Torsion of prismatic orthotropic elastoplastic rods

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Conservation laws were introduced into the theory of differential equations by E. Noether more than 100 years ago and are gradually becoming an important tool for the study of differential equations systems. Not only do they allow you to qualitatively investigate the equation, but, as the authors of this article show, they also enable you to find exact solutions to the boundary value problems. For the equations of the isotropic theory of elasticity, the conservation laws were first calculated by P. Olver. For the equations of the theory of plasticity in the two-dimensional case, the conservation laws were found by one of the authors of this article and used to solve the main boundary value problems of the plasticity equations. Later it turned out that the conservation laws can also be used to find the boundaries between elastic and plastic zones in twisted rods, bent beams, and deformable plates. The proposed work found conservation laws for equations describing the orthotropic elastic state of the twisted straight-line rod. It is assumed that the remaining current depends linearly on the voltage tensor component. In the workit was also found an endless series of laws of preservation, which allows you to find an elastic-plastic boundary, which arises when twisting the orthotropic rod.

Keyword: torsion of rods, boundary value problems, conservation laws.

Кручение призматических ортотропных упругопластических стержней

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Законы сохранения введены в теорию дифференциальных уравнений Э. Нетер более 100 лет назад и постепенно становятся важным инструментом исследования систем дифференциальных уравнений. Они не только позволяют качественно исследовать уравнение, но, и как показано авторами этой статьи, позволяют найти точные решение краевых задач. Для уравнений изотропной теории упругости законы сохранения впервые вычислены П. Олвером. Для уравнений

теории пластичности в двумерном случае законы сохранения найдены одним из авторов этой статьи и использованы для решения основных краевых задач уравнений пластичности. Позднее оказалось, что законы сохранения можно использовать и для нахождения границ между упругими и пластическими зонами в скручиваемых стержнях, изгибаемых балках и деформируемых пластинах. В предлагаемой работе найдены законы сохранения для уравнений описывающих ортотропное упругопластическое состояние скручиваемого прямолинейного стержня. Предполагается, что сохраняющийся ток зависит линейно от компонент тензора напряжений. В работе найдена бесконечная серия законов сохранения, которая позволяет найти упругопластическую границу, возникающую при кручении ортотропного стержня.

Ключевые слова: кручение стержней, краевые задачи, законы сохранения.

Introduction. Conservation laws were introduced into the theory of differential equations by E. Neter more than 100 years ago and are gradually becoming an important tool for the study of differential equation systems. Not only do they allow you to qualitatively investigate equations, but, as the authors of this article show, they allow us to find accurate solutions to boundary value problems. For the equations of isotropic theory of elasticity, the laws of preservation were first calculated by P. Olver. For the equations of the theory of plasticity in a two-dimensional case, the laws of preservation are found by one of the authors of this article and used to solve the main boundary value problems. Later it turned out that the laws of preservation can be used to find boundaries between elastic and plastic zones in twisted rods, plates and curved beams. It is assumed that the remaining current depends linearly on the voltage tensor component. As a result, an endless series of conservation laws have been found, which allows us to find the elastic boundary that arises when the orthotrope rod is twisted.

Problem setting. Consider an elastic orthotropic prismatic rod with an arbitrary cross-section. The side surface is stress-free, and forces equivalent to a torque of M are applied to the ends.

Let the origin be at the center of gravity of the end section, and the z-axis be parallel to the generatrix of the rod. The boundary conditions are written as follows:

$$\sigma_x l + \tau_{xy} m = 0,$$
$$\tau_{xy} l + \sigma_y m = 0, \tag{1}$$
$$\tau_{xz} l + \tau_{yz} m = 0,$$

and on the ends of the rod $(z = 0, \ z = l)$

$$\iint_\Omega \tau_{xz} dx dy = 0, \iint_\Omega \tau_{yz} dx dy = 0,$$
$$\iint_\Omega \sigma_z dx dy = 0, \iint_\Omega x\sigma_z dx dy = 0, \tag{2}$$
$$\iint_\Omega y\sigma_z dx dy = 0,$$

$$\iint_\Omega \left(x\tau_{yz} - y\tau_{xz} \right) dx dy = M, \tag{3}$$

where Ω is a cross-section.

As usual in the theory of torsion, we believe that

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \tag{4}$$

The rest of the stress tensor components satisfy the equilibrium equations, which will be written as follows:

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \frac{\partial \tau_{yz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{5}$$

Guk's generalized law for orthotropic environments will be written as follows [16].

$$a_{55}\tau_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad a_{44}\tau_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad a_{33}\sigma_{zz} = \frac{\partial w}{\partial z}.$$

a_{ii} is an elastic constant here.

From the equations of joint deformations we get

$$\frac{\partial^2 \sigma_z}{\partial x^2} = \frac{\partial^2 \sigma_z}{\partial y^2} = \frac{\partial^2 \sigma_z}{\partial z^2} = \frac{\partial^2 \sigma_z}{\partial x \partial y} = 0,$$

$$a_{13}\frac{\partial^2}{\partial y \partial z}\sigma_z = \frac{\partial}{\partial x}\left(-a_{44}\frac{\partial}{\partial x}\tau_{yz} + a_{55}\frac{\partial}{\partial y}\tau_{xz}\right), \tag{6}$$

$$a_{23}\frac{\partial^2}{\partial x \partial z}\sigma_z = \frac{\partial}{\partial y}\left(-a_{55}\frac{\partial}{\partial y}\tau_{xz} + a_{44}\frac{\partial}{\partial x}\tau_{yz}\right).$$

From (6) and boundary conditions (1) we obtain that $\sigma_z = 0$ in all cross-sections.

Of the last two equations (6) it follows (6)

$$-a_{44}\frac{\partial}{\partial x}\tau_{yz} + a_{55}\frac{\partial}{\partial y}\tau_{xz} = \text{const.} \tag{7}$$

Since

$$a_{55}\tau_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

$$a_{44}\tau_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$

then we have

$$a_{55}\frac{\partial}{\partial y}\tau_{xz} - a_{44}\frac{\partial}{\partial x}\tau_{yz} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -2\frac{\partial \omega_z}{\partial z} = -2\theta.$$

where ω_z is the third component of the vector $rot(u, v, w)$. Therefore θ is a corner of twisting, per unit of length. It's called a twist.

The problem of elastic torsion of the prism rod was reduced to the integration of equations.

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \quad a_{55}\frac{\partial \tau_{xz}}{\partial y} - a_{44}\frac{\partial \tau_{yz}}{\partial x} = -2\theta, \tag{8}$$

and the boundary condition

$$\tau_{xz}l + \tau_{yz}m = 0. \tag{9}$$

It is easy to see that the system of equations (8) is reduced to a linear equation of the second order of the elliptical type.

In the plastic area, the equations (8) should be added to the condition of plasticity, which has the following form:

$$2\alpha_{13}\tau_{xz}^2 + 2\alpha_{23}\tau_{yz}^2 = 1.$$

Here α_{13}, α_{23} are the constatnts, characterizing the current state of plastic anisotropy.

The result is the following task: to find conservation laws for equations (8) that allow you to solve the problem (9). These laws will find a boundary between elastic and plastic areas.

Conservation laws for orthotropic elasticity equations. This part will provide conservation laws for equations (8) to be used further to solve elastic problems.

For simplicity of further layouts, we will write down the system (8) in the form of:

$$
\begin{aligned}
F_1 &= u_x + v_y - f^1 = 0, \\
F_2 &= \alpha u_y - \beta v_y - f^2 = 0,
\end{aligned}
\tag{10}
$$

where $u = \tau_{xz}$, $v = \tau_{yz}$, $\alpha = a_{55}$, $\beta = a_{44}$, the index at the bottom denotes a derivative on the corresponding variable.

Let's call the vector (A, B) as a persistence current for equations (10) if

$$A_x + B_y = \Pi_1(F_1) + \Pi_2(F_2) = 0, \tag{11}$$

executed on all smooth system solutions (11). Here Π_i are some non-identical linear differential operators.

In this case (11) is the law of preservation for the system (10).

Let's set a task to find the laws of preservation for (10) if the remaining current depends only on x, y, u, v.

Note. Nothing also prevents us from finding conservation laws with a persistence current dependent on any number of derivatives, but we will limit ourselves to these, since other conservation laws have not been used for boundary problems yet.

Let

$$A = \alpha^1 u + \beta^1 v + \gamma^1, \quad B = \alpha^2 u + \beta^2 v + \gamma^2, \tag{12}$$

where $\alpha^i, \beta^i, \gamma^i$ are the functions only from x, y

Of (11) we have

$$
\begin{aligned}
&\alpha_x^1 u + \alpha^1 u_x + \beta_x^1 v + \beta^1 v_x + \gamma_x^1 + \alpha_y^2 u + \alpha^2 u_y + \beta_y^2 v + \beta^2 v_y + \gamma_y^2 = \\
&= \delta^1 \left(u_x + v_y - f^1 \right) + \delta^2 \left(\alpha u_y - \beta v_x - f^2 \right).
\end{aligned}
\tag{13}
$$

From (13) we get

$$
\begin{aligned}
&\alpha_x^1 + \alpha_y^2 = 0, \beta_x^1 + \beta_y^2 = 0, \alpha^1 = \beta^2 = \delta^1, \beta^1 = -\delta^2\beta, \\
&\alpha^2 = \delta^2\alpha, \gamma_x^1 + \gamma_y^2 = -\delta^1 f^1 - \delta^2 f^2, {\beta^1}\big/{\beta} = -{\alpha^2}\big/{\alpha}.
\end{aligned}
\tag{14}
$$

Finally we have

$$\alpha_x^1 + \alpha_y^2 = 0, \frac{\beta}{\alpha}\alpha_x^2 - \alpha_y^1 = 0, \gamma_x^1 + \gamma_y^2 = -\alpha^1 f^1 - \frac{\alpha^2}{\alpha}f^2.$$

For simplicity we consider that $\gamma^2 = 0$. Then we get:

$$\alpha^1 = \beta^2, \beta^1 = -\frac{\beta}{\alpha}\alpha^2, \gamma^1 = -\int\left(\alpha^1 f^1 - \frac{\alpha^2}{\alpha}f^2\right)dx, \qquad (15)$$

and the coefficienst α^1 and α^2 are linked by equations:

$$\alpha_x^1 + \alpha_y^2 = 0, \; \beta\alpha_x^2 - \alpha\alpha_y^1 = 0. \qquad (16)$$

We find two special solutions to this system.

They have the following form.

First solution:

$$\alpha_1^1 = \frac{x}{x^2 + \frac{\beta}{\alpha}y^2}, \; \alpha_1^2 = \frac{y}{x^2 + \frac{\beta}{\alpha}y^2}.$$

Second solution:

$$\alpha_1^1 = -\frac{\frac{\beta}{\alpha}y}{x^2 + \frac{\beta}{\alpha}y^2}, \; \alpha_1^2 = \frac{x}{x^2 + \frac{\beta}{\alpha}y^2}.$$

For the simplicity of further calculations, let's put $\frac{\beta}{\alpha} = \frac{a_{44}}{a_{55}} = q$.

Next, we note that equation (16) admits following symmetries

$$x' = x + x_0, \; y' = y + y_0,$$

where x_0, y_0 - are arbitrary constants.

Therefore, the obtained solutions can be written as:

$$\alpha_1^1 = \frac{x - x_0}{(x - x_0)^2 + q(y - y_0)^2}, \; \alpha_1^2 = \frac{y - y_0}{(x - x_0)^2 + q(y - y_0)^2}, \qquad (17)$$

$$\alpha_2^1 = -\frac{q(y - y_0)}{(x - x_0)^2 + q(y - y_0)^2}, \; \alpha_2^2 = \frac{x - x_0}{(x - x_0)^2 + q(y - y_0)^2}. \qquad (18)$$

We obtain two remaining currents.

$$A_i = \alpha_i^1 u - q\alpha_i^2 v + \gamma_i^1, \; B_i = \alpha_i^2 u + \alpha_i^1 v,$$

then, two conservation laws are obtained

$$\partial_x A_i + \partial_y B_i = 0. \qquad (19)$$

From (19) we get:

$$\oint_\Gamma A_i dy - B_i dx = 0, \qquad (20)$$

where Γ_1 is the contour that does not cover the point (x_0, y_0).

Now let the point (x_0, y_0) lie inside the region D, the boundary of which is the contour Γ . In this case, the formula (20) can not be applied directly. Therefore, we use the standard technique: we describe an ellipse around the point of the following form: $(x - x_0)^2 + q(y - y_0)^2 = \varepsilon^2$ (Fig. 1).
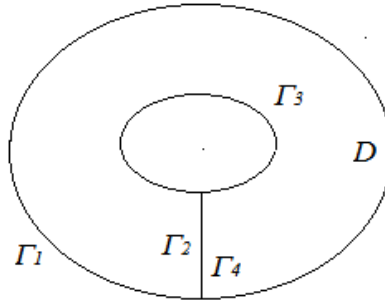

Рис. 1. Вычисление контурного интеграла вокруг особой точки

Fig. 1. Calculation of the contour integral around a singular point

Let's denote this ellipse Γ_1 . Then we can easily get

$$\oiint_{\Gamma} A_i dy - B_i dx = -\oiint_{\Gamma} \left(A_i dy - B_i dx \right),$$
(21)

We calculate the integral in the right part of formula (21) for $i = 1, 2$.

Let $i = 1$. We have:

$$\oiint_{\Gamma_1} = \oiint_{\Gamma_1} \left(\frac{(x - x_0)}{(x - x_0)^2 + q(y - y_0)^2} u - \frac{q(y - y_0)}{(x - x_0)^2 + q(y - y_0)^2} v + \gamma_1^1 \right) dy -$$

$$- \left(\frac{(y - y_0)}{(x - x_0)^2 + q(y - y_0)^2} u + \frac{(x - x_0)}{(x - x_0)^2 + q(y - y_0)^2} v \right) dx = \qquad .$$

$$= \oiint_{\Gamma_1} \left(\frac{(x - x_0)}{\varepsilon^2} u - \frac{q(y - y_0)}{\varepsilon^2} v + \gamma_1^1 \right) dy - \left(\frac{(y - y_0)}{\varepsilon^2} u + \frac{(x - x_0)}{\varepsilon^2} v \right) dx.$$

Let's introduce the notation $x - x_0 = \varepsilon \cos\theta$, $y - y_0 = \varepsilon \sin\theta$.

So we have:

$$\oiint_{\Gamma_1} = \oiint_{\Gamma_1} \left(qu - (1 - q) \frac{\sin 2\theta}{2} v + \gamma_1^1 \right) d\theta = q \int_0^{2\pi} u(x_0 + \varepsilon \cos\theta, y_0 + \varepsilon \sin\theta) d\theta.$$

In the resulting expression, we aim at zero and, using the mean theorem, we get

$$\oiint_{\Gamma_1} = 2\pi q u(x_0, y_0).$$

Now from formula (21) we have

$$2\pi q u(x_0, y_0) = -\int_{\Gamma} A_1 dy - B_1 dx .$$
(22)

Let's consider the case $i = 2$. Similarly, we get:

$$\oint\limits_{\Gamma_1} = \oint\limits_{\Gamma_1} \left(\frac{-q(y-y_0)}{(x-x_0)^2 + q(y-y_0)^2} u - \frac{q(x-x_0)}{(x-x_0)^2 + q(y-y_0)^2} v + \gamma_2^1 \right) dy -$$

$$- \left(\frac{(x-x_0)}{(x-x_0)^2 + q(y-y_0)^2} u + \frac{q(y-y_0)}{(x-x_0)^2 + q(y-y_0)^2} v \right) dx = \qquad .$$

$$= \oint\limits_{\Gamma_1} \left(-\frac{q(y-y_0)}{\varepsilon^2} u - \frac{q(x-x_0)}{\varepsilon^2} v + \gamma_2^1 \right) dy - \left(\frac{(x-x_0)u - q(y-y_0)v}{\varepsilon^2} \right) dx$$

We enter the coordinates $x - x_0 = \varepsilon\cos\theta$, $y - y_0 = \varepsilon\sin\theta$.

Now we have

$$\oint\limits_{\Gamma_1} = \oint\limits_{\Gamma_1} \left[\frac{(-q+1)\sin 2\theta}{2} u - qv \right] d\theta =$$

$$= -q \int\limits_0^{2\pi} v(x_0 + \varepsilon\cos\theta, y_0 + \varepsilon\sin\theta) d\theta = -2\pi v(x_0, y_0) q. \tag{23}$$

Now we specify these conservation laws to twist the prism rod.

We have

$$2\pi q \tau_{xz}(x_0, y_0) = \oint\limits_{\Gamma} \left(\alpha_1^1 \tau_{xz} - q\alpha_1^2 \tau_{yz} + 2\theta \int \alpha_1^1 dx \right) dx - \left(\alpha_1^2 \tau_{xz} + \alpha_1^1 \tau_{yz} \right) dy,$$

$$2\pi q \tau_{yz}(x_0, y_0) = \oint\limits_{\Gamma} \left(\alpha_2^1 \tau_{xz} - q\alpha_2^2 \tau_{yz} + 2\theta \int \alpha_2^1 dx \right) dx - \left(\alpha_2^2 \tau_{xz} + \alpha_2^1 \tau_{yz} \right) dy. \tag{24}$$

Elastic-plastic boundary in a twisted rectilinear orthotropic rod.

Consider the elastic torsion of the orthotropic straight rod, the cross-section of which is limited by the convex outline of Γ.

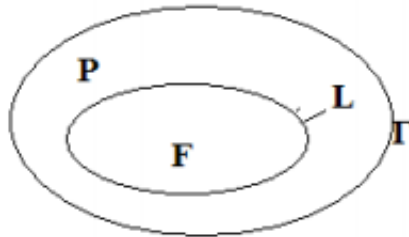At a fairly high value of torque, the rod forms an elastic F zone and a P plastic zone (Figure 2).

Рис. 2. Поперечное сечение скручиваемого стержня

Fig. 2. Cross-section of the twisted rod

It is known that the plastic zone begins to form on the outer contour of Γ. Suppose that the plastic zone completely covers the outer contour. Let L be the boundary of the division of elastic and plastic zones.

The purpose of this paragraph is to build the L border in an explicit form with the help of conservation laws built in previous paragraphs.

Problem setting. In the elastic zone, the stress tensor components satisfy equations (8)

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \ , \ a_{55} \frac{\partial \tau_{xz}}{\partial y} - a_{44} \frac{\partial \tau_{yz}}{\partial x} = -2\theta \ , \tag{25}$$

on the outer contour Γ to the boundary condition

$$\tau_{xz} l + \tau_{yz} m = 0 \ . \tag{26}$$

and the condition of plasticity

$$2\alpha_{13}\tau_{xz}^2 + 2\alpha_{23}\tau_{yz}^2 = 1 \ . \tag{27}$$

From ratios (27) --28) you can identify the components of the tensor τ_{xz}, τ_{yz} on the Γ contour.

We have now

$$\tau_{xz}\big|_{\Gamma} = \frac{m}{\sqrt{2\alpha_{13}m^2 + 2\alpha_{23}l^2}} \ ,$$

$$\tau_{yz}\big|_{\Gamma} = -\frac{l}{\sqrt{2\alpha_{13}m^2 + 2\alpha_{23}l^2}} \ . \tag{28}$$

Here the signs are chosen according to the torque (3).

In (24) we receive the following conservation laws.

$$2\pi q \tau_{xz}\left(x_0, y_0\right) = -\iint\limits_{\Gamma} A_1 dy - B_1 dx \ , \tag{29}$$

и

$$2\pi q \tau_{yz}\left(x_0, y_0\right) = -\iint\limits_{\Gamma} A_2 dy - B_2 dx \ . \tag{30}$$

To calculate $\tau_{xz}\left(x_0, y_0\right)$ and $\tau_{yz}\left(x_0, y_0\right)$ we use formulas (29)-(30). From these formulas we get τ_{xz} and τ_{yz} at all points inside the rod. Now let's check the condition (27).

At those points where the expression in the first part (27) less than one will fall into the elastic zone, and the rest of the dots in the plastic.

Conclusion. These calculations allow us to restore the desired boundary of L. with any accuracy. For the isotropic case, this problem was solved for the first time in [12; 14]. Examples of constructing elastic-plastic boundaries for different types of rolled profiles are considered in [10].

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