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SOLVABILITY OF AN INITIAL-BOUNDARY PROBLEM FOR A LOADED WAVE EQUATION

Solvability of an initial-boundary problem for a loaded wave equation is proved. The proof of the solution uniqueness is based on the a priori estimate of the solution. A sequence of Galerkin approximations is constructed for the solution existence to be proved.

Keywords: loaded wave equation, uniqueness, solution existence.

A loaded equation is an equation with partial derivatives that contain the values of functionals of the sought function. Such equations are encountered in modeling certain physical processes [1] and also in solving inverse problems of mathematical physics [2; 3].

Let us consider the following initial-boundary problem for the function $u(x,t)$, $(x,t)\in\bar{Q}:[0,l]\times[0,T]$:

$$u_{tt}-u_{xx}-b(x,t)\int_0^T(T-\tau)u_{xx}(x,\tau)d\tau=c(x,t), \quad (1)$$

$$u(x,0)=p(x),\ u_t(x,0)=n(x), \quad (2)$$

$$u(0,t)=u(l,t)=0, \quad (3)$$

$$p(0)=n(0)=p(l)=n(l)=0. \quad (4)$$

Here, $b(x,t)$, $c(x,t)$, $p(x)$, and $n(x)$ are specified functions.

To obtain an *a priori* estimate of the solution, we multiply Eq. (1) by u_t and integrate the result with respect to x from 0 to l:

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|u_x\|^2\right)=(U_x,bu_t)+(c,u_t). \quad (5)$$

Here, $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the scalar product in $L_2(0,l)$,

$$U=U(x)=\int_0^T(T-\tau)u_x(x,\tau)d\tau. \quad (6)$$

By virtue of conditions (3), we have

$$(U_x,bu_t)=-(U,b_xu_t+bu_{tx}). \quad (7)$$

Integrating Eq. (5) with allowance for Eqs. (2) and (7) from 0 to t, we obtain

$$\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|u_x\|^2=C_1+\int_0^t(c,u_t)dt-$$
$$-\int_0^t(U,b_xu_t)dt-\int_0^t(U,bu_{tx})dt. \quad (8)$$

Here,

$$C_1=\frac{1}{2}\|n(x)\|^2+\frac{1}{2}\left\|\frac{dp(x)}{dx}\right\|.$$

The following equality is valid:

$$\int_0^t(U(x),b(x,\tau)u_{tx}(x,\tau))d\tau=(U(x),u_x(x,t)b(x,t))-$$
$$-(U(x),u_x(x,0)b(x,0))-\int_0^t(U(x),u_x(x,\tau)b_t(x,\tau))d\tau. \quad (9)$$

The following estimates are obtained:

$$|U(x)|<3^{-1/2}T^{3/2}\left(\int_0^T u_x^2(x,\tau)d\tau\right)^{1/2},$$

$$\left|\int_0^t(c,u_t)dt\right|\le\int_0^T\left(\frac{1}{2\varepsilon}\|c\|^2+\frac{\varepsilon}{2}\|u_t\|^2\right)dt, \quad (10)$$

$$|(U(x),u_x(x,t)b(x,t))|\le B\left(\frac{1}{2T}\|U\|^2+\frac{T}{2}\|u_x\|^2\right).$$

Here, ε is an arbitrary positive number and B is the maximum value of $|b(x,t)|$ in $\bar{Q}$. Taking into account Eqs. (9, 10), and (8), we obtain

$$\|u_t\|^2+\|u_x\|^2<C_2+BT\|u_x\|^2+$$
$$+\alpha T^2\int_0^T\|u_t\|^2dt+\beta T^2\int_0^T\|u_x\|^2dt. \quad (11)$$

Here, the positive constants α, β, and C_2 depend on the maximum values of $|b(x,t)|$, $|b_t(x,t)|$, and $|b_x(x,t)|$ in $\bar{Q}$. Integrating Eq. (11) with respect to t from 0 to T, we obtain

$$(1-\alpha T^3)\int_0^T\|u_t\|^2dt+(1-BT-\beta T^3)\int_0^T\|u_x\|^2dt<C_2T. \quad (12)$$

Under the conditions

$$(1-\alpha T^3)\ge\delta_1>0,\ 1-BT-\beta T^3\ge\delta_2>0, \quad (13)$$

where δ_1 and δ_2 are arbitrarily small fixed positive numbers, we obtain the sought *a priori* estimate

$$\int_0^T\|u_t\|^2dt<\text{const},\ \int_0^T\|u_x\|^2dt>\text{const}. \quad (14)$$

This estimate means that u is bounded in the space $W_2^1(Q)$ of functions that have generalized first-order derivatives integrated with a square.

Estimate (14) immediately yields the theorem of the solution uniqueness, because the constants in Eq. (14) should be equal to zero for different solutions to exist.

The solution existence is proved with the Galerkin method. We seek for the Galerkin approximation in the form

$$u^m(x,t)=\sum_{k=1}^{m} q_{mk}(t)\,v_k(x),$$

where $v_k(x)$, $k=1, 2, ...$ is the basis and $q_{mk}(0)=p_k$, $q'_{mk}(0)=n_k$, where m_k, n_k, $k=1, 2, ...$ are the coefficients of the expansion of the functions $p(x)$ and $n(x)$ in the basis $v_k(x)$. The basis is found by solving the problem

$$v''_k(x)+\mu_k v_k(x)=0,\ v_k(0)=v_k(l)=0,$$

where μ_k are eigenvalues. Obviously, this basis has the form

$$v_k(x)=\sin\frac{\pi kx}{l},\ \mu_k=\frac{\pi^2k^2}{l^2},\ k=1, 2, ...$$

From the condition of orthogonality, we obtain a system of ordinary differential equations for the functions $q_{mk}(t)$

$$\left(u_{tt}^m-u_{xx}^m-\int_0^T(T-\tau)u_{xx}^m(x,\tau)\,d\tau-c, v_k\right)=0, \qquad (15)$$

which has to be solved under the conditions

$$q_{mk}(0)=p_k,\ q'_{mk}(0)=n_k. \qquad (16)$$

After the same considerations as those performed for obtaining the *a priori* estimate (14), we obtain

$$\int_0^T\left\|u_t^m\right\|dt<C_1,\ \int_0^T\left\|u_x^m\right\|dt<C_2,$$

where the constants C_1 and C_2 are independent of m. From this estimate, there follows that the system of ordinary equations (15) with conditions (16) is solvable and that it is possible to choose a subsequence from the Galerkin approximations, which converges to a certain function, which is a solution of the initial problem.

Thus, a theorem follows from here. Let

$$c(x,t)\in L_2(Q),\ b(x,t)\in L^\infty(Q),$$

$$b_x(x,t)\in L^\infty(Q),\ b_t(x,t)\in L^\infty(Q),$$

$$m(x)\in \overset{\circ}{W}{}_2^1(Q),\ n(x)\in L_2(Q)$$

and condition (13) be satisfied. Then, there exists a unique solution of problem (1–4), and

$$u\in L_2(0,T),\overset{\circ}{W}{}_2^1(0,l),\ u_t\in L_2(Q).$$

To conclude, we should note that a similar theorem is also valid if we take for $0\le t\le T$ $u=u(x,y,t)$, $(x,y)\in G$; $u=0$, $(x,y)\in\partial G$, replace u_{xx} in Eq. (1) by the Laplace operator Δu, and replace the functions $b(x,t)$ and $c(x,t)$ by $B(x,y,t)$ and $C(x,y,t)$. The basis $v_k=v_k(x,y)$ and eigenvalues μ_k, $k=1,2,...$ are determined in this case by solving the following problem:

$$\Delta v_k+\mu v_k=0,\ (x,y)\in G,\ v_k\big|_{\partial G}=0.$$

An initial-boundary problem for a one-dimensional (in terms of the spatial variable) loaded wave equation is considered. This equation contains a functional of the sought function. An a priori estimate of the solution is obtained, which is used to prove the solution uniqueness. A sequence of Galerkin approximations is constructed; a converging subsequence that is a solution of the initial problem is chosen from this sequence. The results of this paper can be used to justify the correctness of models of some physical processes and to solve inverse problems of mathematical physics.

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РАЗРЕШИМОСТЬ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ, ОПИСЫВАЮЩЕГО ВОЛНУ НАГРУЖЕНИЯ

Доказана разрешимость начально-краевой задачи для уравнения, описывающего волну нагружения. Доказательство единственности базируется на априорных оценках решения. Построена последовательность Галеркина, которая позволила доказать существование решения.

Ключевые слова: уравнение, описывающее волну нагружения, единственность и существование решения.