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The method of fictitious discrete models in the calculation of bodies with an inhomogeneous regular structure

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When the strength of elastic composite structures (plates, beams, shells) widely used in aviation, rocket and space technology is calculated with the finite element method (FEM), it is important to know the solution error. To analyze the solution error, it is necessary to use a sequence of approximate solutions constructed according to the FEM using the grinding procedure for basic discrete models (BMs), which take into account an inhomogeneous microheterogeneous structure of bodies within the microapproach. Discrete models obtained by grinding BMs have a high dimension, which makes it difficult to use the FEM for them. In addition, there are BMs of composite solids (CSs), for example, BMs of bodies with a microheterogeneous structure, which have such a high dimension that the implementation of the FEM for such BMs is practically impossible due to limited computer resources. To solve these problems, it is proposed to use fictitious discrete models in the calculations of CSs according to the FEM.

In this paper we propose a method of fictitious discrete models (MFDM) for calculating the strength of elastic bodies with an inhomogeneous microheterogeneous regular structure. The MFDM is implemented with the help of the FEM using corrected strength conditions, which take into account the error of approximate solutions. The method is based on the following provision. We believe that BMs of CSs generate solutions that slightly differ from the exact ones. Such BMs always exist for CSs due to the convergence of the FEM. The calculation of CSs according to the MFDM is reduced to the construction and calculation of the strength of fictitious discrete models (FMs), the dimensions of which are smaller than the dimension of the BMs. FMs reflect: the shape, characteristic dimensions, fastening, loading and the type of the inhomogeneous structure of CSs and the distribution of the elastic moduli corresponding to the BM of the CS. The sequence consisting of the FM converges to the BM, i.e., the limiting FM coincides with the BM. The convergence of such a sequence ensures uniform convergence of the FM stresses to the corresponding BM stresses. The implementation of the FEM for FMs with the use of multigrid finite elements leads to a large saving of computer resources, which makes it possible to use the MFDM for strength calculations of bodies with a microheterogeneous regular structure. Calculation of the CS strength according to the MFDM requires $10^3 \div 10^6$ *times less computer memory volume than a similar calculation using the BM of the CS, and does not contain the procedure for grinding the BM. The given example of calculating the strength of a beam with an inhomogeneous regular fibrous structure according to the MFDM shows its high efficiency. Applying the adjusted strength conditions allows using approximate solutions with larger errors in CS strength calculations, which leads to improving the efficiency of the MFDM.*

Keywords: elasticity, composites, adjusted strength conditions, fictitious discrete models, multigrid finite elements.

Метод фиктивных дискретных моделей в расчетах тел с неоднородной регулярной структурой

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В расчетах на прочность упругих композитных конструкций (пластины, балки, оболочки), которые широко применяются в авиационной и ракетно-космической технике, с помощью метода конечных элементов (МКЭ) важно знать погрешность решения. Для анализа погрешности решения необходимо использовать последовательность приближенных решений, построенных по МКЭ с применением процедуры измельчения для базовых дискретных моделей (БМ), которые учитывают в рамках микроподхода неоднородную, микронеоднородную структуру конструкций (тел). Дискретные модели, полученные путем измельчения БМ, имеют высокую размерность, что затрудняет для них применение МКЭ. Кроме того, существуют БМ композитных тел (КТ), например, БМ тел с микронеоднородной структурой, которые имеют такую высокую размерность, что реализация МКЭ для таких БМ, в силу ограниченности ресурсов ЭВМ, практически невозможна. Для решения данных проблем здесь предлагается в расчетах КТ по МКЭ использовать фиктивные дискретные модели.

В данной работе предлагается метод фиктивных дискретных моделей (МФДМ) для расчета на прочность упругих тел с неоднородной, микронеоднородной регулярной структурой. МФДМ реализуется с помощью МКЭ с применением скорректированных условий прочности, которые учитывают погрешность приближенных решений. В основе метода лежит следующее положение. Считаем, что БМ КТ порождают решения, которые мало отличаются от точных. В силу сходимости МКЭ такие БМ для КТ всегда существуют. Расчет КТ по МФДМ сводится к построению и расчету на прочность фиктивных дискретных моделей (ФМ), размерности которых меньше размерности БМ. ФМ отражают: форму, характерные размеры, крепление, нагружение и вид неоднородной структуры КТ и распределение модулей упругости, отвечающее БМ КТ. Последовательность, состоящая из ФМ, сходится к БМ, т. е. предельная ФМ совпадает с БМ. Сходимость такой последовательности обеспечивает равномерную сходимость напряжений ФМ к соответствующим напряжениям БМ. Реализация МКЭ для ФМ с применением многосеточных конечных элементов приводит к большой экономии ресурсов ЭВМ, что позволяет использовать МФДМ для расчетов на прочность тел с микронеоднородной регулярной структурой. Расчет на прочность КТ по МФДМ требует в $10^3 \div 10^6$ раз меньше объема памяти ЭВМ, чем аналогичный расчет с использованием БМ КТ, и не содержит процедуру измельчения БМ. Приведенный пример расчета на прочность балки с неоднородной регулярной волокнистой структурой по МФДМ показывает его высокую эффективность. Применение скорректированных условий прочности позволяет использовать в расчетах КТ на прочность приближенные решения с большой погрешностью, что приводит к повышению эффективности МФДМ.

Ключевые слова: упругость, композиты, скорректированные условия прочности, фиктивные дискретные модели, многосеточные конечные элементы.

Introduction

Composite structures (plates, beams, shells) especially those with a microheterogeneous fibrous structure are widely used in modern aviation, rocket and space technology. Calculation of the structure strength is one of the most important at the stage of preliminary design, which is a feasibility study of a structure project. As a rule, the static strength calculation of an elastic structure (body) of a certain class (for example, aircraft structures) is carried out according to the safety margins [1–3] and comes down to determining the maximum equivalent stress of the structure. In this case for the body V_0 the specified conditions (in terms of safety margins) have the form $n_1 \le n_0 \le n_2$, where n_1, n_2 are specified; n_0 is the body V_0 safety factor, $n_0 = \sigma_T / \sigma_0$; σ_T is a yield point (ultimate stress) [1]; σ_0 is the maximum equivalent body stress corresponding to the exact solution of the elasticity problem (constructed for the body V_0). For stresses that are determined approximately, the corrected strength conditions are used [4], taking into account the stress error. When analyzing the stress-strain state (SSS) of elastic bodies, the finite element method (FEM) is actively used [5–11]. Basic discrete models (BMs) of bodies, which take into account their inhomogeneous, microheterogeneous structure within the micro-approach [12], have a very high dimension.

Let us consider the main difficulties in composite solids (CSs) calculation using the FEM. To analyze the error of the approximate solution, it is necessary to use a sequence of solutions constructed according to the FEM using the grinding procedure (within the microapproach) of composite discrete models. The use of the grinding procedure leads to a sharp increase in the dimensions of discrete models. The multigrid finite element method (MFEM) [13–19] which uses multigrid finite elements (MFEs) [24–29] is effectively used to solve problems of the elasticity theory [20–23]. Since n nested grids ($n \ge 2$) are used instead of one grid when constructing a n-grid finite element (FE), the MFEM can be considered to be a generalization of the FEM, i.e., the FEM is a special case of the MFEM. From here it follows that if the MFEs are used in the calculations of bodies according to the FEM, then in this case, in fact, the MFEM is implemented. Inhomogeneous, microheterogeneous structures in multigrid discrete models are taken into account within the microapproach. MFEs generate discrete models of small dimension. However, for example, BMs of bodies with a microheterogeneous regular structure have such a high dimension that the implementation of the FEM for such BMs with the use of MFEs is difficult due to limited computer resources. To solve this problem, it is proposed to use fictitious discrete models when calculating the strength of CSs according to the FEM. Let us note that the existing approximate approaches and methods for calculating CSs have complex formulations, are laborious and difficult to implement for CSs of complex shapes [30–38].

In this paper, we propose the method of fictitious discrete models (MFDM) for calculating the strength of bodies with an inhomogeneous, microheterogeneous regular structure, which is implemented with the help of the MFEM using the corrected strength conditions. Let us introduce the following definition.

Definition 1. Discrete models constructed for the CS V will be called fictitious models (FMs) if these FMs have the following properties.

1. Inhomogeneous FM structures differ from the inhomogeneous structure of the CS V BM.

2. FMs reflect the shape, characteristic dimensions, fastening, loading and type of the inhomogeneous structure of the CS V, as well as the distribution of elastic moduli corresponding to the CS V BM.

3. The sequence consisting of FMs converges to the CS V BM, that is, the limiting FM of the sequence coincides with the CS V BM.

4. The dimensions of the FM are smaller than the dimension of the CS V BM, except for the limiting FM, the dimension of which is equal to the dimension of the CS V BM.

Let us note that properties 3, 4 are important for practice.

Scaled composite discrete models, the dimensions of which are smaller than the dimension of the CS BM, are considered as FMs in this paper. The proposed FMs formed with a scaled regular CS cell have the same characteristic dimensions, shape, fastening, and loading as BMs, but the inhomogeneous FM structures differ from the inhomogeneous BM structure. The considered FMs reflect the form of the BM inhomogeneous structure and the distribution of the elastic moduli corresponding to the BM. The FM sequence that converges to the BM is used in the calculations, i.e., the limiting FM of this sequence coincides with the BM. The convergence of such a sequence (see property 3 in definition 1) ensures the convergence of the FM stresses to the corresponding BM stresses. Calculations show a uniform monotonic convergence of the maximum equivalent stress of the FM to the maximum equivalent stress of the CS BM. The implementation of the MFDM requires $10^3 \div 10^6$ times less computer memory than a similar calculation using the CS BM, and does not require grinding the CS BM. The implementation of the FEM for FMs with the use of MFEs leads to a large saving of computer resources, which makes it possible to use the MFDM for strength calculations of bodies with a micro-heterogeneous regular structure. The given example of calculating a beam with an inhomogeneous regular fibrous structure according to MFDM shows its high efficiency. The use of the corrected strength conditions allows using the approximate solutions with a large error in the CS strength calculations, which leads to an increase in the MFDM efficiency. When calculating a CS of a complex shape according to the MFDM, it is advisable to use FMs with variable characteristic dimensions.

1. The main provisions of the method of fictitious discrete models. The MFDM is applied for CSs that satisfy the following basic provisions.

<u>*Provision 1*</u>. CSs consist of isotropic homogeneous bodies of different modulus, connections between which are ideal, i.e., the functions of displacements and stresses are continuous on the common boundaries of different-modulus isotropic homogeneous bodies.

<u>*Provision 2.*</u> Displacements, deformations and stresses of different-modulus isotropic homogeneous bodies correspond to the Cauchy relations and Hooke's law of the three-dimensional linear problem of the elasticity theory [39].

<u>*Provision 3.*</u> Approximate solutions that correspond to the CS BM differ little from the exact ones. Such approximate solutions will be considered to be exact ones. Let us note that such BMs for CSs always exist due to the convergence of the FEM.

2. The theorem of the method of fictitious discrete models. Corrected strength conditions which take into account the error of approximate solutions are used in the MFDM.

Theorem. Let the strength conditions be given for the safety factor n_0 of the elastic body V_0

$$n_1 \le n_0 \le n_2, \tag{1}$$

where n_1, n_2 are given; $n_1 > 1$, $n_0 = \sigma_T / \sigma_0$; σ_T is ultimate stress of the body V_0; σ_0 is the maximum equivalent body V_0 stress, which corresponds to the exact solution of the problem of the elasticity theory, constructed for the body V_0.

Let the safety factor n_b of the body V_0, corresponding to the approximate solution of the problem of the elasticity theory, satisfy the corrected strength conditions

$$\frac{n_1}{1-\delta_\alpha} \le n_b \le \frac{n_2}{1+\delta_\alpha}. \quad (2)$$

Then the safety factor n_0 of the body V_0, which corresponds to the exact solution of the problem of the elasticity theory, satisfies the given strength conditions (1), where $n_b = \sigma_T / \sigma_b$; σ_b is the maximum equivalent stress of the body V_0, corresponding to the approximate solution of the problem of the elasticity theory, constructed for the body V_0, and found with such an error δ_b that

$$|\delta_b| \le \delta_\alpha < C_\alpha = \frac{n_2 - n_1}{n_1 + n_2}, \quad (3)$$

where δ_α is the upper estimate of the relative error δ_b; δ_α is given, the error δ_b for the stress σ_b is determined by the formula $\delta_b = (\sigma_0 - \sigma_b)/\sigma_0$.

Let us note that if the body V_0 consists of plastic materials, then σ_T is the yield point. From (3) it follows that if $n_2 - n_1$ is small, then it is necessary to determine σ_b with a small error δ_b. The proof of the theorem is presented in [4].

3. Implementation of the method of fictitious discrete models. For the sake of simplicity, without losing the generality of judgments, we will consider the main procedures for implementing the MFDM using the example of the beam V_0 with an inhomogeneous regular structure with dimensions $H \times L \times H$, where $H = 96h$, $L = 1152h$, h is given, the beam is located in the Cartesian rectangular coordinate system (Fig. 1).

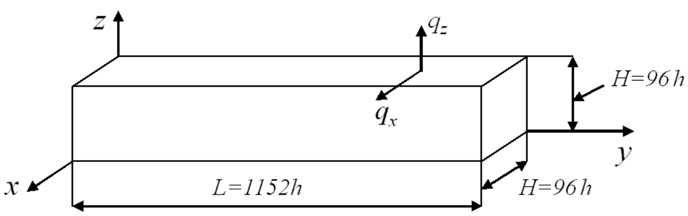

Fig. 1. The dimensions of the beam (body) V_0 (model R_n)

Рис. 1. Размеры балки (тела) V_0 (модели R_n)

The regular cell G_0 of the beam V_0 has a cubic shape with the side $6h$ (Fig. 2). The cell G_0 is located in the local Cartesian rectangular coordinate system $Oxyz$, $i, j, k = 1,...,7$. Fibers with the cross-section $h \times h$ are located along the axis Oy, the cross-sections of the fibers in the plane Oxz are colored (Fig. 2). So, the beam is reinforced with longitudinal continuous fibers. When $y = 0$ the beam is fixed, when $z = H$ it has the loading q_x, q_z. Strength conditions are specified for the beam V_0 (1).

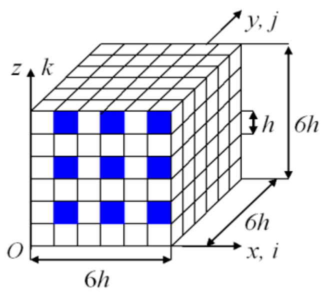

Fig. 2. The regular cell G_0

Рис. 2. Регулярная ячейка G_0

Isotropic homogeneous fibers have the same elastic moduli. It is believed that if the thickness of the fibers is less than 0.5 mm, then these fibers form a microheterogeneous fibrous structure.

3.1. Basic discrete model of the composite body V_0. The BM R_0 of the CS V_0, which consists of one-grid finite elements (1gFEs) V_j^h of the 1st order of a cubic shape with the side h (in which a three-dimensional SSS is realized [39]), takes into account the inhomogeneous structure of the CS V_0 within the microapproach and generates a uniform (basic) grid with the step h of the dimension $97\times1153\times97$ with the total number of nodal unknowns of the FEM equal to $N_0 = 32517504$, the bandwidth of the FEM simultaneous equations (SE) is equal to $b_0 = 28524$. Since the BM R_0 has a high dimension (over 32 million of unknown FEMs) and taking into account that $h/H << 1$ ($h/H = h/(96h) = 0{,}0104 << 1$), we believe that the maximum equivalent stress corresponding to the BM R_0 differs little from the exact one, provision 3 MFDM for BM R_0 is performed (see item 1). Fig. 2 shows the basic grid of the regular cell G_0.

3.2. Scaled composite discrete models. Following the MFDM, (see Fig. 1) we determine the FM sequence for the CS V_0. We use scaled composite discrete models R_n that form the sequence $\{R_n\}_{n=1}^{16}$ as FMs. The model R_n, $n=1,...,16$, has the same characteristic dimensions, shape, fastening and loading as the BM R_0 (Fig. 1). The discrete model R_n, consisting of 1gFEs of the 1st order of a cubic shape with the side h_n (a three-dimensional SSS is implemented in 1gFE V_e^n), has a uniform grid with the step h_n of the dimension $n_1^{(n)} \times n_2^{(n)} \times n_3^{(n)}$, where

$$n_1^{(n)} = 6n+1, \quad n_2^{(n)} = 12\times 6n+1, \quad n_3^{(n)} = 6n+1, \quad n=1,...,16. \tag{4}$$

The steps of the nodal grid of the model R_n along the axes Ox, Oy, Oz respectively, are equal to $h_x^{(n)} = H/(6n)$, $h_y^{(n)} = L/(72n)$, $h_z^{(n)} = H/(6n)$. Since $L = 12H$, then $h_n = h_x^{(n)} = h_y^{(n)} = h_z^{(n)}$. By virtue of (4), we have

$$h_n = \beta_n h, \quad n=1,...,16, \tag{5}$$

where β_n is the scale factor, $\beta_n = 16/n$, for $n=1,...,15$ we have $\beta_n > 1$, i.e. $h_n > h$, for $n \to 16$ we have $\beta_n \to 1$, $\beta_{16} = 1$, $h_{16} = h$.

According to (4), the model R_n consists of a finite number of bodies G_n of the same shape with dimensions $6h_n \times 6h_n \times 6h_n$, $n=1,...,16$ (Fig. 3). The CS G_n is located in the local Cartesian rectangular coordinate system $Oxyz$. The body G_n has the same number of fibers (with the cross-section $h_n \times h_n$) and the same mutual arrangement of these fibers as the regular cell G_0 (Fig. 2). In Fig. 3 the fiber sections of the cell G_n in the plane Oxz are coloured, $i, j, k = 1,...,7$. The fibers and the binder of the CSs G_n and G_0 have the same modulus of elasticity.

Let us introduce the following definitions, which are used in the construction of scaled composite discrete models.

Definition 2. We will say that the three-dimensional elastic body G is formed by scaling the elastic three-dimensional body G^0 with the scale factor $p > 0$ if any point $A \in G^0$ corresponds to such a single point $B \in G$ that $x_B = px_A$, $y_B = py_A$, $z_B = pz_A$, where x_A, y_A, z_A (x_B, y_B, z_B) are the coordinates of the point A (point B) corresponding to the Cartesian rectangular coordinate system $Oxyz$. And vice versa, if any point $B \in G$ corresponds to such a single point $A \in G^0$ that $x_A = x_B / p$, $y_A = y_B / p$, $z_A = z_B / p$. The elastic moduli at the points $A \in G^0$, $B \in G$ are the same.

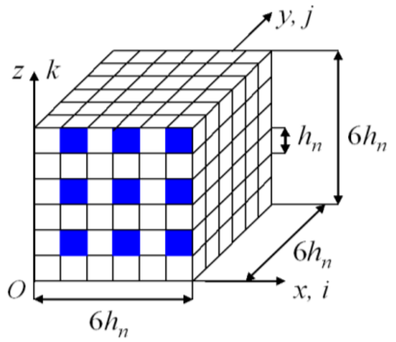

Fig. 3. The regular cell G_n

Рис. 3. Регулярная ячейка G_n

Definition 3. The three-dimensional elastic body G obtained by scaling the given (basic) elastic three-dimensional body G^0 with the given scale factor p will be called a scaled one. The relationship between the scaled body G and the base body G^0 is represented as $G = p\, G^0$, where p is the scale factor.

So, by virtue of (5), the CS G_n is formed by scaling the regular cell G_0 of the CS V_0 BM with the scale factor β_n (see Definition 2), that is, the body G_n is a scaled regular cell (see Definition 3). The shapes and inhomogeneous structures of the bodies G_n and G_0 are geometrically similar, that is, they differ only in scale (Fig. 2, 3, where $h_n > h$, at $n = \overline{1,15}$). Then, taking into account (5) and that the fibers and the binder of the CSs G_n and G_0 have the same elastic moduli, the connection between the bodies G_n, G_0 is represented in the form (see definition 3).

$$G_n = \beta_n G_0, \qquad (6)$$

where $\beta_n = 16/n$; $n = 1,...,16$, at $n \to 16$ we have $\beta_n \to 1$, $\beta_{16} = 1$.

Since the inhomogeneous structure is taken into account in the regular cell G_0, by virtue of (6) and in the CS G_n, the inhomogeneous structure is also taken into account with the help of a 1gFE V_e^n of a cubic shape with the side h_n. The model R_n, which by virtue of (5), (6) is formed using the scaled regular cell G_n, will be called a scaled one. We note that the CS G_n is, in fact, a regular cell of the model R_n. Since the inhomogeneous structure is taken into account in the regular cell G_n, therefore, the inhomogeneous structure is also taken into account in the model R_n. For the model R_n, we note the following properties, which show the main advantages of the MFDM.

1. The dimension of the model R_n at $n \le 15$ due to (4) is smaller than the dimension of the BM R_0. Therefore, the implementation of the FEM for the model R_n (at $n \le 15$) requires less computer resources than for the BM R_0.

2. When constructing scaled composite discrete models R_n, the procedure of grinding the BM of the CS is not used.

We note that the models R_n, $n=\overline{1,15}$ are, in fact, fictitious discrete models.

3.3. Convergence of a sequence of scaled discrete models. Let us show that the sequence $\{R_n\}_{n=1}^{16}$ consisting of scaled discrete models R_n converges to BM R_0 at $n\to 16$. According to (5), (6) at $n=16$ ($h_{16}=h$, $\beta_{16}=1$, $G_{16}=G_0$) the discrete models R_{16}, R_0 coincide, that is, $R_{16}=\mathrm{R}_0$. Since the model R_{16}, like the BM R_0, has a high dimension, that is, it has $N_0=32517504$ nodal unknown FEMs, and taking into account that $h<<H$ $(h/H=h/(96h)=0{,}0104)$, we assume that the maximum equivalent stress σ_{16} of the model R_{16} differs little from the exact stress σ_0 of the CS V_0. Then we assume $\sigma_0=\sigma_{16}$, that is, provision 3 of the MFDM for the BM R_0 is satisfied (see item 1). By virtue of (5), (6) at $n\to 16$ (at $\beta_n\to 1$) we have $G_n\to G_0$. Hence, taking into account that CSs G_n, G_0 are regular cells of the models R_n, R_0, respectively, and that these models have the same shape and characteristic sizes, we obtain

$$R_n\to \mathrm{R}_0 \quad \text{for} \quad n\to 16. \tag{7}$$

According to (7), for $n\to 16$ (taking into account that $R_{16}=\mathrm{R}_0$) we have $\sigma_n\to\sigma_{16}$ or (taking into account the equality $\sigma_0=\sigma_{16}$) $\sigma_n\to\sigma_0$, where σ_n is the maximum equivalent stress of the discrete model R_n. Let $\delta_\sigma=|\sigma_n-\sigma_{n-1}|/\sigma_n$ be a small value and $|\delta_n|\le\delta_\alpha$, where δ_n is the relative error for the stress σ_n, that is, $\delta_n=(\sigma_0-\sigma_n)/\sigma_0$, δ_α is given, $\delta_\alpha<C_\alpha$ (see (3)), $n=2, 3, \ldots$. Then we accept $\sigma_b=\sigma_n$. Let the safety factor n_b (where $n_b=\sigma_T/\sigma_b$, taking into account that $\sigma_b=\sigma_n$, we have $n_b=\sigma_T/\sigma_n$), corresponding to the approximate solution of the elasticity problem, satisfies the adjusted strength conditions (2). Then the safety factor n_0 of the CS V_0 corresponding to the exact solution of the elasticity problem satisfies the given strength conditions (1) (see the theorem in item 2). MFEs are used to reduce the dimension of the model R_n.

4. The results of numerical experiments. Let us consider a model problem of calculating the strength of a cantilever beam V_0 with an inhomogeneous regular fibrous structure with dimensions $96h\times1152h\times96h$ (Fig. 1). The regular cell G_0 of the beam is shown in Fig. 2. For the safety factor n_0 of the beam, the strength conditions are specified

$$1{,}8\le n_0\le 3{,}4. \tag{8}$$

For the model problem we have the following initial data:

$$h=0{,}2083;\ \sigma_T=4{,}5;\ E_c=1,\ E_v=10,\ \nu_c=\nu_v=0{,}3, \tag{9}$$

where E_c, E_v (ν_c,ν_v) are Young's moduli (Poisson's ratios) of the binder and fiber, respectively; σ_T is the fiber yield point; loads $q_z=q_x=0{,}00075$ act on the surface $z=H$, $0{,}5L\le y\le L$ (Fig. 1).

We use two-grid FEs (2gFEs) in the calculations. We will consider the main provisions of the construction of 2gFEs using the example of the 2gFE $V_d^{(2)}$ with dimensions $6h\times6h\times6h$ (Fig. 4), which consist of one regular cell G_0 (Fig. 2). The two-grid FE $V_d^{(2)}$ is located in the local Cartesian rectangular coordinate system $Oxyz$. When constructing the 2gFE $V_d^{(2)}$, we use two nested grids: a uniform fine grid h_d with the step h of the dimension $7\times7\times7$ and a coarse one H_d with dimensions $2\times3\times2$. The grid H_d has the step $6h$ along the axes Ox , Oz and the step $3h$ along the axis Oy . Fig. 4 shows the grids h_d and H_d , the nodes of the coarse grid H_d are marked with dots (12 nodes). The fine grid h_d is generated by the basic partition R_d of the 2gFE $V_d^{(2)}$, which consists of 1gFE V_j^h of the 1st order of a cubic shape with the side h (in which three-dimensional SSS is realized, $j=1,...,M$, M is the total number of 1gFE V_j^h, $M=216$) and takes into account the inhomogeneous structure of the 2gFE $V_d^{(2)}$.

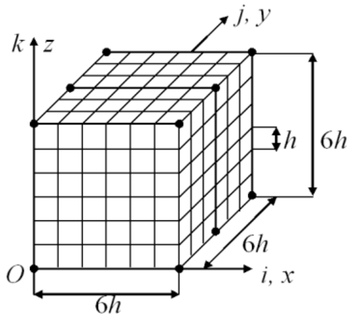

Fig. 4. Fine and coarse grids 2gFE $V_d^{(2)}$

Рис. 4. Мелкая и крупная сетки 2сКЭ $V_d^{(2)}$

We construct a superelement V_S on the partition R_d using the condensation method [10]. We represent the total potential energy Π_d of the partition R_d of 2gFE $V_d^{(2)}$ in the form

$$\Pi_d = \frac{1}{2}\mathbf{q}_S^T[K_S]\mathbf{q}_S - \mathbf{q}_S^T\mathbf{F}_S \,, \tag{10}$$

where T is the transposition; $[K_S]$ is the stiffness matrix (dimensions 654×654); $\mathbf{F}_S$, $\mathbf{q}_S$ are the vectors of nodal forces and displacements (of the dimension 654) of the superelement V_S .

We write the basis function $N_{ijk}(x,y,z)$ for the node i,j,k of the coarse grid H_d using Lagrange polynomials in the form $N_{ijk}=L_i(x)L_j(y)L_k(z)$, where

$$L_i(x)=\prod_{\alpha=1,\alpha\neq i}^{2}\frac{x-x_\alpha}{x_i-x_\alpha}, \quad L_j(y)=\prod_{\alpha=1,\alpha\neq j}^{3}\frac{y-y_\alpha}{y_j-y_\alpha}, \quad L_k(z)=\prod_{\alpha=1,\alpha\neq k}^{2}\frac{z-z_\alpha}{z_k-z_\alpha},$$

where x_i,y_j,z_k are the coordinates of the node i,j,k of the grid H_d in the coordinate system $Oxyz$; i,j,k are the coordinates of the integer coordinate system ijk introduced for the nodes of the coarse grid H_d ; $i,k=1,2$, $j=1,2,3$ (fig. 4).

Let us denote: $N_\beta=N_{ijk}$, $u_\beta=u_{ijk}$, $v_\beta=v_{ijk}$, $w_\beta=w_{ijk}$, where u_{ijk}, v_{ijk} , w_{ijk} are the values of displacements u, v, w in the node i,j,k of the grid H_d ; $i,k=1,2$; $j=1,2,3$; $\beta=1,...,12$. Then the approximating functions of displacements $u^{(2)}$, $v^{(2)}$, $w^{(2)}$ of the 2gFE can be written in the form

$$u^{(2)}=\sum_{\beta=1}^{12}N_\beta u_\beta\,, \quad v^{(2)}=\sum_{\beta=1}^{12}N_\beta v_\beta\,, \quad w^{(2)}=\sum_{\beta=1}^{12}N_\beta w_\beta\,. \tag{11}$$

Let us denote the vector of nodal displacements of the grid H_d (of dimension 36), that is, the vector of nodal unknowns 2gFE $V_d^{(2)}$ by $\mathbf{q}_d$. Using (11), the vector $\mathbf{q}_S$ of nodal displacements of the superelement V_S is expressed through the vector $\mathbf{q}_d$, that is

$$\mathbf{q}_S = [A_S^d]\,\mathbf{q}_d\ , \qquad (12)$$

where $[A_S^d]$ is the rectangular matrix (of dimension 654×36).

Substituting (12) into (10), from the condition $\partial\Pi_d / \partial\mathbf{q}_d = 0$, we obtain $[K_d]\,\mathbf{q}_d = \mathbf{F}_d$, where

$$[K_d] = [A_S^d]^T[K_S][A_S^d]\ ,\ \mathbf{F}_d = [A_S^d]^T\,\mathbf{F}_S\ , \qquad (13)$$

where $[K_d]$ is the stiffness matrix (of dimension 36×36) and $\mathbf{F}_d$ is the vector of nodal forces (of dimension 36) of 2gFE $V_d^{(2)}$.

The solution built for a coarse grid H_d of 2gFE $V_d^{(2)}$ is projected onto the super element V_S grid using formula (12), and then, according to the condensation method [10], is projected onto the fine grid h_d, which makes it possible to calculate stresses in any 1gFE V_j^h of the basic partition R_d of 2gFE $V_d^{(2)}$.

On the basis of the model R_n, we construct a two-grid discrete model R_n^o, which consists of composite 2gFEs of the type $V_d^{(2)}$ with dimensions $6h_n\times 6h_n\times 6h_n$, $n=1,...,12$. For the two-grid model R_n^o, we determine (according to the 4th theory of strength [1]) the maximum equivalent stress σ_n^o, $n=\overline{1,12}$. The calculation results are presented in table 1, where σ_n^o is the maximum equivalent stress of the model R_n^o; N_n^o and b_n^o are the dimension and the bandwidth of the FEM SE of the model R_n^o, $n=5,...,12$, the relative error δ_n (in percent) is determined by the formula

$$\delta_n(\%) = 100\ \%\times|\sigma_n^o - \sigma_{n-1}^o|/\sigma_n^o, \quad n=6,...,12. \qquad (14)$$

The analysis of the results shows uniform monotonic convergence of stresses σ_n^0, $n=5,...,12$, and relative errors $\delta_n(\%)$, $n=6,...,12$.

Table 1

Calculation results for models R_5^o – R_{12}^o

n	R_n^o	σ_n^o	δ_n (%)	N_n^o	b_n^o	n	R_n^o	σ_n^o	δ_n (%)	N_n^o	b_n^o
5	R_5^o	1,476	–	12960	240	9	R_9^o	1,819	4,01	64800	636
6	R_6^o	1,576	6,34	21168	321	10	R_{10}^o	1,888	3,65	87120	765
7	R_7^o	1,665	5,34	32256	414	11	R_{11}^o	1,952	3,28	114048	906
8	R_8^o	1,746	4,64	46656	519	12	R_{12}^o	2,012	2,98	146016	1059

Let us note that the BM R_0 generates the maximum equivalent stress σ_0 of the CS V_0, which differs little from the exact one. The stress σ_0 is considered to be accurate (see provision 3, item 1). According to calculations, $\sigma_{16}^o = 2{,}140$ where σ_{16}^o is the maximum equivalent stress of the model R_{16}^o. We have $R_{16} = \mathrm{R}_0$ (see Section 3.3). The two-grid model R_{16}^o is built on the basis of the model R_{16} using 2gFE $V_d^{(2)}$ (Fig. 4). Since the dimensions of the 1gFE of the BM R_0 are small, the dimensions of the 2gFE model R_{16}^o are also small, so we accept $\sigma_{16}^o = \sigma_0 = 2{,}140$.

Calculations show that if $\delta_n(\%) \le 3\%$ (see (14)), then the error of the maximum equivalent stress σ_n^o of the model R_n^o is not more than $10\ \%$. Since the stresses $\sigma_{12}^o = 2{,}012$ and $\sigma_{11}^o = 1{,}952$ differ by $\delta_{12}(\%) = 2{,}98\ \%$ (see Table 1), that is, we have $\delta_{12}(\%) \le 3\ \%$, the stress error σ_{12}^o is not more than 10%. We note that the stress σ_{12}^o differs from the stress σ_0 by 5.98%. We will assume that the upper estimate for the stress error σ_{12}^o is 10%. Then we accept $\delta_\alpha = 0{,}1$, $\sigma_b = \sigma_{12}^o = 2{,}012$. Condition (3) is satisfied, that is, we have the inequality $\delta_\alpha = 0{,}1 < C_\alpha = 0{,}3$. Substituting $\delta_\alpha = 0{,}1$, $n_1 = 1{,}8$, $n_2 = 3{,}4$ in (2), we obtain the corrected strength conditions for the CS V_0 in the form

$$2 \le n_b \le 3, \tag{15}$$

where n_b is the safety factor of the CS V_0 corresponding to the approximate solution of the elasticity problem,

$$n_b = \sigma_T / \sigma_b. \tag{16}$$

Using in (16) $\sigma_T = 4{,}5$, $\sigma_b = 2{,}012$, we find the safety factor n_b for the CS V_0.

$$n_b = \sigma_T / \sigma_b = 4{,}5 / 2{,}012 = 2{,}24. \tag{17}$$

So, the safety factor $n_b = 2{,}24$ of the CS V_0 (corresponding to the approximate solution of the elasticity problem) satisfies the corrected strength conditions (15). Then, according to the theorem of item 2, the safety factor n_0 of the CS V_0 (corresponding to the exact solution of the elasticity problem) satisfies the given strength conditions (8). We note that the BM R_0 of the CS V_0 has over 32 million nodal unknown FEMs, which makes it difficult to implement FEM using 1gFE of the 1st order of a cubic shape with the side h for constructing the solution for the BM R_0, which we consider to be accurate (see provision 3, item 1 and item 3.1). In calculating the strength according to the MFDM of the composite beam V_0 (see Fig. 1) we use the model R_{12}^o that has $N_{12}^o = 146016$ nodal unknowns of the FEM and the bandwidth of the FEM SE of which is equal to $b_{12}^o = 1059$ (see Table 1). The discrete model R_{12}^o requires $\mathrm{k}_1 = \dfrac{N_0 \times b_0}{N_{12}^o \times b_{12}^o} = \dfrac{32517504 \times 28524}{146016 \times 1059} = 5998{,}34$ times less computer memory, that is, almost 6×10^3 times less than the BM R_0 (see item 3.1), which shows the high efficiency of the MFDM.

5. The application of approximate solutions with a large error in the MFDM. Let us consider the case of calculating a CS for strength according to the MFDM, when it is possible to use elastic approximate solutions with a large error on the example of calculating the CS V_0 (see section 4). Calculations show that if $\delta_n(\%) \le 5\%$ (see (14)), then the error of the maximum equivalent stress σ_n^o of the model R_n^o is not more than 25%. Since the stresses $\sigma_8^o = 1{,}746$ and $\sigma_7^o = 1{,}665$ differ by $\delta_8(\%) = 4{,}64\ \%$ (see Table 1), that is, $\delta_8(\%) \le 5\ \%$, the stress error σ_8^o is not more than $25\ \%$. In fact, the stress σ_8^o is different from the stress $\sigma_0 = 2{,}140$ by $18{,}41\ \%$. We will assume that the upper estimmate for the stress error σ_8^o is $25\ \%$. Then we accept $\delta_\alpha = 0{,}25$, $\sigma_b = \sigma_8^o = 1{,}746$. Condition (3) is satisfied, that is, we have $\delta_\alpha = 0{,}25 < C_\alpha = 0{,}3$. Substituting $\delta_\alpha = 0{,}25$, $n_1 = 1{,}8$, $n_2 = 3{,}4$ in (2), we obtain the following corrected strength conditions for the CS V_0

$$2{,}4 \le n_b \le 2{,}7. \tag{18}$$

Using $\sigma_T = 4{,}5$, $\sigma_b = 1{,}746$ in (16), we find the safety factor n_b for the CS V_0

$$n_b = \sigma_T / \sigma_b = 4{,}5 / 1{,}746 = 2{,}58. \qquad (19)$$

The safety factor $n_b = 2{,}58$ of the CS V_0 (corresponding to the approximate solution of the elasticity problem) satisfies the corrected strength conditions (18). Then the safety factor n_0 of the CS V_0 (corresponding to the exact solution of the elasticity problem) satisfies the given strength conditions (8) (see item 2). In this case, when calculating the strength of the CS V_0 according to the MFDM, we use the model R_8^o that has $N_8^o = 46656$ of unknown FEMs and the bandwidth of the FEM SE of which is equal to $b_8^o = 519$. The model R_8^o requires $\mathrm{k}_2 = \frac{N_0 \times b_0}{N_8^o \times b_8^o} = \frac{32517504 \times 28524}{46656 \times 519} = 38304{,}76$ times less computer memory, that is, almost 38×10^3 times less than the BM R_0.

So, it has been shown that when calculating the CS V_0, it is possible to use elastic approximate solutions with a large error. In this case, in the calculations we use the stress σ_8^o of the model R_8^o, the error $\varepsilon_8 = 18{,}41$ % of which is greater than the error $\varepsilon_{12} = 5{,}98$ % of the stress σ_{12}^o of the model R_{12}^o, which leads to an increase in the efficiency of the MFDM (the coefficient k_2 is 6,38 times greater than the coefficient k_1). This is due to the fact that the dimension and the bandwidth of the FEM SE of the model R_8^o are smaller than the dimension and the bandwidth of the FEM SE of the model R_{12}^o (see Table 1). The following conclusion can be drawn on the basis of the results obtained in the given example. The use of discrete CS models in MFDM, the maximum equivalent stresses of which have a large error, leads to an increase in the MFDM efficiency.

6. Fictitious models with variable characteristic dimensions. When calculating CSs of complex shapes according to the MFDM, it is advisable to use FMs with variable characteristic dimensions. For the sake of simplicity, let us consider the brief essence of such FMs without losing the generality of reasoning, using the example of the beam $V_0^{(1)}$ of a complex shape, that is, with a constant cross-section of a complex shape (such as an I-beam) (Fig. 5). The beam $V_0^{(1)}$ is located in the Cartesian rectangular coordinate system $Oxyz$, the axis Oy is parallel to the beam axis. Let the beam be reinforced with continuous longitudinal fibers with the cross-section $h \times h$, that is, which are parallel to the axis Oy, where $h = L_0 / N$, N is given; L_0 is the length of the beam $V_0^{(1)}$. The BM $R_0^{(1)}$ of the beam $V_0^{(1)}$ consists of the FE V_e of the 1st order of a cubic shape with the side h that takes into account the inhomogeneous structure of the beam and generates an approximate solution that does not differ much from the exact one. We consider such an approximate solution to be exact (see provision 3, item 1). The FM $R_n^{(1)}$ of the beam differs from its BM $R_0^{(1)}$ only by one (variable) characteristic dimension L_n (along the axis Oy) (Fig. 5). The FM $R_n^{(1)}$ has fastening and the same loading pattern as the BM $R_0^{(1)}$ of the beam $V_0^{(1)}$.

We determine the characteristic dimension L_n of the FM $R_n^{(1)}$ by the formula

$$L_n = L_0 n / N = hn, \tag{20}$$

where $n = n_0, ..., N$; $n_0 > 2$, n_0 is given.

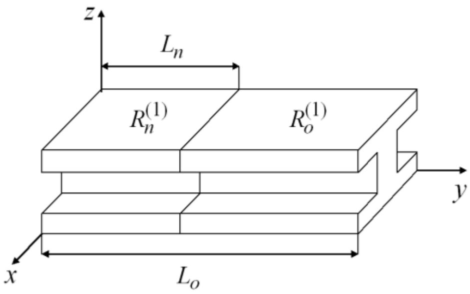

Fig. 5. The beam $V_0^{(1)}$

Рис. 5. Балка $V_0^{(1)}$

The FM $R_n^{(1)}$ has the same inhomogeneous structure as the BM $R_0^{(1)}$, that is, the FM $R_n^{(1)}$ is reinforced with continuous longitudinal fibers with the cross section $h \times h$ and has the same fiber distribution in the cross section as the BM $R_0^{(1)}$ of the beam $V_0^{(1)}$. Inhomogeneous structures of the FM $R_n^{(1)}$ and the BM $R_0^{(1)}$ are taken into account using the FE V_e of the first order of a cubic shape with the side h. From the above, taking into account that according to (20) $L_n \to L_0$ at $n \to N$, it follows

$$R_n^{(1)} \to R_0^{(1)} \quad \text{at}\, n \to N. \tag{21}$$

From the fulfillment of (21) we obtain

$$\sigma_n^{(1)} \to \sigma_0^{(1)} \quad \text{at}\, n \to N, \tag{22}$$

where $\sigma_n^{(1)}$ ($\sigma_0^{(1)}$) is the maximum equivalent stress corresponding to the FM $R_n^{(1)}$ (corresponding to the BM $R_0^{(1)}$ of the beam $V_0^{(1)}$).

Since the FM $R_n^{(1)}$ and the BM $R_0^{(1)}$ beams consist of the FE V_e of the 1-st order of a cubic shape with the side h and the cross sections of these models are the same, then the sections of the FM $R_n^{(1)}$ and the BM $R_0^{(1)}$ contain the same number of nodes, which we denote by N_0. Then the total number of nodes M_0 of the BM $R_0^{(1)}$ is equal to $M_0 = N_0(N+1)$, the total number of nodes M_n of the FM $R_n^{(1)}$ is $M_n = N_0(n+1)$. When $n_0 \le n < N$ we get that $M_n < M_0$, that is, the dimension of the FM $R_n^{(1)}$ is smaller than the dimension of the BM $R_0^{(1)}$. For $n = N$ we have $M_N = M_0$, that is, the dimensions of the FM $R_N^{(1)}$ and the BM $R_0^{(1)}$ coincide. So, it is shown that when calculating the composite beam $V_0^{(1)}$ (Fig. 5) of a complex shape according to the MFDM, it is advisable to use the FM $R_n^{(1)}$ with the variable characteristic dimension L_n, which leads to saving computer resources.

Conclusion

The method of fictitious discrete models is proposed for calculating the static strength of elastic bodies with an inhomogeneous, microheterogeneous regular structure. The proposed method is reduced to constructing and calculating the strength of fictitious discrete models, the dimensions of which are smaller than the dimensions of the basic discrete models of composite solids, and is implemented with the help of the FEM using corrected strength conditions that take into account the error of approximate solutions. The FEM implementation for fictitious discrete models with the use of multigrid finite elements provides a great economy of computer resources, which makes it possible to use the proposed method for calculating the strength of bodies with microheterogeneous regular structure. The implementation of the method of fictitious discrete models requires less computer resources than the implementation of the FEM for basic discrete models. When constructing fictitious discrete models, the grinding procedure for basic models is not used. The calculations show the high efficiency of the proposed method in calculating the strength of bodies with an inhomogeneous regular fibrous structure. The use of the corrected strength conditions makes it possible to use approximate solutions with a large error in the calculations, which leads to an increase in the efficiency of the method of fictitious discrete models.

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