UDC 539.374
Doi: 10.31772/2587-6066-2020-21-3-303-306

For citation: Annin B. D., Senashov S. I., Gomonova O. V. Solving boundary value problems of equations of two-dimensional elasticity theory using conservation laws. *Siberian Journal of Science and Technology.* 2020, Vol. 21, No. 3, P. 303–306. Doi: 10.31772/2587-6066-2020-21-3-303-306

Для цитирования: Аннин Б. Д., Сенашов С. И., Гомонова О. В. Решение краевых задач уравнений двумерной теории упругости с помощью законов сохранения // Сибирский журнал науки и технологий. 2020. Т. 21, № 3. С. 303–306. Doi: 10.31772/2587-6066-2020-21-3-303-306

SOLVING BOUNDARY VALUE PROBLEMS OF EQUATIONS OF TWO-DIMENSIONAL ELASTICITY THEORY USING CONSERVATION LAWS

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The plane problem for elasticity equations is well studied. It can be explained by its importance for applications and by the fact that the equations can be reduced to the Cauchy-Riemann system. In spite of this importance, exact solutions that would describe the stress-strain state of bodies of finite dimensions are not numerous. Conservation laws for differential equations have been appeared more than a hundred years ago, but, as a rule, they were not used to solve specific problems, but were of purely academic interest. The situation changed with the development of the technique of construction of conservation laws for arbitrary systems of differential equations, and then with the use of conservation laws to solve boundary value problems of the theory of plasticity and elastic-plasticity. In this article, new conservation laws are constructed for the equations of the plane theory of elasticity in the stationary case. These laws form an infinite series, which is closely related to the elasticity equations solving. This fact made possible to reduce solving of boundary value problems, in terms of displacements, to the calculation of contour integrals along the boundary of a domain bounded by the studying elastic body. As it follows from the proposed technique, the studied area can be multiply connected, and the considered boundary can be piecewise-smooth.

Keywords: conservation laws, boundary value problem, elasticity equations.

РЕШЕНИЕ КРАЕВЫХ ЗАДАЧ УРАВНЕНИЙ ДВУМЕРНОЙ ТЕОРИИ УПРУГОСТИ С ПОМОЩЬЮ ЗАКОНОВ СОХРАНЕНИЯ

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Плоская задача для уравнений упругости достаточно хорошо изучена. Это объясняется ее важностью для приложений и тем, что уравнения сводятся к системе Коши – Римана. Несмотря на это, точных решений, которые описывали бы напряженно-деформированное состояние тел конечных размеров, не так много. Законы сохранения для дифференциальных уравнений появились более ста лет назад, но, как правило, они не использовались для решения конкретных задач, а представляли «чисто академический» интерес. Ситуация изменилась с развитием техники построения законов сохранений для произвольных систем дифференциальных уравнений, а затем – с использованием законов сохранения для решения краевых задач теории пластичности и упруго-пластичности. В этой статье построены новые законы сохранения для уравнений плоской теории упругости в стационарном случае. Эти законы образуют бесконечную серию, которая тесно связана с решениями уравнений упругости. Именно этот факт позволил свести решение краевых задач в терминах перемещений к вычислению контурных интегралов по границе области, ограниченной изучаемым упругим телом. Из данной методики следует, что область может быть многосвязной, а граница – кусочно-гладкой.

Ключевые слова: законы сохранения, краевая задача, уравнения упругости.

Introduction. Usually, the equations of two-dimensional static elasticity are solved using the complex variable theory methods. These methods were developed by G. V. Kolosov and N. I. Muskhelishvili. The use of complex variable theory methods for solving elasticity problems requires a good knowledge of the complex variable theory technique and, as a rule, is used for areas faceted with smooth contours [1; 2]. This article proposes a method for solving the same problems using conservation laws. Apparently, for the first-time conservation laws for elasticity equations were calculated in [3; 4]. These calculations were based on the theorem of E. Noether. The found laws did not find any practical application and were of purely academic interest [5]. In [6–10] it is shown that conservation laws can be successfully used in solving boundary value problems of the theory of plasticity and elastic-plasticity. G.P. Cherepanov [11; 12] noticed that the complex variable theory methods used in solving plane problems actually go back to conservation laws. In the presented work, an endless series of new conservation laws was found. These conservation laws made it possible to obtain formulas with the help of which one can effectively reduce the boundary value problems to quadratures. The latter are curvilinear integrals over a closed contour and can be easily solved numerically for given boundary conditions in displacements.

1. Formulation of the problem. Consider the equations of two-dimensional elasticity in the stationary case

$$F_1 = (\lambda+2\mu)u_{xx} + (\lambda+\mu)v_{xy} + \mu u_{yy} = 0,$$
$$F_2 = (\lambda+2\mu)v_{yy} + (\lambda+\mu)u_{xy} + \mu v_{xx} = 0, \qquad (1)$$

where λ, μ are constants called Lamé parameters, u, v are components of the deformation vector, the indices below indicate the derivatives with respect to the corresponding components.

Let us find conservation laws of a special form for (1). The conservation law for the system of equations (1) is an expression of the form

$$A_x(x, y, u, v) + B_y(x, y, u, v) = \omega^1 F_1 + \omega^2 F_2, \qquad (2)$$

where ω_i some functions of x, y, which are simultaneously not identically equal to zero. The magnitudes A, B will be called the components of the conserved current. We are looking for the components of the conserved current in the form:

$$A = \alpha^1(\lambda+2\mu)u_x + \beta^1 u_y + \gamma^1\mu v_x + \delta^1 v_y,$$
$$B = \alpha^2 u_x + \beta^2 u_y + \gamma^2 v_x + \delta^2 v_y, \qquad (3)$$

where γ^i, δ^i are some functions depended on x, y only.

Substitute expressions (3) into (2). As a result, one can obtain a polynomial of the first degree in the variables $u_x, u_y, u_{xx}, u_{xy}, u_{yy}, v_x, v_y, v_{xx}, v_{xy}, v_{yy}$. Equating the coefficients at the derivatives in this polynomial to zero and taking into account that $\alpha^i, \beta^i, \gamma^i, \delta^i$ are functions only from x, y, one can get after exclusion ω^i, $i = 1, 2$

$$\alpha^2 = -\beta^1 + (\lambda+\mu)\gamma^1, \quad \beta^2 = \mu\alpha^1,$$
$$\gamma^2 = -\delta^1 + (\lambda+\mu)\alpha^1, \quad \delta^2 = (\lambda+2\mu)\gamma^1,$$
$$(\lambda+2\mu)\alpha^1_x + \alpha^2_y = 0, \quad \mu\gamma^1_x + \gamma^2_y = 0, \qquad (4)$$
$$\beta^1_x + \beta^2_y = 0, \quad \delta^1_x + \delta^2_y = 0.$$

This implies:

$$(\lambda+2\mu)\alpha^1_x - \beta^1_y + (\lambda+\mu)\gamma^1_y = 0,$$
$$\mu\gamma^1_x - \delta^1_y + (\lambda+\mu)\alpha^1_y = 0, \qquad (5)$$

$$\beta^1_x + \mu\alpha^1_y = 0, \quad (\lambda+2\mu)\gamma^1_y + \delta^1_x = 0. \qquad (6)$$

Substituting (6) into (5) one can obtain

$$(\lambda+2\mu)\alpha^1_{xx} + (\lambda+\mu)\gamma^1_{xy} + \mu\alpha^1_{yy} = 0,$$
$$(\lambda+2\mu)\gamma^1_{yy} + (\lambda+\mu)\alpha^1_{xy} + \mu\gamma^1_{xx} = 0. \qquad (7)$$

It is obvious that (7), up to notation, coincides with system (1). Therefore, α^1, γ^1 is an arbitrary smooth solution to the system of equations (1). From (3) taking into account (4), one obtains the conserved current

$$A = \alpha^1(\lambda+2\mu)(u_x+v_y) + \gamma^1\mu(v_x-u_y),$$
$$B = \gamma^1(\lambda+2\mu)(u_x+v_y) + \alpha^1\mu(v_x-u_y), \qquad (8)$$

where α^1, γ^1 is arbitrary solution of the system of equations (1).

2. Construction of solutions to the boundary value problem. It is known that

$$u = G^1 - \frac{1}{4(1-\nu)}(xG^1 + yG^2 + G^0)_x =$$
$$= G^1 - \omega(xG^1 + yG^2 + G^0)_x, \qquad (9)$$
$$v = G^2 - \omega(xG^1 + yG^2 + G^0)_y,$$

is the solution to the system of equations (1). Here G^i are arbitrary harmonic functions, ν is Poisson's ratio, which can be expressed through the Lamé parameters.

From (9) one gets

$$u_x + v_y = (G^1_x + G^2_y)(1-\omega/2),$$
$$-u_y + v_x = -G^1_y + G^2_x. \qquad (10)$$

Let x_0, y_0 be an arbitrary point of a domain Ω with a smooth boundary Γ. Let us assume $G^1 = \ln\left((x-x_0)^2 + (y-y_0)^2\right) = \ln r^2$, $G^2 = 0$. Then one obtains

$$u_x + v_y = 2\frac{x-x_0}{r^2}\omega, \; -u_y + v_x = 2\frac{y-y_0}{r^2}. \qquad (11)$$

From (2), by Green's formula, follows

$$\iint_\Omega (A_x + B_y)dxdy = \oint_\Gamma -Ady + Bdx =$$
$$= \oint_\Gamma -\left(\alpha^1(\lambda+2\mu)\left(u_x+v_y\right) + \gamma^1\mu\left(v_x-u_y\right)\right)dy + \qquad (12)$$
$$+ \left(\gamma^1(\lambda+2\mu)\left(u_x+v_y\right) + \alpha^1\mu\left(v_x-u_y\right)\right)dx = 0.$$

Now let the contour Γ_1 consist of two contours: Γ and a circle of radius ε: $(x-x_0)^2+(y-y_0)^2=\varepsilon^2$.

Assuming in (12) instead of Γ the contour $\Gamma 1$ and taking into account relation (11), one can obtain

$$\oint_{\Gamma_1} -\left(\alpha^1(\lambda+2\mu)2\frac{x-x_0}{r^2}\omega-\gamma^1\mu 2\frac{y-y_0}{r^2}\right)dy+$$
$$+\left(\gamma^1(\lambda+2\mu)2\frac{x-x_0}{r^2}\omega+\alpha^1\mu\left(v_x-u_y\right)\right)dx= \quad (13)$$
$$=\oint_{\Gamma}+\oint_{(x-x_0)^2+(y-y_0)^2=\varepsilon^2}=0.$$

Let's represent the equation of a circle in parametric form:

$$x-x_0=\varepsilon\cos\varphi,\ y-y_0=\varepsilon\sin\varphi,\ 0\le\varphi\le 2\pi.$$

Then the last of the integrals in formula (13) will be written as follows:

$$\oint_{(x-x_0)^2+(y-y_0)^2=\varepsilon^2}=$$
$$=\int_0^{2\pi}-\left(\alpha^1(\lambda+2\mu)2\omega\cos\varphi-\gamma^1\mu 2\sin\varphi\right)\cos\varphi d\varphi+ \quad (14)$$
$$+\left(\gamma^1(\lambda+2\mu)2\cos\varphi+\alpha^1\mu 2\sin\varphi\right)\sin\varphi d\varphi=$$
$$=2\pi\alpha^1\left(x_0,y_0\right)\left(\mu-(\lambda+2\mu)\omega\right).$$

In formula (14), the mean value theorem was used. Finally, from (13) one obtains

$$\frac{-1}{2\pi(\mu-(\lambda+2\mu)\omega}\oint_{\Gamma}-$$
$$-\left(\alpha^1(\lambda+2\mu)2\omega\left(\frac{x-x_0}{r^2}\right)-2\gamma^1\mu\left(\frac{y-y_0}{r^2}\right)\right)dy+ \quad (15)$$
$$+\left(\gamma^1(\lambda+2\mu)2\omega\left(\frac{x-x_0}{r^2}\right)-2\alpha^1\mu\left(\frac{y-y_0}{r^2}\right)\right)dx=$$
$$=\alpha^1\left(x_0,\,y_0\right).$$

Assuming now in equations (10) $G^2=\ln\left((x-x_0)^2+(y-y_0)^2\right)=\ln r^2,\ G^1=0.$ In this case, similar reasoning and calculations give a formula for $\gamma^1(x_0,y_0)$.

One obtains

$$\frac{-1}{2\pi(\mu+(\lambda+2\mu)\omega}\times$$
$$\times\oint_{\Gamma}\left(\alpha^1(\lambda+2\mu)2\omega\left(\frac{y-y_0}{r^2}\right)+2\gamma^1\mu\left(\frac{x-x_0}{r^2}\right)\right)dy+ \quad (16)$$
$$+\left(\gamma^1(\lambda+2\mu)2\omega\left(\frac{y-y_0}{r^2}\right)+2\alpha^1\mu\left(\frac{x-x_0}{r^2}\right)\right)dx=$$
$$=\gamma^1\left(x_0,\,y_0\right).$$

Formulas (15), (16) make it possible to calculate α^1,γ^1 at any point (x_0,y_0) in the area if the values of these functions are known on the boundary of Γ. And since α^1,γ^1 are a solution to equations (1), one can obtain a method for constructing solutions to system (1) according to their boundary conditions. Namely, let the following boundary value problem be set: there is an area Ω with a boundary Γ. On Γ, the components of the displacement vector are given

$$u|_{\Gamma}=u_0(x,y),\ v|_{\Gamma}=v_0(x,y). \quad (17)$$

Then, based on formulas (15), (16), the solution to problem (17) for the system of equations (1) can be written in the form

$$\frac{-1}{2\pi(\mu-(\lambda+2\mu)\omega}\oint_{\Gamma}-$$
$$-\left(u_0(\lambda+2\mu)2\omega\left(\frac{x-x_0}{r^2}\right)-2v_0\mu\left(\frac{y-y_0}{r^2}\right)\right)dy+ \quad (18)$$
$$+\left(v_0(\lambda+2\mu)2\omega\left(\frac{x-x_0}{r^2}\right)-2u_0\mu\left(\frac{y-y_0}{r^2}\right)\right)dx=$$
$$=u_0\left(x_0,\,y_0\right).$$

$$\frac{-1}{2\pi(\mu+(\lambda+2\mu)\omega}\times$$
$$\times\oint_{\Gamma}\left(u_0(\lambda+2\mu)2\omega\left(\frac{y-y_0}{r^2}\right)+2v_0\mu\left(\frac{x-x_0}{r^2}\right)\right)dy+ \quad (19)$$
$$+\left(v_0(\lambda+2\mu)2\omega\left(\frac{y-y_0}{r^2}\right)+2u_0\mu\left(\frac{x-x_0}{r^2}\right)\right)dx=$$
$$=v_0\left(x_0,\,y_0\right).$$

Conclusion. In the article, conservation laws that depend on derivatives of the first order and linear on them are found. It is shown that two coefficients at the derivatives are solutions of the elasticity equations in the plane case. This made it possible to reduce the solution of boundary value problems in displacements to the calculation of contour integrals along the boundary of the area. The proposed technique can be easily generalized to multiply – connected domains [13–15], as well as to the three-dimensional case.

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