad \psi\big|_{PR} = 0'. \tag{124}$$

Under such assumption the final expression for the coordinate y_R looks like:

$$\int_{SPR} Ddx - Cdy =$$

$$= \int_{SP} \left(Ddx - Cdy \right) + \frac{x\psi}{\mathrm{ctg}\theta}\Bigg|_{y=y_P}^{y=y_R} + \frac{y\varphi}{\mathrm{tg}\theta}\Bigg|_{y=y_R}^{y=y_S} = 0, \tag{125}$$

$$y_R = \int_{SP} \left(Ddx - Cdy \right) + y_S'.$$

It remains to resolve the problems (65), (122) and (65), (124). These problems can be redused to the equation (66). Namely, taking into account that fuctions φ, ψ are related with function ρ in the following way

$$\varphi = \frac{\rho}{\cos\theta}', \quad \psi = \frac{2}{\sin\theta}\frac{\partial\rho}{\partial\xi}', \tag{126}$$

one can obtain the problems

$$\frac{\partial^2\rho}{\partial\xi\partial\eta} - \frac{\rho}{4} = 0', \; \rho\big|_{\xi=\xi_0} = \cos\frac{\eta - \xi_0}{2}', \; \frac{\partial\rho}{\partial\xi}\bigg|_{\eta=\eta_0} = 0 \tag{127}$$

and

$$\frac{\partial^2\rho}{\partial\xi\partial\eta} - \frac{\rho}{4} = 0', \; \rho\big|_{\xi=\xi_0} = \sin\frac{\eta - \xi_0}{2}', \; \frac{\partial\rho}{\partial\xi}\bigg|_{\eta=\eta_0} = 0. \tag{128}$$

A general solution of the first problem (128) (for the coordinate x_R) is the following function:

$$\rho = \rho_1(\xi, \eta) = I_0\left(\sqrt{(\xi - \xi_0)(\eta - \eta_0)} \right)\cos\frac{\eta_0 - \xi_0}{2} -$$

$$-\frac{1}{2}\int_{\eta_0}^{\eta} I_0\left(\sqrt{(\xi - \xi_0)(\eta - \tau)} \right)\sin\frac{\tau - \xi_0}{2} d\tau',$$

moreover

$$\frac{\partial\rho_1}{\partial\xi} = \frac{1}{2}\cos\frac{\eta_0 - \xi_0}{2} I_1\left(\sqrt{(\xi - \xi_0)(\eta - \eta_0)} \right)\sqrt{\frac{\eta - \eta_0}{\xi - \xi_0}} -$$

$$-\frac{1}{4}\int_{\eta_0}^{\eta} I_1\left(\sqrt{(\xi - \xi_0)(\eta - \tau)} \right)\sqrt{\frac{\eta - \tau_0}{\xi - \xi_0}}\sin\frac{\tau - \xi_0}{2} d\tau.$$

The solution of the second problem (128) (for the coordinate y_R) is the function:

$$\rho = \rho_2(\xi, \eta) = I_0\left(\sqrt{(\xi - \xi_0)(\eta - \eta_0)} \right)\sin\frac{\eta_0 - \xi_0}{2} +$$

$$+\frac{1}{2}\int_{\eta_0}^{\eta} I_0\left(\sqrt{(\xi - \xi_0)(\eta - \tau)} \right)\cos\frac{\tau - \xi_0}{2} d\tau',$$

when

$$\frac{\partial \rho_2}{\partial \xi} = \frac{1}{2}\sin\frac{\eta_0 - \xi_0}{2} I_1\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)\sqrt{\frac{\eta-\eta_0}{\xi-\xi_0}} +$$

$$-\frac{1}{4}\int_{\eta_0}^{\eta} I_1\left(\sqrt{(\xi-\xi_0)(\eta-\tau)}\right)\sqrt{\frac{\eta-\tau_0}{\xi-\xi_0}}\cos\frac{\tau-\xi_0}{2}d\tau.$$

In all solutions the function I_0 is the Bessel function of the first kind of an imaginary argument $I_0(0)=1$, $I_0^{'}(0)=0.$

The functions φ, ψ can be found using formulas (126). From the relation (66) one can obtain the components of conservation laws:

$$D = \frac{\psi \mathrm{tg}\theta + \varphi \mathrm{ctg}\theta}{\mathrm{tg}\theta + \mathrm{ctg}\theta} = \psi \sin^2\theta + \varphi\cos^2\theta,$$

$$C = \frac{\psi - \varphi}{t\mathrm{g}\theta + \mathrm{ctg}\theta} = (\psi - \varphi)\sin\theta\cos\theta.$$

Substituting the obtained *C* and *D* in (123) and (124) one can get the coordinates of the point *R*. Thus Cauchy problem for the system of plasticity (56), (57) with starting data (121) is resolved completely.

Conclusion. The small range of problems considered in this article, concerning the construction of elasto-plastic boundaries reveals good prospects of the application of conservation laws for the the boundary problems solving. By now, the authors have solved some other problems of solid mechanics and prepare them to publish.

More results in the study of equations of elasticity and plasticity can be found in articles [14–17].

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