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Метод сумм Хаара численного решения системы кинематических уравнений Пуассона, определяющих эволюцию положения космического аппарата

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В представленной работе предложен метод численного решения системы кинематических уравнений Пуассона, определяющих эволюцию положения космического аппарата (КА), по которой определяют матрицу перехода от связанной с КА системы координат в выбранный момент времени t_1 к связанной с КА системе координат в текущий момент времени t_2. Указанная матрица перехода используется в ходе решения задачи определения трехосной ориентации КА по показаниям магнитометра с использованием информации о его угловых скоростях. Предложенный метод основан на замене производных искомых функций в кинематических уравнениях Пуассона на частичные суммы рядов по масштабированной системе Хаара. Эти суммы представляют собой обобщенные многочлены по масштабированной системе Хаара и, следовательно, являются ступенчатыми (кусочно-постоянными) функциями. Выведены оценки погрешности предложенного метода, показывающие, что в случае коэффициентов уравнений, представляющих собой функции, удовлетворяющие условию Липшица, абсолютная погрешность вычисления каждого из элементов матрицы перехода от одной системы координат к другой есть величина $O(N^{-1})$ при $N \to \infty$, где N – число разбиений отрезка $[t_1, t_2]$ при построении сетки узлов, задействованных в данном методе. Доказано, что трудоемкость построенного алгоритма приближенного решения системы кинематических уравнений Пуассона незначительно превышает трудоемкость решения указанной системы методом Эйлера, который имеет первый порядок точности. Приведены результаты численных экспериментов, показывающие, что в определенных случаях метод сумм Хаара дает погрешность, значительно меньшую, чем метод Эйлера, и практически идентичную погрешностям методов Эйлера – Коши и Рунге – Кутты 2-го порядка, трудоемкость которых примерно в два раза превосходит трудоемкость метода сумм Хаара.

Ключевые слова: трехосная ориентация космического аппарата, система координат, связанная с космическим аппаратом, система кинематических уравнений Пуассона, система функций Хаара.

The method of Haar sums for numerical solution of Poisson kinematic equations system determining an evolution of a spacecraft position

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The paper proposes the method for the numerical solution of Poisson kinematic equations system determining the evolution of the spacecraft position. The system of Poisson kinematic equations is used to designate the transition matrix from the coordinate system associated with the spacecraft at the selected time t_1 to the coordinate system associated with the spacecraft at the current time t_2. This matrix is used in the process of solving problems of determining a three-axis orientation of the spacecraft from the readings of the magnetometer using information about its angular velocities. The proposed method is based on replacing the derivatives of the desired functions in the Poisson kinematic equations by partial sums of series in the scaled Haar system. The partial sums of these series are generalized polynomials in the scaled Haar system. Hence, these sums are step (piecewise constant) functions. The estimates of the proposed method error are derived, which reveal that in the case of the coefficients of the equations which are functions matching the Lipschitz condition, the absolute error in calculating each of the elements of the transition matrix from one coordinate system to another is the value $O(N^{-1})$ at $N \to \infty$, where N is the number of partitions of the segment $[t_1, t_2]$ when constructing a grid of nodes involved in this method. It is proved that the complexity of constructing an algorithm for approximating the system of Poisson kinematic properties insignificantly exceeds the complexity of solving this system by Euler method, which has the first order of accuracy. The research presents the results of numerical experiments, showing that in certain cases the Haar sums method gives an error that is much smaller than the Euler method, and is almost identical to the errors of the Euler – Cauchy and Runge – Kutta methods of the 2nd order, the complexity of which is approximately two times greater than the complexity of the Haar sums method.

Keywords: spacecraft three-axis orientation, the coordinate system associated with the spacecraft, system of Poisson kinematic equations, system of Haar functions.

Introduction

[1] proposes a method to determining the triaxial orientation of a spacecraft (SC) based on magnetometer readings using information about its angular velocities. Solving this problem considers two measurements of the vector of the Earth's magnetic field strength (EMF) and the angular velocity of the spacecraft, made at a selected point in time t_1, as well as at the time point t_2, corresponding to the maximum value of the acute angle between these measurements of the EMF intensity vector. Further, taking into account the measured values of the spacecraft angular velocity at the specified times t_1 and t_2, the system of Poisson kinematic equations is integrated [1–6]

$$\begin{cases} d_{11}'(t) = \omega_3(t)d_{21}(t) - \omega_2(t)d_{31}(t), \\ d_{21}'(t) = \omega_1(t)d_{31}(t) - \omega_3(t)d_{11}(t), \\ d_{31}'(t) = \omega_2(t)d_{11}(t) - \omega_1(t)d_{21}(t), \\ d_{12}'(t) = \omega_3(t)d_{22}(t) - \omega_2(t)d_{32}(t), \\ d_{22}'(t) = \omega_1(t)d_{32}(t) - \omega_3(t)d_{12}(t), \\ d_{32}'(t) = \omega_2(t)d_{12}(t) - \omega_1(t)d_{22}(t), \\ d_{13}'(t) = \omega_3(t)d_{23}(t) - \omega_2(t)d_{33}(t), \\ d_{23}'(t) = \omega_1(t)d_{33}(t) - \omega_3(t)d_{13}(t), \\ d_{33}'(t) = \omega_2(t)d_{13}(t) - \omega_1(t)d_{23}(t), \end{cases} \quad (1)$$

designating the rotation matrix D_{12} of the coordinate system associated with the spacecraft relative to the inertial coordinate system from time t_1 to time t_2. System (1) via $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$ denotes the projections of the absolute angular velocity of the spacecraft onto the coordinate axes of abscissa, ordinate and applicate, respectively (due to information from the angular velocity meter), $d_{ij}(t)$ – matrix elements D_{12}, $d_{ij}'(t)$ – their derivatives, $i, j = 1, 2, 3$. Initial matrix value D_{12} (at the time point t_1) is taken to be equal to the identity matrix E.

Methods to solve the system of Poisson kinematic equations (1) were considered, for example, in [7–11]. The current work proposes a method for numerically solving system (1), based on replacing the derivatives of the sought functions in these equations with generalized polynomials in the scaled Haar system. Error estimates are derived for the proposed method

$$\left| d_{ij}(t_2) - d_{ij}^{(n)}(t_2) \right| \le \Sigma_j(n) \quad (i, j = 1, 2, 3),$$

where

$$\Sigma_j(n) \sim T\left[2^{n-1}\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)} - 1\right) + \Omega_S - \omega\left(\omega_j, 2^{-n}\right) \right]$$

in case $n \to \infty$, $j = 1, 2, 3$. Here $d_{ij}^{(n)}(t_2)$ – approximate values if $t = t_2$ of matrix elements D_{12}, obtained as a result of solving the system of equations (1) using this method $(i, j = 1, 2, 3)$, $N = 2^n$ – number of segment partitions $[t_1, t_2]$ when constructing a grid of nodes involved in the proposed method, $\omega_j(t)$ – continuous on a segment $[t_1, t_2]$ of the function $(j = 1, 2, 3)$, $T = t_2 - t_1$, $\Omega = \max\{\Omega_1, \Omega_2, \Omega_3\}$, where

$$\Omega_j = \max_{t \in [t_1, t_2]} \left| \omega_j(t) \right| \quad (j = 1, 2, 3),$$

where $\omega(f, \delta)$ – function continuity modulus f, that is

$$\omega(f, \delta) = \sup_{|t' - t''| \le \delta} \left| f(t') - f(t'') \right|,$$

and value Ω_S is defined by equality

$$\Omega_S = \sum_{j=1}^{3} \omega\left(\omega_j, 2^{-n}\right).$$

If the function $\omega_j(t)$ satisfies the Lipschitz condition with constants $L_j \geq 0,$ then

$$\Sigma_j(n) \sim NT\left[2\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)}-1\right)+L-L_j\right]$$

if $N \to \infty,$ j = 1, 2, 3, $L = L_1 + L_2 + L_3$, it follows that in this case the absolute error in calculating each element of the transition matrix D_{12} while transiting from one coordinate system to another there is the value $O(N^{-1})$ at $N \to \infty$.

System of equations (1) with initial conditions resulting from equality $D_{12} = E,$ is divided into three Cauchy problems. It has been proven that the numerical solution of each of these three Cauchy problems requires $\Lambda_X(N) \sim 17N$ (with $N \to \infty$) arithmetic operations, which slightly exceeds the complexity of solving each of these Cauchy problems using the Euler method.

The results of numerical experiments are presented, showing that in certain cases the Haar sums method gives an error that is significantly smaller than the Euler method, and almost identical to the errors of the Euler-Cauchy and Runge-Kutta methods of the 2nd order, the complexity of which is approximately twice that of the Haar sums method.

1. The problem statement. Basic definitions

To determine the orientation of a spacecraft, it is necessary to take into account its angular motion in the inertial coordinate system. To do this, in the time interval from t_1 to t_2 (t_1 and t_2 correspond to two positions of the spacecraft in orbit) the system of Poisson equations (1) is approximately solved, which determines the evolution of the spacecraft position from t_1 moment to moment t_2. It is obvious that the system of equations (1), taking into account the initial value of the matrix D_{12} ($D_{12} = E$) reduces to the following three Cauchy problems for the systems of equations:

$$\begin{cases} d_{1j}{}'(t) = \omega_3(t)d_{2j}(t) - \omega_2(t)d_{3j}(t), \\ d_{2j}{}'(t) = \omega_1(t)d_{3j}(t) - \omega_3(t)d_{1j}(t), \\ d_{3j}{}'(t) = \omega_2(t)d_{1j}(t) - \omega_1(t)d_{2j}(t); \end{cases} \tag{2}$$

$$d_{ij}(t_1) = \delta_{ij} = \begin{cases} 1 & \text{при } i = j, \\ 0 & \text{при } i \neq j, \end{cases} \quad i = 1,2,3; \tag{3}$$

j = 1, 2, 3. We assume that $\omega_1(t),$ $\omega_2(t),$ $\omega_3(t)$ are continuous at the segment $[t_1, t_2]$ of the function.

We could construct an algorithm to solve the Cauchy problem (2)–(3) and derive error estimates using methods similar to those given in [12] in the case of solving the Cauchy problem for a first-order linear differential equation.

We present the definition of a system of Haar functions and the relating concept of binary intervals, introduced in [13].

A binary interval $l_{m,j}$ is the interval with ends at points $(j-1)/2^{m-1}$, $j/2^{m-1}$ (m = 1,2, ..., j = 1,2, ..., 2^{m-1}). We consider the binary intervals to be closed on the left and open on the right. If the right end of a binary interval coincides with 1, then we will consider this interval to be closed on the right as well. Here are the labeling:

$$l_{m,j}^{-} = l_{m+1,2j-1}, \quad l_{m,j}^{+} = l_{m+1,2j}.$$

It is obvious, that

$$l_{m,j}^{-} \cup l_{m,j}^{+} = l_{m,j}.$$

The system of Haar functions is constructed in groups: group number m contains 2^{m-1} of functions $\chi_{m,j}(x)$, where $m = 1,2, ..., j = 1,2, ..., 2^{m-1}$. The Haar function $\chi_{m,j}(x)$ is defined as:

$$\chi_{m,j}(x) = \begin{cases} 2^{(m-1)/2} & \text{with } x \in l_{m,j}^{-}, \\ -2^{(m-1)/2} & \text{with } x \in l_{m,j}^{+}, \\ 0 & \text{with } x \notin l_{m,j}, \end{cases}$$

$m = 1,2,\dots,\ j = 1,2,\dots,2^{m-1}$. Together with double numeration, simple numeration is also used:

$$\chi_{m,j}(x) = \chi_k(x),$$

where $k = 2^{m-1} + j$, $k = 2, 3, \dots$ The system of Haar functions also function $\chi_1(x) \equiv 1$, that is outside the groups.

2. Constructing an algorithm for numerical solution of Cauchy problems (2)–(3)

We introduce the notation $T = t_2 - t_1$. We are searching for an approximate solution

$$\left(d_{1j}^{(n)}(t), d_{2j}^{(n)}(t), d_{3j}^{(n)}(t)\right)$$

to each of the three Cauchy problems (2)–(3), representing the derivatives

$$d_{1j}^{(n)}{}'(t),\ d_{2j}^{(n)}{}'(t),\ d_{3j}^{(n)}{}'(t)$$

in the form of generalized polynomials in the scaled Haar system $\left\{\chi_k\left((t-t_1)/T\right)\right\}$ of orders not higher than 2^n:

$$d_{ij}^{(n)}{}'(t) = \sum_{k=0}^{2^n-1} C_{ij}^{(n,k)} \chi_k((t-t_1)/T),\ C_{ij}^{(n,k)} \in \mathbf{R},$$

$k = 0, 1, \dots, 2^n - 1$, $i = 1, 2, 3$, $j = 1, 2, 3$. Such generalized polynomials are step functions:

$$d_{ij}^{(n)}{}'(t) = d_{ij}^{(n,k)} \text{ if } t_1 + 2^{-n}Tk \le t \le t_1 + 2^{-n}T(k+1), \tag{4}$$

$k = 0, 1, \dots, 2^n - 1$, $i = 1, 2, 3$, $j = 1, 2, 3$.

We can restore the functions $d_{ij}^{(n)}(t)$ on their derivatives:

$$d_{ij}^{(n)}(t) = \delta_{ij} + 2^{-n}T\sum_{l=0}^{k-1} d_{ij}^{(n,l)} + \left(t - t_1 - 2^{-n}Tk\right)d_{ij}^{(n,k)} \text{ with } t_1 + 2^{-n}Tk \le t \le t_1 + 2^{-n}T(k+1), \tag{5}$$

$k = 0, 1, \dots, 2^n - 1$, $i = 1, 2, 3$, $j = 1, 2, 3$. We consider in (5) at $k = 0$

$$\sum_{l=0}^{k-1} d_{ij}^{(n,l)} = 0.$$

Functions (5) are piecewise-linear with nodes at points of the set

$$\left\{t_{n,k} : t_{n,k} = t_1 + 2^{-n}Tk, k = 0,1,\dots,2^n - 1\right\}. \tag{6}$$

We assume that the angular velocity meter determines the values of the projections of the absolute angular velocities $\omega_1(t)$, $\omega_2(t)$, $\omega_2(t)$, precisely at the points of set (6). We require that functions (5) satisfy systems of equations (2) on this set. Then we obtain:

$$\begin{cases} d_{1j}^{(n)\prime}\left(t_{n,k}\right) = \omega_3\left(t_{n,k}\right) d_{2j}^{(n)}\left(t_{n,k}\right) - \omega_2\left(t_{n,k}\right) d_{3j}^{(n)}\left(t_{n,k}\right), \\ d_{2j}^{(n)\prime}\left(t_{n,k}\right) = \omega_1\left(t_{n,k}\right) d_{3j}^{(n)}\left(t_{n,k}\right) - \omega_3\left(t_{n,k}\right) d_{1j}^{(n)}\left(t_{n,k}\right), \\ d_{3j}^{(n)\prime}\left(t_{n,k}\right) = \omega_2\left(t_{n,k}\right) d_{1j}^{(n)}\left(t_{n,k}\right) - \omega_1\left(t_{n,k}\right) d_{2j}^{(n)}\left(t_{n,k}\right), \end{cases}$$

$k = 0, 1, \ldots, 2^n - 1$, $j = 1, 2, 3$. Taking into account functions (5) and (4), defining them for brevity as,

$$\omega_j\left(t_{n,k}\right) = \omega_j^{(n,k)} \quad (j = 1, 2, 3),$$

we could get:

$$\begin{cases} d_{1j}^{(n,k)} = \omega_3^{(n,k)}\left(\delta_{2j} + 2^{-n} T \sum_{l=0}^{k-1} d_{2j}^{(n,l)}\right) - \omega_2^{(n,k)}\left(\delta_{3j} + 2^{-n} T \sum_{l=0}^{k-1} d_{3j}^{(n,l)}\right), \\ d_{2j}^{(n,k)} = \omega_1^{(n,k)}\left(\delta_{3j} + 2^{-n} T \sum_{l=0}^{k-1} d_{3j}^{(n,l)}\right) - \omega_3^{(n,k)}\left(\delta_{1j} + 2^{-n} T \sum_{l=0}^{k-1} d_{1j}^{(n,l)}\right), \\ d_{3j}^{(n,k)} = \omega_2^{(n,k)}\left(\delta_{1j} + 2^{-n} T \sum_{l=0}^{k-1} d_{1j}^{(n,l)}\right) - \omega_1^{(n,k)}\left(\delta_{2j} + 2^{-n} T \sum_{l=0}^{k-1} d_{2j}^{(n,l)}\right), \end{cases} \tag{7}$$

$k = 0, 1, \ldots, 2^n - 1$, $j = 1, 2, 3$.

Therefore, we obtain the following algorithm for the numerical solution of Cauchy problems (2)–(3).

1. For each j = 1, 2, 3, we perform the calculations.

1.1. We find the values

$$s_{1j}^{(n,0)},\ s_{2j}^{(n,0)},\ s_{3j}^{(n,0)}$$

due to the formulae derived from (7) with $k = 0$:

$$\begin{cases} s_{1j}^{(n,0)} = \delta_{2j}\omega_3^{(n,0)} - \delta_{3j}\omega_2^{(n,0)}, \\ s_{2j}^{(n,0)} = \delta_{3j}\omega_1^{(n,0)} - \delta_{1j}\omega_3^{(n,0)}, \\ s_{3j}^{(n,0)} = \delta_{1j}\omega_2^{(n,0)} - \delta_{2j}\omega_1^{(n,0)} \end{cases}$$

(for k = 0, the sums in (7) are considered equal to zero).

1.2. We consistently find the values of the quantities

$$\begin{gathered} s_{1j}^{(n,1)},\ s_{2j}^{(n,1)},\ s_{3j}^{(n,1)}; \\ s_{1j}^{(n,2)},\ s_{2j}^{(n,2)},\ s_{3j}^{(n,2)}; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{1j}^{\left(n,2^n-1\right)},\ s_{2j}^{\left(n,2^n-1\right)},\ s_{3j}^{\left(n,2^n-1\right)}, \end{gathered}$$

according to the formulae

$$s_{1j}^{(n,k)} = s_{1j}^{(n,k-1)} + d_{1j}^{(n,k)},\ s_{2j}^{(n,k)} = s_{2j}^{(n,k-1)} + d_{2j}^{(n,k)},\ s_{3j}^{(n,k)} = s_{3j}^{(n,k-1)} + d_{3j}^{(n,k)}, \tag{8}$$

pre-calculating for each $k = 1, 2, \ldots, 2^n - 1$ values

$$\begin{cases} d_{1j}^{(n,k)} = \omega_3^{(n,k)} \cdot \left(\delta_{2j} + \tau \cdot s_{2j}^{(n,k-1)}\right) - \omega_2^{(n,k)} \cdot \left(\delta_{3j} + \tau \cdot s_{3j}^{(n,k-1)}\right), \\ d_{2j}^{(n,k)} = \omega_1^{(n,k)} \cdot \left(\delta_{3j} + \tau \cdot s_{3j}^{(n,k-1)}\right) - \omega_3^{(n,k)} \cdot \left(\delta_{1j} + \tau \cdot s_{1j}^{(n,k-1)}\right), \\ d_{3j}^{(n,k)} = \omega_2^{(n,k)} \cdot \left(\delta_{1j} + \tau \cdot s_{1j}^{(n,k-1)}\right) - \omega_1^{(n,k)} \cdot \left(\delta_{2j} + \tau \cdot s_{2j}^{(n,k-1)}\right), \end{cases} \tag{9}$$

where $\tau = 2^{-n}T$. Formulae (8), (9) follow from recurrence relations (7).

2. Using the formulae following from (5), for j = 1, 2, 3, we calculate the values

$$d_{1j}^{(n)}\left(t_2\right)=\delta_{1j}+\tau\cdot s_{1j}^{(n,2^n-1)},\; d_{2j}^{(n)}\left(t_2\right)=\delta_{2j}+\tau\cdot s_{2j}^{(n,2^n-1)},\; d_{3j}^{(n)}\left(t_2\right)=\delta_{3j}+\tau\cdot s_{3j}^{(n,2^n-1)}.$$

3. We compose a transition matrix from the coordinate system associated with the spacecraft at time t_1 to the coordinate system associated with the spacecraft at time t_2:

$$D_{12}=\begin{pmatrix} d_{11}^{(n)}\left(t_2\right) & d_{12}^{(n)}\left(t_2\right) & d_{13}^{(n)}\left(t_2\right) \\ d_{21}^{(n)}\left(t_2\right) & d_{22}^{(n)}\left(t_2\right) & d_{23}^{(n)}\left(t_2\right) \\ d_{31}^{(n)}\left(t_2\right) & d_{32}^{(n)}\left(t_2\right) & d_{33}^{(n)}\left(t_2\right) \end{pmatrix}.$$

We estimate the number of arithmetic operations required to implement this algorithm.

At the k-th step (k = 1, 2, …, 2^n) of the process of calculating the quantities

$$d_{1j}^{(n,k)},\; d_{2j}^{(n,k)},\; d_{3j}^{(n,k)} \;\; (j\in\{1,2,3\} \text{ hold fixed})$$

there are 17 arithmetic operations performed: 3 arithmetic operations are required to find intermediate values (8), 13 arithmetic operations are required to find $d_{1j}^{(n,k)}$, $d_{2j}^{(n,k)}$, $d_{3j}^{(n,k)}$ based on formulae (9) (among three quantities δ_{1j}, δ_{2j}, δ_{3j}, only one is equal to 1, two rest are equal to 0) and only 1 operation is to move to the next grid node ($t_{n,k+1} = t_{n,k} + \tau$). After performing the (2^n–1)th step of calculations using formulae (8), (9) we find

$$d_{1j}^{(n)}\left(t_2\right)=\delta_{1j}+\tau\cdot s_{1j}^{(n,2^n)},\; d_{2j}^{(n)}\left(t_2\right)=\delta_{2j}+\tau\cdot s_{2j}^{(n,2^n)},\; d_{3j}^{(n)}\left(t_2\right)=\delta_{3j}+\tau\cdot s_{3j}^{(n,2^n)},\; j\in\{1,2,3\}.$$

Therefore, if N is the number of partitions of the segment $\left[t_1,t_2\right]$ ($N=2^n$), then the numerical solution of each of the three Cauchy problems (2)–(3) (for each j = 1, 2, 3) requires $\Lambda_{\text{X}}(N)$ arithmetic operations, where the quantity $\Lambda_{\text{X}}(N)$ satisfies the relation

$$\Lambda_{\text{X}}(N)\sim 17N \text{ with } N\to\infty.$$

We could compare the complexity of the composed algorithm with the complexity of numerically solving Cauchy problems (2)–(3) using the Euler method [14; 15], the error of which, just like the method presented in this research (in the case of functions satisfying the Lipschitz condition $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$), is a value limited in comparison with N^{-1} with $N\to\infty$.

It is easy to check that for the numerical solution of each of the three Cauchy problems (2)–(3) (for each j = 1, 2, 3) by the Euler method, $\Lambda_{\text{Э}}(N)$ arithmetic operations are required, where the quantity $\Lambda_{\text{Э}}(N)$ satisfies the relation

$$\Lambda_{\text{Э}}(N)\sim 16N \text{ with } N\to\infty.$$

Therefore, the complexity of solving problems (2)–(3) using the algorithm composed in this work slightly surpasses the complexity of solving these problems using the Euler method.

3. Deriving a method of error estimation

If $t_1+2^{-n}Tk<t<t_1+2^{-n}T\left(k+1\right)$ (k = 0, 1, …, 2^n – 1), for the difference between the derivatives of the functions-component of the exact and approximate solutions, we will have:

$$d_{1j}{}'(t)-d_{1j}^{(n)}{}'(t)=\omega_3(t)\int_{t_1}^{t}\left[d_{2j}{}'(\tau)-d_{2j}^{(n)}{}'(\tau)\right]d\tau+\left(\omega_3(t)-\omega_3^{(n,k)}\right)\int_{t_1}^{t}d_{2j}^{(n)}{}'(\tau)d\tau+$$

$$+\omega_3^{(n,k)}\int_{t_{n,k}}^{t}d_{2j}^{(n)}{}'(\tau)d\tau-\omega_2(t)\times\int_{t_1}^{t}\left[d_{3j}{}'(\tau)-d_{3j}^{(n)}{}'(\tau)\right]d\tau-\left(\omega_2(t)-\omega_2^{(n,k)}\right)\int_{t_1}^{t}d_{3j}^{(n)}{}'(\tau)d\tau-$$

$$-\omega_2^{(n,k)}\int_{t_{n,k}}^{t}d_{3j}^{(n)}{}'(\tau)d\tau+\delta_{2j}\left(\omega_3\left(t\right)-\omega_3^{(n,k)}\right)-\delta_{3j}\left(\omega_2\left(t\right)-\omega_2^{(n,k)}\right).$$

Then for $t_1+2^{-n}Tk<t<t_1+2^{-n}T\left(k+1\right)$ ($k = 0, 1, \ldots, 2^n - 1$) we obtain:

$$\left|d_{1j}{}'(t)-d_{1j}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{2j}{}'(\tau)-d_{2j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{2j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{3j}{}'(\tau)-d_{3j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{3j}^{(n)}{}'(\tau)\right|d\tau\right]+$$

$$+\omega\left(\omega_3,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{2j}^{(n)}{}'(\tau)\right|d\tau+\omega\left(\omega_y,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{3j}^{(n)}{}'(\tau)\right|d\tau+\delta_{2j}\omega\left(\omega_3,2^{-n}\right)+\delta_{3j}\omega\left(\omega_2,2^{-n}\right),$$

where

$$\Omega=\max\left\{\Omega_1,\Omega_2,\Omega_3\right\},\ \ \Omega_j=\max_{t\in[t_1,t_2]}\left|\omega_j(t)\right|\ \ (j=1,\,2,\,3),$$

and $\omega(f,\delta)$ is the continuity modulus of function *f*, that is

$$\omega(f,\delta)=\sup_{|t'-t''|\le\delta}\left|f\left(t'\right)-f\left(t''\right)\right|.$$

Similarly, to $t_1+2^{-n}Tk<t<t_1+2^{-n}T\left(k+1\right)$ ($k = 0, 1, \ldots, 2^n - 1$) we are having:

$$\left|d_{2j}{}'(t)-d_{2j}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{3j}{}'(\tau)-d_{3j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{3j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{1j}{}'(\tau)-d_{1j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{1j}^{(n)}{}'(\tau)\right|d\tau\right]+$$

$$+\omega\left(\omega_1,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{3j}^{(n)}{}'(\tau)\right|d\tau+\omega\left(\omega_3,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{1j}^{(n)}{}'(\tau)\right|d\tau+\delta_{3j}\omega\left(\omega_1,2^{-n}\right)+\delta_{1j}\omega\left(\omega_3,2^{-n}\right),$$

$$\left|d_{3j}{}'(t)-d_{3j}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{1j}{}'(\tau)-d_{1j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{1j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{2j}{}'(\tau)-d_{2j}^{(n)}{}'(\tau)\right|d\tau+\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{2j}^{(n)}{}'(\tau)\right|d\tau\right]+$$

$$+\omega\left(\omega_2,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{1j}^{(n)}{}'(\tau)\right|d\tau+\omega\left(\omega_1,2^{-n}\right)\int_{t_1}^{t_{n,k+1}}\left|d_{2j}^{(n)}{}'(\tau)\right|d\tau+\delta_{1j}\omega\left(\omega_2,2^{-n}\right)+\delta_{2j}\omega\left(\omega_1,2^{-n}\right),$$

$k = 0, 1, \ldots, 2^n - 1$.

For $t_1+2^{-n}Tk<t<t_1+2^{-n}T\left(k+1\right)$ ($k = 0, 1, \ldots, 2^n - 1$) the inequalities are valid

$$\left|d_{i1}^{(n)}{}'(t)\right|\le\alpha_i^{(n,k)}\ \ (i=1,\,2,\,3), \tag{10}$$

where

$$\alpha_1^{(n,k)}=2^{-n}\Omega\sum_{l=0}^{k-1}\left(\left|d_{21}^{(n,l)}\right|+\left|d_{31}^{(n,l)}\right|\right),\ \alpha_2^{(n,k)}=\Omega+2^{-n}\Omega\sum_{l=0}^{k-1}\left(\left|d_{11}^{(n,l)}\right|+\left|d_{31}^{(n,l)}\right|\right),\ \alpha_3^{(n,k)}=$$

$$=\Omega+2^{-n}\Omega\sum_{l=0}^{k-1}\left(\left|d_{11}^{(n,l)}\right|+\left|d_{21}^{(n,l)}\right|\right), \tag{11}$$

$k = 1, 2, \ldots, 2^n - 1$. Here we still assume that the sums in (11) equal to zero with $k = 0$.

If $k = 0$, the truth of inequalities (10) follows from the equalities,

$$d_{11}^{(n,0)} = 0,\ d_{21}^{(n,0)} = -\omega_3^{(n,0)},\ d_{31}^{(n,0)} = \omega_2^{(n,0)}, \tag{12}$$

coming from (7). Applying inequalities (10) to estimate the quantities,

$$\left|d_{i1}^{(n,k-1)}\right| \ (i = 1, 2, 3),$$

we obtain

$$\alpha_i^{(n,k)} = \alpha_i^{(n,k-1)} + 2^{-n}\Omega\left(\Delta^{(k-1)} - \left|d_{i1}^{(n,k-1)}\right|\right) \le \alpha_i^{(n,k-1)} + 2^{-n}\Omega\left(\Gamma^{(k-1)} - \alpha_i^{(n,k-1)}\right), \tag{13}$$

$i = 1, 2, 3$, where

$$\Delta^{(k-1)} = \sum_{i=1}^{3}\left|d_{i1}^{(n,k-1)}\right|,\ \Gamma^{(k-1)} = \sum_{i=1}^{3}\alpha_i^{(n,k-1)},$$

$k = 1, 2, \ldots, 2^n - 1$. We derive the inequalities by induction from (13)

$$\alpha_i^{(n,k)} \le \frac{1}{3}\left[\left(1 + 2^{-n+1}\Omega\right)^k \Gamma^{(0)} + \left(3\alpha_i^{(n,0)} - \Gamma^{(0)}\right)\left(1 - 2^{-n}\Omega\right)^k\right],$$

$i = 1, 2, 3$, $k = 1, 2, \ldots, 2^n - 1$. The equalities (12) succeed

$$\alpha_1^{(n,0)} = 0,\ \alpha_2^{(n,0)} = \Omega,\ \alpha_3^{(n,0)} = \Omega,$$

whence, taking into account (11), we obtain the inequalities for
$t_1 + 2^{-n}Tk < t < t_1 + 2^{-n}T(k+1)$ ($k = 0, 1, \ldots, 2^n - 1$):

$$\left|d_{11}^{(n)}{}'(t)\right| \le \frac{2\Omega}{3}\left[\left(1 + 2^{-n+1}\Omega\right)^k - \left(1 - 2^{-n}\Omega\right)^k\right],$$

$$\left|d_{i1}^{(n)}{}'(t)\right| \le \frac{\Omega}{3}\left[2\left(1 + 2^{-n+1}\Omega\right)^k + \left(1 - 2^{-n}\Omega\right)^k\right] \ (i = 2, 3). \tag{14}$$

(14) succeed the inequalities

$$\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{11}^{(n)}{}'(\tau)\right|d\tau \le \frac{2^{-n+1}\Omega}{3}\left[\left(1 + 2^{-n+1}\Omega\right)^k - \left(1 - 2^{-n}\Omega\right)^k\right],$$

$$\int_{t_{n,k}}^{t_{n,k+1}}\left|d_{i1}^{(n)}{}'(\tau)\right|d\tau \le \frac{2^{-n}\Omega}{3}\left[2\left(1 + 2^{-n+1}\Omega\right)^k + \left(1 - 2^{-n}\Omega\right)^k\right]$$

($i = 2, 3$), based on them, we get:

$$\int_{t_1}^{t_{n,k+1}}\left|d_{11}^{(n)}{}'(\tau)\right|d\tau \le \frac{\Omega T(k+1)}{3 \cdot 2^{n-1}}\left[\left(1 + \frac{\Omega}{2^{n-1}}\right)^k - \left(1 - \frac{\Omega}{2^n}\right)^k\right],$$

$$\int_{t_1}^{t_{n,k+1}}\left|d_{i1}^{(n)}{}'(\tau)\right|d\tau \le \frac{\Omega T(k+1)}{3 \cdot 2^n}\left[2\left(1 + \frac{\Omega}{2^{n-1}}\right)^k + \left(1 - \frac{\Omega}{2^n}\right)^k\right]$$

($i = 2, 3$), thus, for $t_1 + 2^{-n}Tk < t < t_1 + 2^{-n}T(k+1)$ ($k = 0, 1, \ldots, 2^n - 1$) will realise

$$\left|d_{11}{}'(t)-d_{11}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{21}{}'(\tau)-d_{21}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{31}{}'(\tau)-d_{31}^{(n)}{}'(\tau)\right|d\tau\right]+\frac{2^{-n+1}\Omega^2}{3}\times$$

$$\times\left[2\left(1+2^{-n+1}\Omega\right)^k+\left(1-2^{-n}\Omega\right)^k\right]+\frac{2^{-n}\Omega T\left(k+1\right)}{3}\left[2\left(1+2^{-n+1}\Omega\right)^k+\left(1-2^{-n}\Omega\right)^k\right]\times$$

$$\times\left(\omega\left(\omega_3,2^{-n}\right)+\omega\left(\omega_2,2^{-n}\right)\right). \tag{15}$$

Similarly, the inequalities are derived

$$\left|d_{21}{}'(t)-d_{21}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{31}{}'(\tau)-d_{31}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{11}{}'(\tau)-d_{11}^{(n)}{}'(\tau)\right|d\tau\right]+$$

$$+\frac{\Omega^2}{3\cdot2^n}\left[4\left(1+\frac{\Omega}{2^{n-1}}\right)^k-\left(1-\frac{\Omega}{2^n}\right)^k\right]+\frac{\Omega T\left(k+1\right)}{3\cdot2^n}\left[2\left(1+\frac{\Omega}{2^{n-1}}\right)^k+\left(1-\frac{\Omega}{2^n}\right)^k\right]\omega\left(\omega_1,2^{-n}\right)+$$

$$+\left[\frac{\Omega T\left(k+1\right)}{3\cdot2^{n-1}}\left(\left(1+\frac{\Omega}{2^{n-1}}\right)^k-\left(1-\frac{\Omega}{2^n}\right)^k\right)+1\right]\omega\left(\omega_3,2^{-n}\right), \tag{16}$$

$$\left|d_{31}{}'(t)-d_{31}^{(n)}{}'(t)\right|\le\Omega\left[\int_{t_1}^{t}\left|d_{11}{}'(\tau)-d_{11}^{(n)}{}'(\tau)\right|d\tau+\int_{t_1}^{t}\left|d_{21}{}'(\tau)-d_{21}^{(n)}{}'(\tau)\right|d\tau\right]+$$

$$+\frac{\Omega^2}{3\cdot2^n}\left[4\left(1+\frac{\Omega}{2^{n-1}}\right)^k-\left(1-\frac{\Omega}{2^n}\right)^k\right]+\frac{\Omega T\left(k+1\right)}{3\cdot2^n}\left[2\left(1+\frac{\Omega}{2^{n-1}}\right)^k+\left(1-\frac{\Omega}{2^n}\right)^k\right]\omega\left(\omega_1,2^{-n}\right)+$$

$$+\left[\frac{\Omega T\left(k+1\right)}{3\cdot2^{n-1}}\left(\left(1+\frac{\Omega}{2^{n-1}}\right)^k-\left(1-\frac{\Omega}{2^n}\right)^k\right)+1\right]\omega\left(\omega_2,2^{-n}\right), \tag{17}$$

$t_1+2^{-n}Tk<t<t_1+2^{-n}T\left(k+1\right)$, $k=0,1,\ldots,2^n-1$. Summing inequalities (15)–(17) term by term, we obtain:

$$\left|d_{11}{}'(t)-d_{11}^{(n)}{}'(t)\right|+\left|d_{21}{}'(t)-d_{21}^{(n)}{}'(t)\right|+\left|d_{31}{}'(t)-d_{31}^{(n)}{}'(t)\right|\le$$

$$\le2\Omega\int_{t_1}^{t}\left(\left|d_{11}{}'(\tau)-d_{11}^{(n)}{}'(\tau)\right|+\left|d_{21}{}'(\tau)-d_{21}^{(n)}{}'(\tau)\right|+\left|d_{31}{}'(\tau)-d_{31}^{(n)}{}'(\tau)\right|\right)d\tau+A(n,k),$$

where

$$A(n,k)=2^{-n+2}\Omega^2\left(1+2^{-n+1}\Omega\right)^k+\frac{2^{-n+1}\Omega T\left(k+1\right)}{3}\left[2\left(1+2^{-n+1}\Omega\right)^k+\left(1-2^{-n}\Omega\right)^k\right]\omega\left(\omega_1,2^{-n}\right)+$$

$$+\left[\frac{2^{-n}\Omega T\left(k+1\right)}{3}\left(4\left(1+2^{-n+1}\Omega\right)^k-\left(1-2^{-n}\Omega\right)^k\right)+1\right]\left[\omega\left(\omega_2,2^{-n}\right)+\omega\left(\omega_3,2^{-n}\right)\right].$$

The following statement was proven in [12].

Lemma 1. If non-negative function $f(x)$, $x_0\le x\le X$ has only a finite number of discontinuity points of the first kind $\{x_1,x_2,\ldots,x_N\}\subset(x_0,X)$, where $f(x_k)\le\max\{f(x_k-0),f(x_k+0)\}$, $k=1,2,\ldots,N$, and with some constants $\alpha,\beta>0$, it meets the condition

$$f(x) \le \alpha + \beta \int_{x_0}^{x} f(t)\,dt$$

at least at all points of continuity (and then in general at all points of the segment [x0, X]), then with $x_0 \le x \le X$, inequality is realised

$$f(x) \le \alpha e^{\beta(x - x_0)}.$$

Applying this lemma, we get the following inequality:

$$\left| d_{11}{}'(t) - d_{11}^{(n)}{}'(t) \right| + \left| d_{21}{}'(t) - d_{21}^{(n)}{}'(t) \right| + \left| d_{31}{}'(t) - d_{31}^{(n)}{}'(t) \right| \le A(n,k) e^{2\Omega(t - t_1)}.$$

Therefore, to $t_1 + 2^{-n} Tk < t < t_1 + 2^{-n} T(k+1)$ $(k = 0, 1, \ldots, 2^n - 1)$

$$\left| d_{i1}{}'(t) - d_{i1}^{(n)}{}'(t) \right| \le A(n,k) e^{2\Omega(t - t_1)}, \tag{18}$$

$i = 1, 2, 3$. Equality is correct

$$d_{i1}(t_{n,k}) - d_{i1}^{(n)}(t_{n,k}) = \delta_{i1} + \int_{t_1}^{t_{n,k}} d_{i1}{}'(\tau) d\tau - \delta_{i1} - 2^{-n} \sum_{l=0}^{k-1} d_{i1}^{(n,l)} = \int_{t_1}^{t_{n,k}} d_{i1}{}'(\tau) d\tau - 2^{-n} \sum_{l=0}^{k-1} d_{i1}^{(n,l)},$$

where $t_{n,k}$ – set points (6), $k = 0, 1, \ldots, 2^n - 1$, $i = 1, 2, 3$. Thus, we obtain:

$$d_{i1}(t_2) - d_{i1}^{(n)}(t_2) = \int_{t_1}^{t_2} d_{i1}{}'(\tau) d\tau - 2^{-n} T \sum_{l=0}^{2^n - 1} d_{i1}^{(n,l)} = \sum_{l=0}^{2^n - 1} \int_{t_{n,l}}^{t_{n,l+1}} d_{i1}{}'(\tau) d\tau - 2^{-n} T \sum_{l=0}^{2^n - 1} d_{i1}^{(n,l)} =$$

$$= 2^{-n} T \sum_{l=0}^{2^n - 1} \left(d_{i1}{}'(\tau_l) - d_{i1}^{(n)\prime}(t_{n,l}) \right) = 2^{-n} T \sum_{l=0}^{2^n - 1} \left[\left(d_{i1}{}'(\tau_l) - d_{i1}^{(n)\prime}(\tau_l) \right) + \left(d_{i1}^{(n)\prime}(\tau_l) - d_{i1}^{(n)\prime}(t_{n,l}) \right) \right] =$$

$$= 2^{-n} T \sum_{l=0}^{2^n - 1} \left(d_{i1}{}'(\tau_l) - d_{i1}^{(n)\prime}(\tau_l) \right), \tag{19}$$

where τ_l are interval points $(t_{n,l}, t_{n,l+1})$, $l = 0, 1, \ldots, 2^n - 1$, $i = 1, 2, 3$. Here we use the mean value theorem for a definite integral, followed by that there are τ_l points in the interval $(t_{n,l}, t_{n,l+1})$, which means: for continuous functions $d_{i1}{}'(\tau)$, the equalities are realised

$$\int_{t_{n,l}}^{t_{n,l+1}} d_{i1}{}'(\tau) d\tau = d_{i1}{}'(\tau_l)\left(t_{n,l+1} - t_{n,l}\right) = 2^{-n} d_{i1}{}'(\tau_l),$$

where $l = 0, 1, \ldots, 2^n - 1$, $i = 1, 2, 3$, as well as at each interval,

$$\left(t_{n,l}, t_{n,l+1}\right) = \left(t_1 + 2^{-n} Tl, t_1 + 2^{-n} T(l+1)\right)$$

the Haar polynomials $d_{i1}^{(n)\prime}(t)$ obtain a constant value, it results in

$$d_{i1}^{(n)\prime}(\tau_l) = d_{i1}^{(n)\prime}(t_{n,l}),$$

$l = 0, 1, \ldots, 2^n - 1$, $i = 1, 2, 3$. Using the triangle inequality and inequalities (16), derived from (17), we get:

$$\left| d_{i1}(t_2) - d_{i1}^{(n)}(t_2) \right| \le 2^{-n} T \sum_{l=0}^{2^n - 1} \left| d_{i1}{}'(\tau_l) - d_{i1}^{(n)\prime}(\tau_l) \right| \le$$

$$\leq 2^{-n} T \sum_{l=0}^{2^n-1} A(n,l) e^{2\Omega(\tau_l - t_1)} < 2^{-n} T \sum_{l=0}^{2^n-1} A(n,l) e^{2^{-n+1}\Omega T(l+1)}, \tag{20}$$

$i = 1, 2, 3$. The equalities take place

$$\sum_{l=0}^{N-1} q^l = \left(q^N - 1\right)\left(q-1\right)^{-1}, \quad \sum_{l=0}^{N-1} (l+1) q^l = \left(Nq^{N+1} - (N+1)q^N + 1\right)\left(q-1\right)^{-2},$$

their correctness is easily verified by induction. Using these equalities, we calculate the sums:

$$\sigma_1(n) = \sum_{l=0}^{2^n-1} e^{2^{-n+1}\Omega T(l+1)} \left(1 + 2^{-n+1}\Omega\right)^l = e^{2^{-n+1}\Omega T} \frac{e^{2\Omega T}\left(1+2^{-n+1}\Omega\right)^{2^n} - 1}{e^{2^{-n+1}\Omega T}\left(1+2^{-n+1}\Omega\right) - 1},$$

$$\sigma_2(n) = \sum_{l=0}^{2^n-1} e^{2^{-n+1}\Omega T(l+1)} (l+1)\left(1 + 2^{-n+1}\Omega\right)^l =$$

$$= e^{2^{-n+1}\Omega T} \frac{e^{2\Omega T} 2^n e^{2^{-n+1}\Omega T}\left(1+2^{-n+1}\Omega\right)^{2^n+1} - e^{2\Omega T}\left(2^n+1\right)\left(1+2^{-n+1}\Omega\right)^{2^n} + 1}{\left(e^{2^{-n+1}\Omega T}\left(1+2^{-n+1}\Omega\right) - 1\right)^2},$$

$$\sigma_3(n) = \sum_{l=0}^{2^n-1} e^{2^{-n+1}\Omega T(l+1)} (l+1)\left(1 - 2^{-n}\Omega\right)^l =$$

$$= e^{2^{-n+1}\Omega T} \frac{e^{2\Omega T} 2^n e^{2^{-n+1}\Omega T}\left(1-2^{-n}\Omega\right)^{2^n+1} - e^{2\Omega T}\left(2^n+1\right)\left(1-2^{-n}\Omega\right)^{2^n} + 1}{\left(e^{2^{-n+1}\Omega T}\left(1-2^{-n}\Omega\right) - 1\right)^2}.$$

Inequality (20) results in:

$$\left|d_{i1}(t_2) - d_{i1}^{(n)}(t_2)\right| \leq 2^{-n} T \left\{ 2^{-n+2}\Omega^2 \sigma_1(n) + \frac{2^{-n+1}\Omega T}{3}\left[2\sigma_2(n) + \sigma_3(n)\right]\omega\left(\omega_1, 2^{-n}\right) + \right.$$

$$\left. + \left[\frac{2^{-n}\Omega T}{3}\left(4\sigma_2(n) - \sigma_3(n)\right) + 1\right]\left[\omega\left(\omega_2, 2^{-n}\right) + \omega\left(\omega_3, 2^{-n}\right)\right]\right\}, \tag{21}$$

$i = 1, 2, 3$. It should be noted

$$\lim_{n\to\infty} 2^{-n+2}\sigma_1(n) = 2\Omega^{-1}(T+1)^{-1}\left(e^{2\Omega(T+1)} - 1\right), \ \lim_{n\to\infty} 2^{-n+1}\sigma_2(n) = 0, \ \lim_{n\to\infty} 2^{-n+1}\sigma_3(n) = 0. \tag{22}$$

Then, inequality (20) results in the evaluation

$$\left|d_{i1}(t_2) - d_{i1}^{(n)}(t_2)\right| \leq \Sigma_1(n), \tag{23}$$

$i = 1, 2, 3$, where

$$\Sigma_1(n) \sim T\left[2^{-n+1}\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)} - 1\right) + \omega\left(\omega_2, 2^{-n}\right) + \omega\left(\omega_3, 2^{-n}\right)\right]$$

in case $n \to \infty$. If, in this case, the functions $\omega_2(t)$ and $\omega_3(t)$ satisfy the Lipschitz condition, that is, there exist such constants $L_2 \geq 0$ and $L_3 \geq 0$ that for any number $\delta > 0$, the inequalities are realised

$$\omega\left(\omega_2, \delta\right) \leq L_2\delta, \ \ \omega\left(\omega_3, \delta\right) \leq L_3\delta,$$

then for the quantity $\Sigma_1(n)$, occuring in inequality (23), there is a relation

$$\Sigma_1(n) \sim 2^{-n} T\left[2\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)}-1\right)+L_2+L_3\right]$$

in case $n \to \infty$.

Similarly to obtaining inequality (21), we derive the relations

$$\left|d_{ij}(t_2)-d_{ij}^{(n)}(t_2)\right| \le 2^{-n} T\left\{2^{-n+2}\Omega^2\sigma_1(n)+\frac{2^{-n+1}\Omega T}{3}\left[2\sigma_2(n)+\sigma_3(n)\right]\omega\left(\omega_j,2^{-n}\right)+\right.$$

$$\left.+\left[\frac{2^{-n}\Omega T}{3}\left(4\sigma_2(n)-\sigma_3(n)\right)+1\right]\left[\Omega_S-\omega\left(\omega_j,2^{-n}\right)\right]\right\}, \qquad (24)$$

$i = 1, 2, 3, j = 2, 3$, where Ω_S is determined by equality

$$\Omega_S=\sum_{j=1}^{3}\omega\left(\omega_j,2^{-n}\right).$$

Taking into consideration (22) и (24), we can estimate

$$\left|d_{ij}(t_2)-d_{ij}^{(n)}(t_2)\right| \le \Sigma_j(n),$$

$i = 1, 2, 3, j = 2, 3$, where

$$\Sigma_j(n) \sim T\left[2^{-n+1}\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)}-1\right)+\Omega_S-\omega\left(\omega_j,2^{-n}\right)\right]$$

in case $n \to \infty$, $j = 2, 3$. If the functions $\omega_j(t)$ satisfy the Lipschitz condition with constants $L_j \ge 0$ ($j = 1, 2, 3$), then

$$\Sigma_j(n) \sim 2^{-n} T\left[2\Omega(T+1)^{-1}\left(e^{2\Omega(T+1)}-1\right)+L-L_j\right]$$

in case $n \to \infty$, $j = 2, 3$, where $L = L_1 + L_2 + L_3$.

4. Results of numerical experiments

Example 1. We consider the Cauchy problem (2)–(3) for $j = 1$ with parameter values $t_1 = 0$, $t_2 = 1$ and coefficients

$$\omega_1(t)=\cos 1{,}5t;\ \ \omega_2(t)=\frac{1}{2}\sin 1{,}5t+\frac{3\sqrt{3}}{4};\ \ \omega_3(t)=\frac{\sqrt{3}}{2}\sin 1{,}5t-0{,}75.$$

It is easy to verify that there is the exact solution of such a Cauchy problem

$$d_{11}(t)=\cos 1{,}5t;\ \ d_{21}(t)=\frac{1}{2}\sin 1{,}5t;\ \ d_{31}(t)=\frac{\sqrt{3}}{2}\sin 1{,}5t;$$

and $d_{11}(t_2) \approx -0{,}07074$, $d_{21}(t_2) \approx 0{,}49875$, $d_{31}(t_2) \approx 0{,}86386$.

Table 1 shows the values of the square root of the mean square-root error of the components of the solution to the Cauchy problem under consideration at a point t_2 with the values of parameters and coefficients used in Example 1, obtained by the methods of Haar, Euler, Euler-Cauchy sums [16] and Runge-Kutta 2nd order sums [14; 15] for N segment partitions $[t_1, t_2]$, where $N = 2^{15}, 2^{16}, \ldots, 2^{24}$.

In this case, the errors in solving the Cauchy problem (2)–(3) using the Haar and Euler sum methods are almost identical, but the errors for the 2nd order Euler–Cauchy and Runge–Kutta methods are significantly smaller than the errors obtained by applying the first two methods.

Table 1

The square root value out of the mean square-root error of the components for the solution at t_2 point for the parameter values and coefficients used in example 1

N (number of segment partitions $[t_1, t_2]$)	Value $\sqrt{\frac{1}{3}\sum_{i=1}^{3}\left(d_{i1}\left(t_2\right)-d_{i1}^{(n)}\left(t_2\right)\right)^2}$			
	Haar sum method	Euler method	Euler–Cauchy method	Runge–Kutta Method of the 2nd order
2^{15}	$1.98221\cdot10^{-5}$	$2.14845\cdot10^{-5}$	$2.90010\cdot10^{-10}$	$2.90010\cdot10^{-10}$
2^{16}	$9.91096\cdot10^{-6}$	$1.07423\cdot10^{-5}$	$7.25045\cdot10^{-11}$	$7.25045\cdot10^{-11}$
2^{17}	$4.95546\cdot10^{-6}$	$5.37117\cdot10^{-6}$	$1.81151\cdot10^{-11}$	$1.81152\cdot10^{-11}$
2^{18}	$2.47772\cdot10^{-6}$	$2.68559\cdot10^{-6}$	$4.53666\cdot10^{-12}$	$4.53667\cdot10^{-12}$
2^{19}	$1.23886\cdot10^{-6}$	$1.34279\cdot10^{-6}$	$1.13229\cdot10^{-12}$	$1.13229\cdot10^{-12}$
2^{20}	$6.19430\cdot10^{-7}$	$6.71397\cdot10^{-7}$	$2.89097\cdot10^{-13}$	$2.89107\cdot10^{-13}$
2^{21}	$3.09715\cdot10^{-7}$	$3.35699\cdot10^{-7}$	$6.36173\cdot10^{-14}$	$6.36283\cdot10^{-14}$
2^{22}	$1.54857\cdot10^{-7}$	$1.67849\cdot10^{-7}$	$1.62080\cdot10^{-14}$	$1.62080\cdot10^{-14}$
2^{23}	$7.74287\cdot10^{-8}$	$8.39247\cdot10^{-8}$	$5.05499\cdot10^{-14}$	$5.05598\cdot10^{-14}$
2^{24}	$3.87144\cdot10^{-8}$	$4.19624\cdot10^{-8}$	$2.36189\cdot10^{-14}$	$2.36183\cdot10^{-14}$

Example 2. We consider the Cauchy problem (2)–(3) for j = 1 with parameter values $t_1 = 0$, $t_2 = 2$ and coefficients

$$\omega_1(t)=\sqrt[5]{\mathrm{ch}^9 t},\ \ \omega_2(t)=\frac{\sqrt{2}}{2}\left(\sqrt[5]{\mathrm{ch}^9 t}\ \mathrm{tg}\,t+1\right),\ \ \omega_3(t)=\frac{\sqrt{2}}{2}\left(\sqrt[5]{\mathrm{ch}^9 t}\ \mathrm{tg}\,t-1\right).$$

It is easy to verify that there is the exact solution of such a Cauchy problem

$$d_{11}(t)=\cos t,\ \ d_{21}(t)=d_{31}(t)=\frac{\sqrt{2}}{2}\sin t,$$

and $d_{11}\left(t_2\right)\approx -0,41615,\ \ d_{21}\left(t_2\right)=d_{31}\left(t_2\right)\approx 0,64297.$

Table 2 demonstrates the values of the square root of the mean square-root error of the components of the solution to the considered Cauchy problem at a point t_2 with the values of parameters and coefficients used in Example 2, obtained by the methods of Haar, Euler, Euler - Cauchy sums [16] and Runge – Kutta 2nd order sums [14; 15] for N segment partitions $\left[t_1, t_2\right]$, where $N = 2^{15}, 2^{16},\ldots,2^{24}$.

Table 2

The square root value out of the mean square-root error of the components for the solution at t_2 point for the parameter values and coefficients used in example 2

N (number of segment partitions $[t_1, t_2]$)	Value $\sqrt{\frac{1}{3}\sum_{i=1}^{3}\left(d_{i1}\left(t_2\right)-d_{i1}^{(n)}\left(t_2\right)\right)^2}$			
	Haar sum method	Euler method	Euler–Cauchy method	Runge–Kutta method of the 2nd order
2^{15}	$1.77319\cdot10^{-2}$	$5.86421\cdot10^{-2}$	$1.94818\cdot10^{-2}$	$4.54692\cdot10^{-3}$
2^{16}	$2.27484\cdot10^{-3}$	$1.68334\cdot10^{-2}$	$1.28402\cdot10^{-3}$	$6.33870\cdot10^{-4}$
2^{17}	$3.17159\cdot10^{-4}$	$6.71757\cdot10^{-4}$	$9.98299\cdot10^{-5}$	$1.30330\cdot10^{-4}$
2^{18}	$6.52261\cdot10^{-5}$	$3.35977\cdot10^{-4}$	$1.68894\cdot10^{-5}$	$6.26192\cdot10^{-5}$
2^{19}	$3.12779\cdot10^{-5}$	$2.52341\cdot10^{-4}$	$1.56949\cdot10^{-5}$	$9.12427\cdot10^{-6}$
2^{20}	$4.57051\cdot10^{-6}$	$1.76562\cdot10^{-4}$	$1.35320\cdot10^{-6}$	$2.37992\cdot10^{-6}$
2^{21}	$1.45672\cdot10^{-6}$	$1.43105\cdot10^{-4}$	$3.89106\cdot10^{-7}$	$6.50765\cdot10^{-7}$
2^{22}	$5.97437\cdot10^{-7}$	$6.01678\cdot10^{-5}$	$2.00878\cdot10^{-7}$	$3.41657\cdot10^{-7}$
2^{23}	$3.05874\cdot10^{-7}$	$3.76402\cdot10^{-5}$	$1.86299\cdot10^{-7}$	$3.07409\cdot10^{-7}$
2^{24}	$1.51830\cdot10^{-7}$	$1.30176\cdot10^{-5}$	$1.13470\cdot10^{-7}$	$2.58138\cdot10^{-7}$

In this case, the Haar sum method gives an error that is significantly smaller than the Euler method and is almost identical to the errors in solving the Cauchy problem (2)–(3) using methods of Euler - Cauchy and Runge – Kutta of the 2nd order.

Example 3. We consider the Cauchy problem (2)–(3) for j = 1 with parameter values $t_1 = 0$, $t_2 = 2$ and coefficients

$$\omega_1(t) = \sqrt[8]{|\sec t|},\ \ \omega_2(t) = \frac{3}{5}\sqrt[8]{|\sec t|}\ \mathrm{tg}\, t + \frac{4}{5},\ \ \omega_3(t) = \frac{4}{5}\sqrt[8]{|\sec t|}\ \mathrm{tg}\, t - \frac{3}{5}.$$

The exact solution of such a Cauchy problem has got the form

$$d_{11}(t) = \cos t,\ \ d_{21}(t) = \frac{3}{5}\sin t,\ \ d_{31}(t) = \frac{4}{5}\sin t,$$

but $d_{11}(t_2) \approx -0{,}41615$, $d_{21}(t_2) \approx 0{,}54558$, $d_{31}(t_2) \approx 0{,}72744$.

Table 3 reveals the values of the square root of the mean square-root error of the components of the solution to the considered Cauchy problem at a point t_2 with the values of parameters and coefficients used in Example 3, obtained by the sums methods of Haar, Euler, Euler - Cauchy [16] and Runge - Kutta of 2nd order sums [14; 15] for N segment partitions $[t_1, t_2]$, where $N = 2^{15}, 2^{16}, \ldots, 2^{24}$.

Table 3

The square root value out of the mean square-root error of the components for the solution at t_2 point for the parameter values and coefficients used in example 3

N (number of segment partitions $[t_1, t_2]$)	Value $\sqrt{\frac{1}{3}\sum_{i=1}^{3}\left(d_{i1}(t_2) - d_{i1}^{(n)}(t_2)\right)^2}$			
	Haar sum method	Euler method	Euler–Cauchy method	Runge–Kutta method of the 2nd order
2^{15}	$4{,}09952 \cdot 10^{-5}$	$3{,}64380 \cdot 10^{-2}$	$8{,}14584 \cdot 10^{-5}$	$1{,}60285 \cdot 10^{-5}$
2^{16}	$1{,}83821 \cdot 10^{-5}$	$1{,}47670 \cdot 10^{-3}$	$1{,}05944 \cdot 10^{-5}$	$4{,}83737 \cdot 10^{-6}$
2^{17}	$8{,}94108 \cdot 10^{-6}$	$1{,}00902 \cdot 10^{-3}$	$5{,}50037 \cdot 10^{-6}$	$3{,}37017 \cdot 10^{-6}$
2^{18}	$4{,}46564 \cdot 10^{-6}$	$6{,}25570 \cdot 10^{-4}$	$2{,}39992 \cdot 10^{-6}$	$2{,}41542 \cdot 10^{-6}$
2^{19}	$2{,}41361 \cdot 10^{-6}$	$5{,}37907 \cdot 10^{-4}$	$1{,}77466 \cdot 10^{-6}$	$1{,}31424 \cdot 10^{-6}$
2^{20}	$1{,}16140 \cdot 10^{-6}$	$2{,}79979 \cdot 10^{-4}$	$1{,}47872 \cdot 10^{-6}$	$1{,}05524 \cdot 10^{-6}$
2^{21}	$6{,}26278 \cdot 10^{-7}$	$1{,}65719 \cdot 10^{-4}$	$7{,}68163 \cdot 10^{-7}$	$1{,}01374 \cdot 10^{-6}$
2^{22}	$3{,}14122 \cdot 10^{-7}$	$1{,}80869 \cdot 10^{-4}$	$5{,}62648 \cdot 10^{-7}$	$8{,}27771 \cdot 10^{-7}$
2^{23}	$6{,}62019 \cdot 10^{-7}$	$1{,}51395 \cdot 10^{-4}$	$5{,}56389 \cdot 10^{-7}$	$2{,}44021 \cdot 10^{-7}$
2^{24}	$6{,}36338 \cdot 10^{-7}$	$4{,}78043 \cdot 10^{-5}$	$4{,}17734 \cdot 10^{-7}$	$1{,}36309 \cdot 10^{-7}$

Example 3, like Example 2, shows that in certain cases the Haar sum method gives an error that is significantly smaller than the Euler method and almost identical to the errors of the Euler-Cauchy and Runge – Kutta methods of the 2nd order.

It should be noted that the complexity of the Euler-Cauchy and Runge – Kutta methods of the 2nd order is approximately twice the complexity of the Haar sums method: the number of arithmetic operations $\Lambda_{ЭК}(N)$ and $\Lambda_{РК}(N)$, required to solve each of the three Cauchy problems (2)–(3) by the Euler-Cauchy methods and Runge – Kutta of the 2nd order, respectively, it satisfies the relations

$$\Lambda_{ЭК}(N) \sim 34N \text{ with } N \to \infty,\ \ \Lambda_{РК}(N) \sim 32N \text{ with } N \to \infty.$$

Conclusion

The current research presents a new method to solve the system of Poisson kinematic equations determining the evolution of the spacecraft position from time moment t_1 to moment t_2. Based on the obtained estimates of the method error, it follows that if the functions representing the projections of the absolute angular velocity of the spacecraft onto the coordinate axes satisfy the Lipschitz condition, then the absolute error in calculating each element of the transition matrix from the coordinate system associated with the spacecraft at an instant of time t_1 to the coordinate system associated with the spacecraft at the moment of time t_2, just as in the case of solving the specified system of equations by the Euler method, there is a value $O(N^{-1})$ for $N \to \infty$. where N is the number of segment partitions $[t_1, t_2]$ when constructing the grid of nodes used.

Comparing the algorithms to solve the system of equations under consideration using the proposed method and the Euler method in terms of their computational efficiency has shown that the implementation of each of them requires $O(N)$ arithmetic operations with $N \to \infty$, , while the complexity of the algorithm constructed in this research is slightly higher than the complexity of the algorithm to solve the system by the Euler method.

From the results of numerical experiments presented in the research, it follows that in certain cases the Haar sums method gives an error that is significantly smaller than the Euler method and it is almost identical to the errors of the Euler-Cauchy and Runge-Kutta methods of the 2nd order, the complexity of which is approximately twice that of complexity of the Haar sums method.

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