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CAUCHY PROBLEM SOLUTION FOR A HYPERBOLIC SYSTEM OF THE HOMOGENEOUS 2-DIMENSIONAL QUASILINEAR EQUATIONS*

The method of solving the boundary-value problems for hyperbolic system of the homogeneous quasilinear equations of two independent variables with the help of conservation laws is presented. This method is applied to basic boundary problems for the system of two-dimensional plasticity equations under Sent-Venan–Mises yield criterion, as well for the system under Coulomb's criterion.

Keywords: conservation laws, plane ideal plasticity.

Among hyperbolic systems of the nonlinear equations with partial derivatives the systems of the quasilinear equations of two independent variables, are most studied. These systems in particular describe the unsteady one-dimensional and the supersonic two-dimensional stationary flows of compressible gases and liquids, two-dimensional deformed plastic state of continuous medium, etc. A lot of them are reduced to a hyperbolic system of homogeneous quasilinear equations

$$u_x + A(u,v)u_y = 0,$$
$$v_x + B(u,v)v_y = 0, \quad (1)$$

where $u = u(x, y)$, $v = v(x, y)$, an indices the lower meaning of the derivation with respect to the corresponding variables. This kind of system can be linearized by applying a so-called hodograph transformation if and only if the corresponding Jacobian is not equals to zero in some domain of solution existence.

From the other hand, the conservation laws [1] are one of the fundamental characteristics of any mechanical process. In the section 2 we have described a method of analytical solving of boundary-value problems for the system (1). This method is based on the use of conservations laws of systems under consideration. The main features of the presnted method consist of the possibility of problem linearization without considering the singularity of Jacobian transformation to obtain exact solutions for boundary-valued problems in explicit form.

In section 3 we consider the application of this method to some exact systems of the bi-dimensional plasticity mathematical theory.

Conservation Laws. Let us set Cauchy's problem for the system (1): in something like an arc $a \le \tau \le b$ of a smooth curve L

$$L = \{(x, y) : x = x(\tau), y = y(\tau), \tau \in [a,b]\}$$

in the plane xOy it is required to find a solution of the system (1), that takes given values on L

$$u(x, y)|_L = u(x(\tau), y(\tau)) = u^0(\tau),$$
$$v(x, y)|_L = v(x(\tau), y(\tau)) = v^0(\tau).$$

The characteristics equations of the system (1) look like

$$\frac{dx}{dy} = A, \ \frac{dx}{dy} = B, \quad (2)$$

with the relations in the characteristics $u = u^0$, $v = v^0$ equally.

A conservation law of the equations set (1) is searched in the form of the relation

$$C_x + D_y = 0, \quad (3)$$

where $C = C(u,v)$, $D = D(u,v)$ should vanish from all solutions of the system (1):

$$C_u u_x + C_v v_x + D_u u_y + D_v v_y =$$
$$= -AC_u u_y - BC_v v_y + D_u u_y + D_v v_y = 0,$$

hence

$$D_u - AC_u = 0, \quad D_v - BC_v = 0. \quad (4)$$

Equation (3), if the conditions of Green's theorem are satisfied, is equivalent to a relation

$$\int_\Gamma -Cdy + Ddx = 0,$$

where Γ is an arbitrary smooth closed contour.

In the plane xOy we have considered the closed path MNK, where $M(x_m(a),\ y_m(a)),\ N(x_n(b),\ y_n(b)) \in L$, $K(x_k,\ y_k)$ is the point of an intersection of characteristics $v = v^0$, $u = u^0$, drawing through the points M, N respectively.

Then,

$$\int_{MNK} -Cdy + Ddx = \int_{MN} Ddx - Cdy +$$
$$+ \int_{NK} Ddx - Cdy + \int_{KM} Ddx - Cdy = 0. \quad (5)$$

Taking into account expressions (2)

$$\int_{NK} Ddx - Cdy = \int_{NK} (D - AC)dx =$$
$$= x(D - AC)|_{x_n}^{x_k} - \int_{NK} x\partial_v (D - AC)dv.$$

Similarly,

$$\int_{KM} Ddx - Cdy = \int_{KM} (D - BC)dx =$$
$$= x(D - BC)|_{x_k}^{x_m} - \int_{KM} x\partial_u (D - BC)du.$$

Let us put the following conditions

$$\partial_v \left[(D - AC)|_{u=u^0(x(b),y(b))} \right] =$$
$$= 0, \partial_u \left[(D - BC)|_{v=v^0(x(a),y(a))} \right] = 0. \quad (6)$$

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Let us denote

$$\phi(u,v)=D-AC,\quad \psi(u,v)=D-BC.$$

Then

$$D=(A\psi-B\phi)/(A-B),\ C=(\psi-\phi)/(A-B),$$

where $A\neq B$, because the considered system (1) is a hyperbolic one and has two different characteristics.

In the new variables, the system (4) has the form

$$\phi_u+K_1(\psi-\phi)=0,$$
$$\psi_v+K_2(\psi-\phi)=0, \qquad (7)$$

where $K_1=A_u/(A-B)$; $K_2=B_v/(A-B)$.

Conditions (6) we shall take as follows

$$\phi|_{u=u^0}=\mathrm{Const}_1=1,\ \psi|_{v=v^0}=\mathrm{Const}_2=0. \qquad (8)$$

Coming back to (5) and taking into account (8), we shall obtain

$$\int_{MN} Ddx-Cdy=-(x(D-AC)|_{x_n}^{x_k}+x(D-BC)|_{x_k}^{x_m})=$$
$$=-(x_k\phi|_{u=u^0,v=v^0}-x_n\phi|_{u=u^0}+x_m\psi|_{v=v^0}-x_k\psi|_{u=u^0,v=v^0})=$$
$$=x_n-x_k. \qquad (9)$$

If we find a solution of the linear system (7), that satisfied the boundary conditions (8), then we shall define the coordinate x_k from the equation (9).

On the other hand, for y-coordinate

$$\int_{NK} Ddx-Cdy=\int_{NK}\left(\frac{D}{A}-C\right)dy=$$
$$=y\frac{\phi}{A}\Big|_{y_n}^{y_k}-\int_{NK} y\partial_v\left(\frac{\phi}{A}\right)dv.$$

Similarly,

$$\int_{KM} Ddx-Cdy=\int_{KM}\left(\frac{D}{B}-C\right)dy=$$
$$=y\frac{\psi}{B}\Big|_{y_k}^{y_m}-\int_{KM} y\partial_u\left(\frac{\psi}{B}\right)du.$$

Let's assume

$$\partial_v\left[\frac{\phi}{A}\Big|_{u=u^0(x(b),y(b))}\right]=0,$$
$$\partial_u\left[\frac{\psi}{B}\Big|_{v=v^0(x(a),y(a))}\right]=0. \qquad (10)$$

The conditions (10) we can note are the following

$$\phi|_{u=u^0}=A(u^0,v),\ \psi|_{v=v^0}=0. \qquad (11)$$

From (5) taking into account (10) we have

$$\int_{MN} Ddx-Cdy=-y_k\frac{\phi}{A}\Big|_{u=u^0,v=v^0}+y_n\frac{\phi}{A}\Big|_{u=u^0}-$$
$$-y_m\frac{\psi}{B}\Big|_{v=v^0}+y_k\frac{\psi}{B}\Big|_{u=u^0,v=v^0}=y_n-y_k. \qquad (12)$$

The solution of the problem (7), (11) makes possible to find the coordinate y_k from the equation (12). Thus, we shall define the coordinates of the point K, where the values of the functions u,v can be restored.

Let us notice, that the same steps can be applied for the Riemann problem [2].

Examples of applying the method. In this work [3] we have found all the conservations laws of the bi-dimensional plasticity system under the Sent-Venan–Mises yield criterion:

$$\sigma_x-2k\left(\theta_x\cos 2\theta+\theta_y\sin 2\theta\right)=0,$$
$$\sigma_y-2k\left(\theta_x\sin 2\theta-\theta_y\cos 2\theta\right)=0, \qquad (13)$$

where σ is the hydrostatic pressure; θ is the angle between the first main direction of a stress tensor and the *ox*-axis; k is a constant of plasticity. Here we have described the special class of conservation laws that are used for solution for Cauchy problem system (13).

The system (13) in the form (1) looks like this:

$$\xi_x+\xi_y\,\mathrm{tg}\,\theta=0,\ \eta_x-\eta_y\,\mathrm{ctg}\,\theta=0,$$

where $2\theta=\eta-\xi$; $\sigma=k(\eta+\xi)$. The solution of the problem (7), (8) has a form $\phi=\rho/\cos\theta$, $\psi=2\rho_\xi/\sin\theta$ where:

$$\rho(\xi,\eta)=R(\xi,\xi_0,\eta,\eta_0)\cos\left(\frac{\eta_0-\xi_0}{2}\right)-$$
$$-\frac{1}{2}\int_{\eta_0}^{\eta}R(\xi,\xi_0,\eta,\tau)\sin\left(\frac{\tau-\xi_0}{2}\right)d\tau.$$

Accordingly, the solution of the problem (7), (11) is:

$$\rho(\xi,\eta)=R(\xi,\xi_0,\eta,\eta_0)\sin\left(\frac{\eta_0-\xi_0}{2}\right)+$$
$$+\frac{1}{2}\int_{\eta_0}^{\eta}R(\xi,\xi_0,\eta,\tau)\cos\left(\frac{\tau-\xi_0}{2}\right)d\tau,$$

where $R(\xi,\eta,\xi_0,\eta_0)=I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)$ is the Bessel function of the zero order of an imaginary argument; $I_0(0)=1, I_0'(0)=0$.

The generalization of (13) shows that it is a system of ideal plasticity equations under Coulomb's yield criterion of the form:

$$\sigma_x(1+\cos 2\alpha\cos 2\Theta)+\sigma_y\cos 2\alpha\sin 2\Theta=$$
$$=2(\sigma\cos 2\alpha+k\sin 2\alpha)(\Theta_x\sin 2\Theta-\Theta_y\cos 2\Theta),$$
$$\sigma_x\cos 2\alpha\sin 2\Theta+\sigma_y(1-\cos 2\alpha\cos 2\Theta)=$$
$$=-2(\sigma\cos 2\alpha+k\sin 2\alpha)(\Theta_x\cos 2\Theta+\Theta_y\sin 2\Theta),$$

where $\pi/2-2\alpha$ is a constant angle of internal friction; $\Theta=\theta+\pi/4$. If $\alpha=\pi/4$. Then we have the system (13). In the system described above there is a stress state of granular materials.

This system in the form (1) looks like this:

$$\xi_x+\xi_y\,\mathrm{tg}\,(\Theta-\alpha)=0,\ \eta_x-\eta_y\,\mathrm{tg}\,(\Theta+\alpha)=0,$$

where $\xi=\frac{1}{2}\mathrm{tg}\,2\alpha\ln(\sigma\,\mathrm{ctg}\,2\alpha+k)-\Theta$; $\eta=\frac{1}{2}\mathrm{tg}\,2\alpha\ln(\sigma\,\mathrm{ctg}\,2\alpha+k)+\Theta$. The solution of problem (7), (8) has a form [4]:

$$\phi=\gamma(-\xi,-\eta)V(\xi,\eta)/\cos(\Theta-\alpha),$$
$$\gamma(\xi,\eta)=\exp(-(\xi+\eta)/2\,\mathrm{ctg}\,2\alpha),$$

where

$$V=\gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\cos\left((\eta_0-\xi_0)/2-\alpha\right)-$$

$$-\frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\times$$

$$\times\left[\sin\left((\tau-\xi_0)/2-\alpha\right)-\text{ctg } 2\alpha\cos\left((\tau-\xi_0)/2-\alpha\right)\right]d\tau. \quad (14)$$

The solution of the problem (7), (11) is:

$$V=\gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\sin\left((\eta_0-\xi_0)/2-\alpha\right)-$$

$$-\frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\times$$

$$\times\left[\text{ctg } 2\alpha\sin\left((\tau-\xi_0)/2-\alpha\right)-\cos\left((\tau-\xi_0)/2-\alpha\right)\right]d\tau, \quad (15)$$

where $R(\xi,\eta,\xi_0,\eta_0)=I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}/\sin 2\alpha\right)$. From the equation (9) taken into account (14) we have discovered coordinate x_k. Using (15), and from equations (12) we have received coordinate y_k. This way, we have determined point K, in which the values of functions ξ,η are restored.

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РЕШЕНИЕ ЗАДАЧИ КОШИ ДЛЯ ГИПЕРБОЛИЧЕСКОЙ СИСТЕМЫ ОДНОРОДНЫХ ДВУМЕРНЫХ КВАЗИЛИНЕЙНЫХ УРАВНЕНИЙ

Описан метод решения граничных задач для гиперболической системы однородных квазилинейных уравнений двух независимых переменных с применением законов сохранения. Этот метод применяется к задаче Коши для системы двумерной пластичности с условием Сен-Венана–Мизеса, а также для системы с условием Колумба.

Ключевые слова: законы сохранения, двумерная идеальная пластичность

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УСТОЙЧИВОСТЬ ОРТОТРОПНОЙ ПЛАСТИНЫ С ДВУМЯ СВОБОДНЫМИ КРАЯМИ, НАГРУЖЕННОЙ ИЗГИБАЮЩИМ МОМЕНТОМ В ПЛОСКОСТИ*

Решена задача устойчивости при чистом изгибе ортотропной пластины, у которой два противоположных края свободны, а два других края шарнирно закреплены. Для решения задачи использовался метод конечных разностей.

Ключевые слова: ортотропная пластина, метод конечных разностей.

Задача устойчивости прямоугольной пластины, нагруженной по двум противоположным краям усилиями, распределенными по линейному закону, впервые была решена для изотропной пластины И. Г. Бубновым [1] и С. П. Тимошенко [2]. Для ортотропной пластины эта задача была впервые сформулирована и решена С. Г. Лехницким [3]. Эти классические решения были получены для случая шарнирного закрепления краев пластины в форме двойных тригонометрических рядов. Для определения критического усилия в этих решениях был использован энергетический метод Ритца. Это связано с тем, что дифференциальное уравнение устойчивости пластины имеет переменный коэффициент и поэтому его интегрирование оказывается затруднительным. Метод Ритца был также использован в [4] и [5] для решения рассматриваемой нами задачи применительно к композитным пластинам с шарнирно закрепленными краями, которые нагружены равномерными сжимающими усилиями.

Таким образом, можно утверждать, что устойчивость пластины при изгибе в плоскости наиболее полно исследована только для самого распространенного вида граничных условий – шарнирного закрепления сторон. Это

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