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NEW EXACT SOLUTIONS WHICH DESCRIBE 2-DIMENSIONAL VELOCITY FIELD FOR PRANDTL'S SOLUTION*

New velocity fields are found for the well-known Prandtl's solution which describes pressing of a thin layer of plastic material between two parallel stiff and rough plates. The method of construction of other velocity fields is considered.

Keywords: ideal plasticity, velocity field, Prandtl's solution.

The 2-dimensional ideal plasticity equations in case of steady-state problem have the form:

$$\frac{\partial\sigma}{\partial x}-2k\left(\frac{\partial\theta}{\partial x}\cos 2\theta+\frac{\partial\theta}{\partial y}\sin 2\theta\right)=0,$$

$$\frac{\partial\sigma}{\partial y}-2k\left(\frac{\partial\theta}{\partial x}\sin 2\theta-\frac{\partial\theta}{\partial y}\cos 2\theta\right)=0, \qquad (1)$$

$$\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\operatorname{tg}2\theta+\left(\frac{\partial v_x}{\partial x}-\frac{\partial v_y}{\partial y}\right)=0,$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0, \qquad (2)$$

here σ is hydrostatic pressure, θ is the angle between the first principal direction of stress tensor and axis Ox, k is plasticity constant, $\nu_x \nu_y$ are components of velocity vector of strain field.

Prandtl's solution is one of the practically applied and frequently used in different computations. This solution describes in particular the pressing of a thin layer of plastic material between two parallel stiff and rough plates, and it has the following form:

$$\sigma_x=-p-k\left(x-2\sqrt{1-y^2}\right),$$

$$\sigma_y=-p-kx,\ \ \tau=ky, \qquad (3)$$

p is an arbitrary constant.

It is well known that to describe the plasticity state of material completely one should know velocity field.

Let's substitute the equations (2) into the system (1). We get:

$$y\left(\frac{\partial v_x}{\partial x}-\frac{\partial v_y}{\partial y}\right)=\sqrt{1-y^2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right),$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0. \qquad (4)$$

One can see that because of its linearity the system (4) has an infinite set of solutions which can be used for the analysis of the stress-strained state of a plastic medium.

At the present moment two classes of solutions of this system are known, Nadai's solution [1] and Ivlev–Senashov's solution [2], which are the following:

$$v_x=-\alpha xy+\beta x-\alpha\arcsin y-$$

$$-\alpha y\sqrt{1-y^2}-2\beta\sqrt{1-y^2}+C_1,$$

$$v_y=\alpha\left(\frac{x^2}{2}+\frac{y^2}{2}\right)-\beta y+C_2,$$

here α, β, C_1, C_2 are arbitrary constants (if $\alpha = 0$ Nadai's solution comes out).

Let's point out others solutions of the system (4). Notice that in variables ξ, η, where $\sigma=k(\xi+\eta)$, $2\theta=\xi-\eta$, the equations (2) are written as follows (5):

$$\frac{\partial v_x}{\partial\xi}-\operatorname{tg}\theta\frac{\partial v_y}{\partial\xi}=0,\ \ \frac{\partial v_y}{\partial\eta}+\operatorname{ctg}\theta\frac{\partial v_x}{\partial\eta}=0. \qquad (5)$$

If we put new variables into (5) using the formulas:

$$v_x=u\cos\theta-v\sin\theta,$$

$$v_y=u\sin\theta+v\cos\theta, \qquad (6)$$

we'll get a system (7):

$$\frac{\partial v}{\partial\xi}-\frac{1}{2}u=0,\ \ \frac{\partial u}{\partial\eta}-\frac{1}{2}v=0. \qquad (7)$$

Further we use the following procedure: we solve the system (7), put expressions from Prandtl's solution for ξ and η into this system, make substitution (6) and find a velocity field which corresponds to the solution (3).

Let's do it, for example, using the simplest solution of the system (7). It is obvious that

$$v=u=\exp\frac{1}{2}(\xi+\eta)$$

is a solution of the equations (7). Put it into the (6). We get:

$$v_x=(\cos\theta-\sin\theta)\exp\frac{1}{2}(\xi+\eta),$$

$$v_y=(\cos\theta+\sin\theta)\exp\frac{1}{2}(\xi+\eta).$$

From Prantl's solution (3) one can get easily

$$\xi+\eta=-\frac{p}{k}-x-\sin 2\theta,\ \ \cos 2\theta=y. \qquad (8)$$

And finally we find a new velocity field:

$$v_x=\exp\left(\frac{1}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)(\cos\theta-\sin\theta),$$

$$v_y=\exp\left(\frac{1}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)(\cos\theta+\sin\theta).$$

By using this scheme some other velocity fields are found.

For the equations (7) solutions are given in [3]. Further with respect to these indicated solutions, 5 more classes of new solutions of the equations (5) are built.

1) $u=\cos\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\mu\frac{\xi-\eta}{2}\right)\right].$

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Here A, B, μ, λ are arbitrary constants; $\mu^2 - \lambda^2 = 1$.

In this case

$$\frac{\partial u}{\partial \eta} = -\frac{\lambda}{2}\sin\left(\lambda\frac{\xi+\eta}{2}\right)\times$$

$$\times\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\mu\frac{\xi-\eta}{2}\right)\right]+$$

$$+\frac{\mu}{2}\cos\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\sin\left(\mu\frac{\xi-\eta}{2}\right)-B\cos\left(\mu\frac{\xi-\eta}{2}\right)\right].$$

From the equations (7) we get

$$v = -\lambda\sin\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\mu\frac{\xi-\eta}{2}\right)\right]+$$

$$+\mu\cos\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\sin\left(\mu\frac{\xi-\eta}{2}\right)-B\cos\left(\mu\frac{\xi-\eta}{2}\right)\right].$$

Making substitution for u, v into the equations (6) and taking into account the equalities (8) and $\theta = \frac{\xi-\eta}{2}$ we get

$$v_x = \cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\cos\theta-$$

$$-\left(-\lambda\sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+\right.$$

$$\left.+\mu\cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\sin\theta,$$

$$v_y = \cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\sin\theta+$$

$$+\left(-\lambda\sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+\right.$$

$$\left.+\mu\cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\cos\theta.$$

2) $u = \sin\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\frac{\xi-\eta}{2}\right)\right].$

Here A, B, μ, λ are arbitrary constants; $\mu^2 - \lambda^2 = 1$.

In this case

$$v = -\lambda\cos\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\mu\frac{\xi-\eta}{2}\right)\right]+$$

$$+\mu\sin\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\sin\left(\mu\frac{\xi-\eta}{2}\right)-B\cos\left(\mu\frac{\xi-\eta}{2}\right)\right].$$

And

$$v_x = \sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\cos\theta-,$$

$$-\left(-\lambda\cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+\right.$$

$$\left.+\mu\sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\sin\theta,$$

$$v_y = \sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\sin\theta+$$

$$+\left(\lambda\cos\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+\right.$$

$$\left.+\mu\sin\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\cos\theta.$$

3) $u = \exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\frac{\xi-\eta}{2}\right)\right].$

Here A, B, μ, λ are arbitrary constants; $\mu^2 + \lambda^2 = 1$.

In this case

$$v = \lambda\exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\cos\left(\mu\frac{\xi-\eta}{2}\right)+B\sin\left(\mu\frac{\xi-\eta}{2}\right)\right]+$$

$$+\mu\exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\sin\left(\mu\frac{\xi-\eta}{2}\right)-B\cos\left(\mu\frac{\xi-\eta}{2}\right)\right].$$

$$v_x = \exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times$$

$$\times\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\cos\theta-$$

$$-\left(-\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times\right.$$

$$\times\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+$$

$$\left.+\mu\exp\left(\lambda\left(-\frac{p}{2k}-\frac{x}{2}+\frac{\sqrt{1-y^2}}{2}\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\sin\theta,$$

$$v_y = \exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times$$

$$\times\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]\sin\theta+$$

$$+\left(\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times\right.$$

$$\times\left[A\cos(\mu\theta)-B\sin(\mu\theta)\right]+$$

$$\left.+\mu\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\sin(\mu\theta)-B\cos(\mu\theta)\right]\right)\cos\theta.$$

4) $u = \exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\exp\left(\mu\frac{\xi-\eta}{2}\right)+B\exp\left(\frac{\xi-\eta}{2}\right)\right].$

Here A, B, μ, λ are arbitrary constants; $\lambda^2 - \mu^2 = 1$.

In this case

$$v = \lambda\exp\left(\lambda\frac{\xi+\eta}{2}\right)\times$$

$$\times\left[A\exp\left(\mu\frac{\xi-\eta}{2}\right)+B\exp\left(\mu\frac{\xi-\eta}{2}\right)\right]+$$

$$+\mu\exp\left(\lambda\frac{\xi+\eta}{2}\right)\times$$

$$\times\left[-A\exp\left(\mu\frac{\xi-\eta}{2}\right)+B\exp\left(\mu\frac{\xi-\eta}{2}\right)\right].$$

$$v_x = \exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times$$

$$\times\left[A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]\cos\theta-$$

$$-\left(-\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times\right.$$

$$\times\left[A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]+$$

$$\left.+\mu\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]\right)\sin\theta,$$

$$v_y = \exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times$$

$$\times\left[A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]\sin\theta+$$

$$+\left(\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\times\right.$$
$$\times\left[A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]+$$
$$\left.+\mu\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\exp(-\mu\theta)+B\exp(-\mu\theta)\right]\right)\cos\theta.$$

5) $u=\exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\frac{\xi-\eta}{2}+B\right].$

Here A, B, λ are arbitrary constants; $\lambda^2=1$.
In this case

$$v=\lambda\exp\left(\lambda\frac{\xi+\eta}{2}\right)\left[A\frac{\xi-\eta}{2}+B\right]-A\exp\left(\lambda\frac{\xi+\eta}{2}\right).$$

$$v_x=\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\theta+B\right]\cos\theta-$$
$$-\left(\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\theta+B\right]-\right.$$
$$\left.-A\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\right)\sin\theta,$$

$$v_y=\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\theta+B\right]\sin\theta+$$
$$+\left(\lambda\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\left[-A\theta+B\right]-\right.$$
$$\left.-A\exp\left(\frac{\lambda}{2}\left(-\frac{p}{k}-x-\sin 2\theta\right)\right)\right)\cos\theta.$$

Bibliography

1. Sockolovsky, V. V. Plasticity Theory / V. V. Sockolovsky. M. : Higher School, 1969. (in Russian)
2. Limiting State of Strained Media and Rocks / D. D. Ivlev, L. A. Maximova, R. I. Nepershin et al. M. : Physmathlit, 2008. (in Russian)
3. Polyanin, A. D. Reference Book on Nonlinear Equations of Mathematical Physics / A. D. Polyanin. M. : Physmathlit, 2001. (in Russian)

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THE ASYMPTOTIC PROBABILISTIC GENETIC ALGORITHM*

This paper proposes the modification of probabilistic genetic algorithm, which uses genetic operators, not affecting the particular solutions, but the probabilities distribution of solution vector's components. This paper also compares the reliability and efficiency of the base algorithm and proposed modification using the set of test optimization problems and bank loan portfolio problem.

Keywords: probabilistic genetic algorithm, mutation, selection.

The probabilistic genetic algorithm (PGA) is an attempt to create an algorithm with a scheme similar to that of the traditional genetic algorithm (GA), preserving the basic properties of the genetic operators, but defined in terms of the pseudo-Boolean optimization theory [1].

The probabilistic genetic algorithm explicitly (as opposed to the traditional GA) computes the components of the probability vector and has no crossover operator (it is replaced a by random solution generation operator) but retains the genetic operators of mutation and selection.

The purpose of this study is to develop a probabilistic genetic algorithm modification with mutation and selection operators, effecting not particular individuals, but genes' values distribution as a whole; and to compare efficiency and reliability of basic algorithm and modification.

Asymptotic mutation. PGA uses a standard GA mutation operator, which inverts genes with a given probability (as a rule, this probability is very low). Since genes mutate independently, we can study one particular gene. All following formulas will stand for every gene in the chromosome. Let us suppose that p – denoting the probability of that fact was equal to 1 before mutation. We will determine the probability as equal to 1 for same gene after mutation (p' denotes this probability). The mutation probability is p_m.

The gene can be equal to 1 after mutation in two cases: it was equal to 1 before mutation and has not mutated or it was equal to 0 before mutation and has mutated. If x denotes the gene value before mutation and y – after mutation – the following equality is:

$$P\{y=1\}=P\{x=1\}(1-p_m)+P\{x=0\}\,p_m=$$
$$=p(1-p_m)+(1-p)p_m=p_m+p(1-2p_m).$$

Using the aforementioned designations for genes probabilities before and after mutation we can write down:

$$p'=p_m+p(1-2p_m)\,.$$

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