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SYSTEM ANALYSIS OF DYNAMIC PROBLEMS OF ANISOTROPIC PLASTICITY THEORY

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Dynamic problems are the least studied area of plasticity theory. These problems arise in various fields of engineering and science, but the complexity of the original differential equations do not allow to develop accurate solutions and correctly solve numerical boundary value problems. This is even more typical of dynamic equations of anisotropic plasticity. Anisotropy reduces the group of symmetries allowed by the equations, and therefore narrows the number of invariant solutions. One-dimensional dynamic plasticity problems are well studied, but two-dimensional problems cause insurmountable mathematical difficulties due to the nonlinearity of the basic equations, even in the isotropic case. The study of the symmetries of the plasticity equations allowed us to find some exact solutions. The most known solution was found by B. D. Annin, who described the unsteady compression of a plastic layer made of isotropic material by rigid plates. Annin's solution is linear in two spatial variables, however, it includes arbitrary functions of time. Symmetries are also used in the proposed work. Point symmetries are first calculated for dynamic plasticity equations in the anisotropic case and are presented in the paper. The Lie algebra generated by the found symmetries appeared to be infinite-dimensional. This circumstance made it possible to apply the method of constructing new classes of non-stationary solutions. Symmetry can transform the exact solution of stationary dynamic equations in non-stationary solutions. The framed solutions include arbitrary functions and arbitrary constants. The outline of the article is as follows: according to the method of Lie group of point symmetries allowed by the equations of anisotropic plasticity is calculated. Two classes of new stationary invariant solutions are framed. These stationary solutions, by means of transformations generated by point symmetries, are transformed into new non-stationary solutions. In conclusion, a new self-similar solution of unsteady equations of anisotropic plasticity is framed; Annin's solution is generalized for the anisotropic case. The framed solutions can be used to describe the compression of plastic material between rigid plates, as well as to test programs, designed to solve anisotropic plastic problems.

Keywords: anisotropic plasticity, dynamics, symmetries, exact solutions.

СИСТЕМНЫЙ АНАЛИЗ ДИНАМИЧЕСКИХ ЗАДАЧ АНИЗОТРОПНОЙ ТЕОРИИ ПЛАСТИЧНОСТИ

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Динамические задачи – это наименее изученная область теории пластичности. Динамические задачи возникают в самых разных областях техники и науки, но сложность исходных дифференциальных уравнений не позволяет строить точные решения и корректно численно решать краевые задачи. Это еще в большей степени касается динамических уравнений анизотропной пластичности. Анизотропия уменьшает группу симметрий, допускаемую уравнениями, а, следовательно, и сужает количество инвариантных решений. Неплохо исследованы одномерные динамические задачи пластичности, но уже двумерные задачи вызывают непреодолимые математические сложности из-за нелинейности основных уравнений, даже в изотропном случае.

Изучение симметрий уравнений пластичности позволило построить некоторые точные решения. Б. Д. Аннин построил наиболее известное решение, описывающее сжатие жесткими плитами пластического слоя из изотропного материала. Решение Аннина линейно по двум пространственным переменным, и в него входят произвольные функции времени. В предлагаемой работе также используются симметрии. В статье впервые вычислены точечные симметрии для динамических уравнений пластичности в анизотропном случае. Алгебра Ли, порождаемая найденными симметриями, оказалась бесконечномерной. Это обстоятельство дало возможность применить методику построения новых классов нестационарных решений. Симметрии позволяют преобразовать точные решения стационарных динамических уравнений в нестационарные решения. В построенные решения входят произвольные функции и произвольные постоянные. В статье по методике Ли-Овсяннникова вычисляется группа точечных симметрий, допускаемая уравнениями анизотропной пластичности. Строятся два класса новых стационарных инвариантных решений. Эти стационарные решения, с помощью преобразований, порождаемых точечными симметриями, преобразуются в новые нестационарные решения. В заключении работы построено новое автомодельное решение нестационарных уравнений анизотропной пластичности, а решение Аннина обобщено на анизотропный случай. Приведенные решения можно использовать для описания сжатия пластического материала между жесткими плитами, а также для тестирования программ, предназначенных для исследования анизотропных пластических задач.

Ключевые слова: анизотропная пластичность, динамика, симметрии, точные решения.

Introduction. The theory of plasticity is covered in numerous studies, which is caused by the importance and relevance of the tasks under consideration. These tasks arise in the design of machines and mechanisms, in the technological processes using plastic deformations, in the process of armor drilling with a missile. Modern and classical articles or monographs [1–5] deal mainly with static problems and isotropic materials, the reason of which is not the lack of dynamic tasks importance to applications, but because there are no methods developed to solve dynamic problems. For the first time the spatial solution of dynamic equations was framed by B. D. Annin [6] in 1978. This solution was linear across two spatial variables and contained several arbitrary time-dependent functions. It was found by B. D. Annin on the basis of point symmetry group research, allowed by the equation system of dynamic plasticity theory. Later, based on the equation group properties, exact solutions of some flat dynamic problems were framed [6]. Since then, prior to the work [7], there seems to be no new solutions. Here the authors refer to the unique overview [3], where systems of nonlinear equations of solid media mechanics and their exact solutions are collected. The study of the symmetry group showed that the new method can be applied to anisotropic dynamic plasticity equations – to convert stable solutions of ideal plasticity into unstable ones, as it was done in [7].

Group properties can be applied for different purposes. They are most often used to frame invariant solutions. These are solutions that do not change under continuous transformations allowed, according to Lie, by the given system of differential equations. The invariant solutions of plasticity equations and their framing methods, for example, can be found in more detail in [6] and in the literature references. The works of authors [8–15] show how it is possible to “deform” exact solutions and to reduce one exact solution into another exact solution in the case of flat stationary equations of ideal plasticity by means of point symmetry. In the present paper we use a group of point symmetry to transform new stable solutions into new unstable ones for three-dimensional non-stationary plasticity equations, which was first done to solve plasticity equations in the isotropic case in [7].

Problem setting. Assume that $x = x_1$, $y = x_2$, $z = x_3$ – orthogonal Cartesian coordinate system, $u = v_1$, $v = v_2$, $w = v_3$ – components of the strain velocity vector, e_{ij} – strain rate tensor components, σ_{ij} – stress tensor components. Stress tensor components and velocity vector components satisfy motion equations

$$\partial_t v_i + v_j \partial_j v_i = \partial_i \sigma_{ij}, \; i, j = 1,\; 2,\; 3 . \qquad (1)$$

Duplicate indexes are expected to sum. Stress strain tensor and pine strain rate tensor

$$\sigma_{ij} - \delta_{ij} p = \lambda e_{ij} = \lambda(\partial_j v_i + \partial_i v_j)/2 , \qquad (2)$$

where δ_{ij} – Kronecker's symbol, λ – some non-negative function, $3p = \sigma_{ii}$.

The medium is assumed to be uncompressible, so there is an incompressibility equation

$$\partial_i v_i = 0 . \qquad (3)$$

The equation system (1)–(3) is completed by the Mises plasticity condition

$$a_{11}\left(\sigma_{11} - p\right)^2 + a_{22}\left(\sigma_{22} - p\right)^2 + a_{33}\left(\sigma_{33} - p\right)^2 + \\ + \; 2\left(a_{12}\sigma_{12}^2 + a_{13}\sigma_{13}^2 + a_{23}\sigma_{23}^2\right) = 1, \qquad (4)$$

where a_{ij} – current anisotropy parameters.

1. Group properties of dynamic plasticity theory equations. Let's calculate the group of point symmetries allowed by equations (1)–(4). We will do this according to the known method of Lie-Ovsyannikov, which has been already applied to equations of plasticity in [6]. The point symmetry group is generated by the following operators:

$$X_0 = \partial_t, M = t\partial_t + x_i\partial_{x_i}, \; S = \varphi(t)\partial_p, \\ T_i = f_i(t)\partial_{x_i} - \dot{f}_i(t)\partial_{v_i} - x_i \ddot{f}_i(t)\partial_p, \qquad (5)$$

on not to add together i.

Functions $\varphi(t), f_i(t)$ are arbitrary from the class C^∞, thus operators (5) give rise to an infinitesimal Lie algebra. The point at the top indicates a derivative of the variable *t*.

A remarkable property of point symmetries is that they transfer the solution of the system (1)–(4) back into the exact solutions of the same system. Let the original coordinates be t, x_i, v_i, p then they are converted to new coordinates t', x_i', v_i', p' using transformations corresponding to operators (5):

$$t' = t(a_0 + expa_1), x' = \left(expa_1 + a_{i+2} f_i\left(t\right)\right),$$
$$v_i' = v_i + a_{i+2} \dot{f}_i\left(t\right), p' = p + \sum_{i=1}^{3} \ddot{f}_i(t) a_{i+2}. \quad (6)$$

Here a_i – are group parameters that continuously change in some proximity of zero. Then we use these transformations to frame new solutions of the system (1)–(4).

2. Stable system solutions (1)–(4). Since the system (1)–(4) accept the operator $X_0 = \partial_t$, invariant system solutions can be searched, which are independent of variable *t*. These solutions are determined from the system

$$v_j \partial_j v_i = \partial_i \sigma_{ij},$$
$$\sigma_{ij} - \delta_{ij} p = \lambda e_{ij} = \lambda(\partial_j v_i + \partial_i v_j)/2,$$
$$\partial_i v_i = 0, a_{11}\left(\sigma_{11} - p\right)^2 + a_{22}\left(\sigma_{22} - p\right)^2 + \quad (7)$$
$$+\ a_{33}\left(\sigma_{33} - p\right)^2 + 2\left(a_{12}\sigma_{12}^2 + a_{13}\sigma_{13}^2 + a_{23}\sigma_{23}^2\right) = 1.$$

The system (7) is simpler than the initial equation system as it possesses less independent variables. However, the solution, except the trivial ones, is not available to the knowledge of the authors [1–6].

A) Let us search for the invariant system solution (7) with reference to two-dimensional subalgebra generated by operators $\partial_y + A\partial_p, \partial_z + B\partial_p$ This solution will have the following appearance

$$u = u\left(x\right), v = v\left(x\right), w = w\left(x\right), p = Ay + Bz + p\left(x\right). \quad (8)$$

Here *A*, *B* arbitrary constants, functions *u*, *v*, *w*, *p* are determined from the system (7). Substituting ratios (7) into (8), we obtain

$$u = \text{const},\ \partial_x \sigma_{11} = 0,$$
$$u\partial_x v = \partial_x \sigma_{12} + A,\ u\partial_x w = \partial_x \sigma_{13} + B. \quad (9)$$

From (9) we obtain A system of ordinary differential equations to determine function *v*, *w*

$$ud_x v = d_x \frac{2d_x v}{\sqrt{a_{12}(d_x v)^2 + a_{13}(d_x w)^2}} + A,$$
$$ud_x w = d_x \frac{2d_x w}{\sqrt{a_{12}(d_x v)^2 + a_{13}(d_x w)^2}} + B, \quad (10)$$
$$\sigma_{11} = \sigma_{22} = \sigma_{33} = p = \text{const}.$$

Let us introduce the new required functions according to the formulas

$$\sqrt{a_{12} d_x v} = hsin\varphi,\ \sqrt{a_{13} d_x w} = h\cos. \quad (11)$$

Then, the system (10) will be written as

$$uh\sin\varphi = \varphi'\cos\varphi + A,\ uh\cos\varphi = -\varphi'\sin\varphi + B. \quad (12)$$

From (12) we obtain

$$\frac{d\varphi}{-A\cos\varphi + B\sin\varphi} = dx.$$

While integrating the equation we have

$$\frac{1}{\sqrt{A^2 + B^2}} \ln\left|\text{tg}\frac{\varphi + \theta}{2}\right| = x + C,$$

where $\sin\theta = -\dfrac{A}{\sqrt{A^2 + B^2}}$.

Hence follows

$$\varphi = -\theta + 2\text{arctg}\left(\exp\left(x + C\right)\sqrt{A^2 + B^2}\right),$$

$$h = \begin{cases} \dfrac{1}{u}\left(-2\dfrac{\exp\left(x + C\right)\sqrt{A^2 + B^2}}{1 + \exp^2\left(x + C\right)\sqrt{A^2 + B^2}}\right)\text{tg}\varphi + \\ +\ \dfrac{B}{u\cos\varphi},\ \text{если}\cos\varphi \neq 0, \\ \dfrac{1}{u}\left(2\dfrac{\exp\left(x + C\right)\sqrt{A^2 + B^2}}{1 + \exp^2\left(x + C\right)\sqrt{A^2 + B^2}}\right)\text{ctg}\varphi + \\ +\ \dfrac{A}{u\sin\varphi},\ \text{если}\cos\varphi = 0, \end{cases} \quad (13)$$

$$\sigma_{12} = \sin\varphi / \sqrt{a_{12}},\ \sigma_{13} = \cos\varphi / \sqrt{a_{13}}.$$

We study functions behavior included in formulas (13).

Assuming that x varies from $-\infty$ to $+\infty$ then $\exp(x + C)\sqrt{A^2 + B^2}$ from zero to $+\infty$, $2\text{arctg}\left(\exp(x + C)\sqrt{A^2 + B^2}\right)$ monotonously increases from zero to π. Here in φ varies from $-\theta$ to $-\theta + \pi$. Hereafter based on formulas (13) it becomes clear how the tensor tension components change.

This small study makes it possible to interpret the resulting solution as follows. There are two rigid rough plates $x = x_1 = \text{const}, x = x_2 = \text{const}$. Plastic material is pressed between them. Tangential stresses σ_{12}, σ_{13}.

B) We will search for invariant system solution (7) with reference to single-dimensional subalgebra, created by operators $\frac{1}{\alpha}\partial_x + \frac{1}{\beta}\partial_y - \frac{2}{\gamma}\partial_z$. The solution will have the following appearance

$$u = Ag\left(\alpha x + \beta y + \gamma z\right), v = Bg\left(\alpha x + \beta y + \gamma z\right),$$
$$w = Cg\left(\alpha x + \beta y + \gamma z\right), p = F\left(\alpha x + \beta y + \gamma z\right) \quad (14)$$

Here $A, B, C, \alpha, \beta, \gamma$ are arbitrary parameters, function g, F are calculated from (6). Let's substitute the ratios (14) in (7). Obtained are the following ratios between functions and constant

$$\alpha A + \beta B + \gamma C = 0,\ p = \text{const},$$
$$g \text{ – arbitrary smooth function.} \quad (15)$$

From (14) and (15) it follows that all the components of the tension tensor are constant and have the following appearance:

$$\sigma_{11} = p + \frac{\alpha A}{D},\ \sigma_{22} = p + \frac{\beta B}{D},\ \sigma_{33} = p + \frac{\gamma C}{D},$$

$$\sigma_{12} = \frac{\beta A + \alpha B}{2D},\ \sigma_{13} = \frac{\gamma A + \alpha C}{2D},\ \sigma_{23} = \frac{\gamma B + \beta C}{2D},$$

$$D^2 = (a_{11}(\alpha A)^2 + a_{22}(\beta B)^2 + a_{33}(\gamma C)^2 +$$

$$+ a_{12}/2(\beta A + \alpha B)^2 + a_{13}/2(\gamma A + \alpha C)^2 + \frac{a_{23}}{2}\left(\gamma B + \beta C\right)^2\Big).$$

A similar solution in the absence of convective members was framed in [15].

3. Deformation of system non-stationary solution (1)–(4).

A) Let's consider the stationary solution (8)–(13), framed in the previous paragraph, and by means of transformations (6) deform it into non-stationary solutions of the original system (1)–(4).

We have

$$\varphi = -\theta + 2\mathrm{arctg}\left(\exp\left(x + a_3 f_1(t) + C\right)\sqrt{A^2 + B^2}\right). \quad (16)$$

In this case, the tangential stresses σ_{12}, σ_{13} are no longer constant on the plates, as it was the case in the previous paragraph, but vary depending on the selected function included in the function (16). This solution can be interpreted by the impact of vibration loads on the plates $x = x_1 = \mathrm{const},\ x = x_2 = \mathrm{const}$. At the same time the plates themselves also change their shape $x + a_3 f_1(t) = x_1,\ x + a_1 f_1(t) = x_2$. Here, if a_3 – a group parameter is the one which can be fixed $a_3 = 0$ then we get the initial stationary solution.

B) Let's consider the second stationary solution (14) framed in paragraph 2 B). By means of transformations (5), similar to the previous solution, we deform it into non-stationary solutions of the original system (1)–(4).

To do this, we apply a remarkable property of point symmetry of their ability to convert the system solution (1)–(4) back into the exact solutions of the same system.

The system (1)–(4) allows operators

$$S = \varphi(t)\partial_p,$$

$$T_i = f_i(t)\partial_{x_i} + \dot f_i(t)\partial_{v_i} - x_i \ddot f_i(t)\frac{\partial}{\partial p},\ i = 1,\ 2,\ 3.$$

It means that it does not change under transformation

$$x_i' = x_i + a_{i+2} f_i,\ v'_i = v_i + a_{i+2}\dot f_i(t),$$

$$p' = p - \sum_{i=1}^{3} a_{i+2} x_i \dot f_i(t) + a_2\varphi(t). \quad (17)$$

Here, variables without a prime are original, and variables with a prime are derived from point transformations corresponding to the subalgebra generated by the operators S, T_i with a_i – group parameters that continuously change in some proximity of zero.

Assuming that v_i^1, p^1 – is some system solution (1)–(4) then according to (17) v_i^2, p^2 a new solution for the same system

$$v_1^2 = v_1^1\left(t, x_1 + a_3 f_1(t), x_2 + a_4 f_2(t), x_3 + a_5 f_3(t)\right) + a_3 \dot f_1(t),$$

$$v_2^2 = v_2^1\left(t, x_1 + a_3 f_1(t), x_2 + a_4 f_2(t), x_3 + a_5 f_3(t)\right) + a_4 \dot f_2(t),$$

$$v_3^2 = v_3^1\left(t, x_1 + a_3 f_1(t), x_2 + a_4 f_2(t), x_3 + a_5 f_3(t)\right) + a_5 \dot f_3(t),$$

$$p^2 = p^1\left(t, x_1 + a_3 f_1(t), x_2 + a_4 f_2(t), x_3 + a_5 f_3(t)\right) -$$

$$- \sum_{i=1}^{3} x_i a_{i+2} \ddot f_i(t), \quad (18)$$

is also accurate solution for the same system. We use this property to frame new system solutions (1)–(4). We will apply formula (18) to the solution framed in paragraph 2B).

We have

$$u = Ag\left(\alpha(x + f_1(t)) + \beta(y + f_2(t)) + \gamma(z + f_3(t))\right) + \dot f_1(t),$$

$$v = Bg\left(\alpha\left(x + f_1(t)\right) + \beta\left(y + f_2(t)\right) + \gamma\left(z + f_3(t)\right)\right) + \dot f_2(t),$$

$$w = Cg\left(\alpha\left(x + f_1(t)\right) + \beta\left(y + f_2(t)\right) + \gamma\left(z + f_3(t)\right)\right) + \dot f_3(t),$$

$$p = -x\ddot f_1(t) - y\ddot f_2(t) - z\ddot f_3(t) + \varphi(t) \quad (19)$$

As a result, a new unstable velocity field, which corresponds to the following stressed state, has been built

$$\sigma_{11} = p + \frac{\alpha A}{D},\ \sigma_{22} = p + \frac{\beta B}{D},\ \sigma_{33} = p + \frac{\gamma C}{D},$$

$$\sigma_{12} = \frac{\beta A + \alpha B}{2D},\ \sigma_{13} = \frac{\gamma A + \alpha C}{2D},\ \sigma_{23} = \frac{\gamma B + \beta C}{2D},$$

$$D^2 = (a_{11}(\alpha A)^2 + a_{22}(\beta B)^2 + a_{33}(\gamma C)^2 +$$

$$+ a_{12}/2(\beta A + \alpha B)^2 + a_{13}/2(\gamma A + \alpha C)^2 + \frac{a_{23}}{2}\left(\gamma B + \beta C\right)^2\Big),$$

$$p = -x\ddot f_1(t) - y\ddot f_2(t) - z\ddot f_3(t) + \varphi(t).$$

4. New automatic solution of the equation system (1)–(4). Let us frame the invariant solution based on subalgebra $M = t\partial_t + x_i\partial_{x_i}$. It has the following appearance

$$u = u(\xi, \eta, \zeta), v = v(\xi, \eta, \zeta), w = w(\xi, \eta, \zeta),$$

$$p = p(\xi, \eta, \zeta),\ \xi = \frac{x_1}{t},\ \eta = \frac{x_2}{t},\ \zeta = \frac{x_3}{t}. \quad (20)$$

In the literature, such decisions are commonly referred to as auto-model.

We substitute (20) into the equation system (1)–(4) and obtain

$$(u - \xi)\partial_\xi u + (v - \eta)\partial_\eta u + (w - \zeta)\partial_\zeta u = \partial_\xi\sigma_{11} + \partial_\eta\sigma_{12} + \partial_\zeta\sigma_{13},$$

$$(u - \xi)\partial_\xi v + (v - \eta)\partial_\eta v + (w - \zeta)\partial_\zeta v = \partial_\xi\sigma_{12} + \partial_\eta\sigma_{22} + \partial_\zeta\sigma_{23},$$

$$(u - \xi)\partial_\xi w + (v - \eta)\partial_\eta w + (w - \zeta)\partial_\zeta w = \partial_\xi\sigma_{13} + \partial_\eta\sigma_{23} + \partial_\zeta\sigma_{33},$$

$$\partial_\xi u + \partial_\eta v + \partial_\zeta w = 0,$$

$$a_{11}(\sigma_{11} - p)^2 + a_{22}(\sigma_{22} - p)^2 + a_{33}(\sigma_{33} - p)^2 +$$

$$+ 2\left(a_{12}\sigma_{12}^2 + a_{13}\sigma_{13}^2 + a_{23}\sigma_{23}^2\right) = 1. \quad (21)$$

The equation system (21) is more simplistic than the initial equation system since it contains one independent variable less.

We will search for the equation system solution (21) with the following appearance

$$u=\xi,\ v=\eta,\ w=-2\zeta+f\left(\xi,\eta\right),\qquad p=p\left(\xi,\eta,\zeta\right) \tag{22}$$

where $f\left(\xi,\eta\right)$ is differentiable function.

Substituting (22) into (21) we receive

$$\begin{aligned}&\partial_\xi\sigma_{11}=0,\\&\partial_\eta\sigma_{22}=0.2\left(-f+3\zeta\right)=.\\&=\ \partial_\xi\sigma_{13}+\partial_\eta\sigma_{23}+\partial_\zeta p.\end{aligned} \tag{23}$$

Note that the incompressibility equation is satisfied in the identical manner.

From the system (23) we receive

$$\sigma_{11}=\sigma_{22},\ p=3\zeta^2+\sigma_{11},\qquad -2f=\partial_\xi\sigma_{13}+\partial_\eta\sigma_{23}.$$

Out of the last system equation (23) follows the equation to determine function f

$$-2f=\partial_\xi\frac{f_\xi}{\sqrt{a_{11}+a_{22}-2a_{33}+a_{12}f_\xi^2+a_{13}f_\eta^2}}+\ \partial_\eta\frac{f_\eta}{\sqrt{a_{11}+a_{22}-2a_{33}+a_{12}f_\xi^2+a_{13}f_\eta^2}} \tag{24}$$

Equation (24) is found in the study of equilibrium surfaces in the hydromechanics of weightlessness [16]. It is also found in the theory of plasticity [17] when describing slow non-stationary currents in the cylindrical channel, generating of which are parallel to the z axis.

In general, it is difficult to solve equation (24), so we will consider the particular case where $f=f\left(\xi\right)$. In this case we get an ordinary second-order differential equation that allows order downgrade and is reduced to a first-order equation of the class

$$f'_\xi=\pm\frac{\sqrt{1+(2a_{33}-a_{11}-a_{22})a_{12}^2(f^2+C)^2}}{a_{12}^{\frac{3}{2}}(f^2+C)}, \tag{25}$$

where C – arbitrary variable. Equation (25) is found via quadratures; solutions are written as elliptical integrals of the first and second classes.

For convenience of the framed solution interpretation we will transform variables as follows. Let's enter new independent variables based on formula $x'=-mx,$ $y'=-my,$ $z'=mz,$ $t'=h-mt$. Here m,h – are positive constants. This transformation can be done because the source system allows stretch and transfer operators. In this case the solution is framed as

$$u=\frac{-mx}{h-mt},\ v=\frac{-my}{h-mt},\ w=\frac{2mz}{h-mt}+f\left(\frac{-mx}{h-mt}\right).$$

This solution can be interpreted as a plastic flow of a layer along the oz axis that is compressed by rigid and rough plates in the x and y directions with plates approaching at a constant speed m. Then, $2H=h-mt$, the thickness of the layer at the time t.

5. Generalization of Annin 's solution to the anisotropic case. In this paragraph we will summarize B. D. Annin's solution [3] to the anisotropic case. To do this, let us find an invariant solution based on a two-dimensional sub-algebra generated by operators

$$\left\langle T_1,T_2\right\rangle=\left\langle\begin{array}{l}f_1(t)\partial_x+\dot f_1\left(t\right)\partial_u-x\ddot f_1\left(t\right)\dfrac{\partial}{\partial p},\\ f_2(t)\partial_y+\dot f_2\left(t\right)\partial_v-y\ddot f_2\left(t\right)\dfrac{\partial}{\partial p}\end{array}\right\rangle.$$

It should be looked for as

$$\begin{aligned}&u=\frac{\dot f_1}{f_1}x+B^1\left(t,z\right)=A^1x+B^1,\\&v=\frac{\dot f_2}{f_2}y+B^2\left(t,z\right)=A^2x+B^2,\\&w=A^3\left(t\right)x+B^3\left(t\right),\end{aligned} \tag{26}$$

$$p=-\frac{x^2\ddot f_1}{2f_1}-\frac{y^2\ddot f_2}{2f_2}+B^3\left(t,z\right)=-\frac{x^2}{2}\left(\dot A^1+(A^1)^2\right)-\ -\ \frac{y^2}{2}\left(\dot A^2+(A^2)^2\right)+B^4\left(t,z\right).$$

Here A^i,B^i – functions are calculated from (1)–(4). Adding (26) to equation system (1)–(4) we obtain

$$A^3=-A^1-A^2,$$

$$p=-\frac{x^2}{2}\left(\dot A^1+(A^1)^2\right)-\frac{y^2}{2}\left(\dot A^2+(A^2)^2\right)-\ -\ \frac{z^2}{2}\left(\dot A^3+(A^3)^2\right)+\left(t\right)x+b_2\left(t\right)y+\lambda A^3,$$

$$S_{11}=\lambda A^1,\ S_{22}=\lambda A^2,\ S_{33}=\lambda A^3,\ S_{12}=0,$$

$$S_{13}=\lambda\partial_zB^1,\ S_{23}=\lambda\partial_zB^2,$$

$$\lambda^{-2}=a_{11}(A^1)^2+a_{22}(A^2)^2+a_{33}(A^3)^2+\ +\ 2(a_{12}(\partial_zB^1)^2+a_{13}\left(\partial_zB^2)^2\right).$$

Here b_i – arbitrary function of t. Functions A^i,B^i are calculated from

$$\dot B^1-b_1+A^1B^1+\left(A^3z+B^3\right)\partial_zB^1=\partial_z(\lambda B^1),$$

$$\dot B^2-b_2+A^2B^2+\left(A^3z+B^3\right)\partial_zB^2=\partial_z(\lambda B^2).$$

This solution can be applied when analyzing a plastic flow of a parallelepiped made of anisotropic material compressed between rigid plates.

Conclusion. The paper demonstrates how with the help of symmetries it becomes possible to turn a stationary solution of plasticity equations into a whole class of non-stationary solutions of dynamic plasticity equations. These solutions might be used to analyze dynamic technological processes. In addition, new solutions to dynamic plasticity equations have been framed by the standard methods of group analysis.

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