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ON CLASES OF FUNCTIONS WITH BINARY VARIABLES

The scheme proposed below is often used for solving problems and developing optimization algorithms. To solve a specific problem an efficient algorithm of optimization has been developed. The proposed algorithm combines several classes of problems by generelasing and determining a function class. For this reason establishing correlation among available classes of functions with binary variables in different experiments allows to apply even not perfect optimization algorithms.

In this paper we consider a question on correlation of the function classes based on the different approaches to classification itself. First approach offers classes of separable, modular and submodular function; second one offers function classes based on structural features of the set of binary variables: monotone, unmonotone and weakly unmonotone functions. It has been proven that separable functions are always unimodal and monotone ones. The results obtained in this study will allow to use a more efficient algorithm for optimization of separable and modular functions.

Keywords: pseudoboolean functions, optimization.

In scientific literature different classificiations for several types of pseudoboolean functions can be found. Authors in [1] introduced classification which separates pseudoboolean functions by the number of local minimums and by the characters of function meanings change (monotonecity). Several clases of pseudoboolean functions exist: unimodal and polymodal, monotone and unmonotone, separable, modular, sub- and supermodular and some others. For certain classes of pseudoboolean functions an efficient algorithms of optimization are already proposed. This is the case for a monotone and weakly unmonotone, both unimodal polymodal functions. That is why if we would be able to find in each correlation the different classes of functions, then it would be possible to use effectively a well-known algorithms in application to the other classes of functions.

In [2] the effectiveness of (1+1) evolutionary algorithm in the case of separable pseudoboolean functions is analized

$$f(x) = f(x_1, ..., x_n) = \sum_{1 \le i \le n} g_i(x_I),$$

where $f: \{0, 1\}^n \to R$.

The considered algorithm has shown a convergence by order $O(n \ln n)$. Such result is rather efficient.

A potential possibility of the optimization algorithm for the monotone unimodal pseudoboolean functions to the separable ones offered in [1]. Let us investigate in which correlation classes the separable, monotone and unmonotone pseudoboolean functions are. For subsequent analysys besides the introduced notions let us give a number of used notions here by Stupina [3].

Definition 1. The point set $P(X^0, X^l) = \{X^0, X^1, ..., X^i, ..., X^l\} \subset B_2^n$ we will call the way between the points X^0 and X^l if $\forall i = 1, ... l$ the point X^i is neighbor of X^{i-1}. The way $P(X^0, X^l)$ between the k-neigboring points we shall call the shortest way if $l = k$. The way of the largest decrease of the function f is denoted $P^j(X^0, X^l)$ where $\forall X^{i-1}, X^i \in P^j(X^0, X^l), i = 1, ..., l$

$$f(X^i) = \min_{X_j^1 \in O_2(X^{i-1})} f(X_j^1).$$

Definition 2. A point $X^* \in B_2^n$ for which $f(X^*) < f(X), \forall X \in O_1(X^*)$ is a local minimum of the function f and if the function has only one local minimum on B_2^n, than this function is an unimodale one.

Definition 3. An unimodale pseudoboolean function f is called monotone on B_2^n if $\forall X^k \in O_k(X^*)$, $k = 1, ...,$ $n: f(X^{k-1}) \le f(X^k) \forall X^{k-1} \in O_{k-1}(X^*) \bigcap O_1(X^k)$ and strong monotone, if this condition is fulfilled with the sign of the strong unequality.

For optimization of unimodale pseudoboolean functions the algorithm proposed by Antamoshkin at al. [1] reguers $(n + 1)$ colculations of the objective function values for the exact location of the optimum.

Definition 4. An unimodale nonmonotone function f on B_2^n is called weakly unmonotone, if $\forall X^k \in O_k(X^*)$, $k = 1, ..., n$ the point $X_{\min}^1$ such that:

$$f(X_{\min}^1) = \min_{X_j^1 \in O_1(X^k)} f(X_j^1),$$

belongs to $O_{k-1}(X^*)$.

The class of weakly unmonotone functions will be called *WU*.

In [1] the following theorem has been prooved.

Theorem 1. To define the minimum point X^* of the unimodal different meanings weakly unmonotone on B_2^n function f in average by the original point of search location expects to calculate the meanings of f in T points on B_2^n:

$$T = \frac{n^2 + 4}{2} - \frac{1}{2^n}.$$

Separable and Monotone Pseudoboolean Functions. Show that an arbitrary separable function is an unimodal function on B_2^n. We use the results of Droste at al. [2] showing that any separable function may be presented in the form

$$f(X) = \sum_{i=1}^{n} w_i x_i,$$

where $w_i \in R$ and $w_i > 0 \forall i = 1, ..., n$ (in the general case $q_i(x_i) = w_i x_i + x_i$ but since $x_i \in B_2^n$ the constant term has no influence on the optimixation process). Then, evidently, the function f has a minimum in the point (0, ..., 0). Supposing that f has another local minimum in a point $X^* \neq (0, ..., 0)$ and that this point $X^* \in O_k(0, ..., n)$ i. e. k componends of X^* are unit e. g. $i_1, ..., i_k$.

Hence

$$f(X^*) = \sum_{l=1}^{k} w_{i_l}.$$

Lemma 1.

$$\forall X \in \{0,1\}^n \wedge X^k \in O_k(X),\ k = 0, ..., n:$$

$$card\{O_1(X^k) \cap O_{k-1}(X)\} = k,$$

$$card\{O_1(X^k) \cap O_{k+1}(X)\} = n - k.$$

Then according to this Lemma k neighbouring to X^* points belong to $O_{k-1}(0, ..., 0)$, in particular, the point X' with the unit components $i_2, ..., i_k$. Now as we presupposed that X^* was a local minimum point we obtain:

$$f(X^*) < f(X') \Rightarrow \sum_{l=1}^{k} w_{i_l} < \sum_{l=2}^{k} w_{i_l}.$$

For the positive values for w_i it has no sense and therefore we can conclude that separable pseudoboolean functions here are unimodal functions.

Let us test the monotonicity condition fulfilment for a separable function.

An unimodal function f is called strictly monotone on $\{0,1\}^n$ if $\forall X^k \in O_k(X^*),\ k = 1, ..., n$:

$$f(X^{k-1}) < f(X^k) \forall X^{k-1} \in O_{k-1}(X^*) \cap O_1(X^k),$$

where X^* as before is the minimum point.

Let us take again an arbitrary point $X^k \in O_k(X^*)$. As we have already determined the minimum point of a separable function is the point (0, ..., 0). Then the point X^k has k unit components $i_1, ..., i_k$. Respectively, any point $X_{k-1} \in O_{k-1}(X^*) \cap O_1(X^k)$ will have $k-1$ unit components $i_1, ..., i_{j-1}, i_{j+1}, ..., i_k$, $j = 1, ..., k$ and as w_i are the positive inequality is always fulfilled

$$f(X^{k-1}) = \sum_{l=1}^{j-1} w_{i_l} + \sum_{l=j+1}^{k} w_{i_l} < f(X^k) = \sum_{l=1}^{k} w_{i_l}.$$

This means that any separable pseudoboolean function is a strictly monotone function.

Modular, submodular, monotone and weakly unmonotone pseudoboolean functions. In [3] the effectiveness of evolutionary algorithms on two classes of the pseudoboolean functions – modular and submodular is analized.

Definition 5. A pseudoboolean function is called modular (*MOD*), if

$$f(X \wedge Y) + f(X \vee Y) = f(X) + f(Y)$$

for all $X, Y \in B_2^n$.

Definition 6. A pseudoboolean function is called submodular (*SUB*) if

$$f(X \wedge Y) + f(X \vee Y) \le f(X) + f(Y)$$

for all $X, Y \in B_2^n$.

To solve optimization problems with an objectiv function, which belongs to the given class Individual Based Evolutionary Algorithms is proposed in [2]. One of the representatives of this class of algorithms is (1 + 1) evolutionary algorithm, which was mention above.

As for the modular functions in contrast to data in [3] there is a proof that only linealy pseudoboolean functions belongs to this class, but the linealy functions belong to the class of separable functions, i. e. they are unimodal and monotone. Consequently, it is possible to atribute all that we have shown above,it is also the spead of optimization algorithms convergence.

The condition from the definition 6 includes the equality from definition 5. It means that the class of submodular functions includes the class of modular functions. Consequently, the functions, which belong to the class $MOD \subset SUB$ are monotone unimodale.

Now we consider the submodular functions, which are not modular. i. e. the class of functions *SUB\MOD*. The notion of a weakly unmonotone function was introduce above.

We investigate the functions, which belongs to the class *SUB\MOD* with the purpose of their itersection with weakly unmonotone functions. In the case, if these functions intersect then following correlation possible.

Variants:

a) $SUB \setminus MOD \subset WU$,

b) $WU \subset SUB \setminus MOD$,

c) $WU \cap SUB \ne \varnothing$.

Let us show that the variant c) is fulfilled, i. e. classes *SUB\MOD* and *WU* are only intersected and no one of them include others.

For example take three pseudoboolean functions, which belong to each of 3 classes: $WU \setminus SUB,\ WU \cap SUB,\ SUB \setminus WU$.

1. $f_1(X) \in SUB \setminus WU$.

$$f(0,0,0,0) = 0; f(0,0,0,1) = 8{,}5; f(0,0,1,0) = 9; f(0,1,0,0) = 9{,}5;$$

$$f(1,0,0,0) = 10; f(0,0,1,1) = 14; f(0,1,0,1) = 15;$$

$$f(0,1,1,0) = 16; f(1,0,0,1) = 17; f(1,0,1,0) = 18;$$

$$f(1,1,0,0) = 19; f(0,1,1,1) = 20; f(1,0,1,1) = 22;$$

$$f(1,1,0,1) = 23; f(1,1,1,0) = 24; f(1,1,1,1) = 25.$$

This function is unimodal with an unique local minimum in the point $X^*(0, 0, 0, 0)$

$f_1(X^*) = 0$, in neighbouring to X^* points the function has meaning on the segment [8, 5; 10], i. e.

$$8{,}5 < f_1(X^1) < 10.\ 14 < f_1(X^2) < 19,\ 20 < f_1(X^3) < 24,$$
$$f_1(X^4) = f_1(1,1,1,1) = 25.$$

We have $f_1(X^*) < f_1(X^1) < f_1(X^2) < f_1(X^3) < f_1(X^4)$.

So the function $f_1(X)$ answer the monotone condition, i. e. $f_1(X) \in SUB \setminus WU$.

2. $f_2(X) \in SUB \cap WU$.

$$f(0,0,0,0) = 0; f(0,0,0,1) = 8{,}5; f(0,0,1,0) = 9;$$

$$f(0,1,0,0) = 9{,}5; f(1,0,0,0) = 10; f(0,0,1,1) = 14;$$

$$f(0,1,0,1) = 15; f(0,1,1,0) = 16; f(1,0,0,1) = 17;$$

$$f(1,0,1,0) = 18; f(1,1,0,0) = 19; f(0,1,1,1) = 20;$$

$$f(1,0,1,1) = 22; f(1,1,0,1) = 23;$$

$$f(1,1,1,0) = 24; f(1,1,1,1) = 21.$$

$f(1, 1, 1, 1) = 21$ and in the other points the meanings of function are the same as in the previous case.

This function is unimodal one with the unique local minimum in the point $X^* = (0, 0, 0, 0)$.

In the point $X^4 = (1,1,1,1)$ the monotone condition is broken, so that $f_2(1, 1, 1, 0)$, $f_2(1, 1, 0, 1)$, $f_2(1, 0, 1, 1)$ more than $f_2(X^4)$

But this function is weakly unmonotone, because for all points on B_2^n the monotone condition is fulfilled.

The condition (3) is fulfilled for any two meanings of the function $f_2(X)$ in the same way.

Consequntly, $f_2(X)$ is a submodular function.

3. $f_3(X) \in WU \setminus SUB$.

$$f(0,0,0,0)=0; f(0,0,0,1)=8{,}5; f(0,0,1,0)=9;$$

$$f(0,1,0,0)=9{,}5; f(1,0,0,0)=10; f(0,0,1,1)=14;$$

$$f(0,1,0,1)=15; f(0,1,1,0)=19{,}5; f(1,0,0,1)=17;$$

$$f(1,0,1,0)=18; f(1,1,0,0)=19; f(0,1,1,1)=20;$$

$$f(1,0,1,1)=22; f(1,1,0,1)=23;$$

$$f(1,1,1,0)=24; f(1,1,1,1)=21.$$

$f(0,1,1,0)=19{,}5$ and in the other points the meanings of function are the same as in the first case.

For this function:

– the unimodale condition is fulfilled, so the point $X^* = (0,0,0,0)$ is an unique point of the local minimum;

– the weakly unmonotone condition is fulfilled. For any point $X \in O_k(X^*), k = 1, ..., n$ the point $X^1_{\min} \in O_k(X^*) \bigcap O_1(X_k)$ is so that $f(X^1_{\min}) = \min\limits_{X^1_j \in O_1(X^k)} f(X^1_j)$.

For the points 0100 and 0010 the condition (3) is not fulfilled (the function is not submodular one).

Consequently $f_3(X) \in WU \setminus SUB$.

We have obtained following results.

The set of separable functions belongs to the set of monotone unimodal functions. Our results confirm that we can apply the algorithm of bounding search for the separable functions. This algorithm was propose in [1] for monotone unimodale functions. The main merit of this algorithm is of its high efficiency (in comparison with other available algorithms). It is equal to $n + 1$ calculations of the meaning of an objective function, where n is demention of the optimaizing pseudoboolean function. For example, an evolutionary algorithm in application to the separable functions converge for $n \cdot \ln n$ steps, that considerably more then results, which were obtained by the bounding algorithm.

It has turned out that modular functions are also belonging to the class of monotone unimodale functions and, therefore, all conclusions, which we made for the separable functions, are also fulfilled for the modular functions.

Submodular functions were only intersected with each of classes, which were described above, and do not belong to these clases. Also none of this classes belongs to the class of submodular functions.

The results obtained in this study are looking very promising as they allow to get a considerable gain when applying already known algorithms to the local search or bounding methods of pseudoboolean functions optimization. They also allow (when informations about properties of separable pseudoboolean functions available) to construct new and more qualitative procedures for this class of functions optimization.

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О КЛАССАХ ФУНКЦИЙ С БИНАРНЫМИ ПЕРЕМЕННЫМИ

Предлагаемая схема исторически используется для решения задач и разработки алгоритмов оптимизации. Для решения реальной задачи разработан достаточно эффективный алгоритм оптимизации. Рассматриваемый алгоритм объединяет в себе несколько классов задач, обобщая классы функций. Вследствие этого, определение корреляции уже представленных классов функций с бинарными переменными при помощи различных исследований позволяет применять даже не усовершенствованные алгоритмы оптимизации. Рассматривается вопрос о корреляции классов функций при различных подходах классификации. Первый подход включает классы сепарабельных, модулярных и субмодулярных функций, второй – классы функций, которые основываются на структурных свойствах множества бинарных переменных: монотонные, немонотонные и слабо немонотонные функции. На практике доказано, что сепарабельные функции унимодальные и монотонные. Полученные результаты позволяют использовать для оптимизации сепарабельных и модулярных функций более эффективные алгоритмы по сравнению с ранее используемыми.

Ключевые слова: псевдобулевые функции, оптимизация.