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Решение интегрального уравнения для средней стоимости восстановлений в теории надежности технических систем

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Отказы элементов при работе технических и многих других систем имеют, как правило, случайный характер. Это приводит к различным моделям процесса восстановления, изучаемым в теории вероятностей и математической теории надежности. В процессе восстановления отказавшие элементы восстанавливаются или заменяются на новые, при этом часто происходит изменение стоимостей и качества восстанавливаемых элементов (функций распределения наработок до отказа).

В работе рассматривается функция затрат (средняя стоимость восстановления) в процессе восстановления порядка (k_1, k_2), *в котором по определенному правилу изменяются стоимости каждого восстановления и функции распределения наработок.*

Учитывая, что функция восстановления (среднее число отказов) хорошо изучена в теории надежности, получено решение интегрального уравнения для функции затрат через функцию восстановления рассматриваемой модели.

Для процесса восстановления порядка (k_1, k_2) *получена формула вычисления функции затрат через функцию восстановления простого процесса, образованного сверткой всех функций распределения периодической части. Для практического применения получены явные формулы функции затрат при процессе восстановления, у которого периодическая часть распределена по экспоненциальному закону или закону Эрланга порядка m с одним и тем же показателем* α.

Полученные формулы могут быть использованы для изучения свойств функции затрат и решения оптимизационных задач в стратегиях проведения процесса восстановления в терминах «цена», «качество», «риск», если, например, за качество принимать среднее число отказов, за цену – среднюю стоимость восстановлений, за риск – дисперсии числа отказов или стоимости восстановлений.

Ключевые слова: модели процесса восстановления, функция восстановления, функция затрат, распределение Эрланга.

Solution of the integral equation for the average cost of restoration in the theory of reliability of technical systems

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Failures of elements during the operation of technical and many other systems are, as a rule, random in nature. This leads to various models of the recovery process, studied in probability theory and mathematical reliability theory. During the restoration process, failed elements are restored or replaced with new ones, and there is often a change in the costs and quality of the restored elements (time-to-failure distribution functions).

The work examines the cost function (average cost of restoration) in the process of restoration of order (k_1, k_2), *in which, according to a certain rule, the costs of each restoration and the distribution functions of operating time change.*

Considering, that the recovery function (average number of failures) is well studied in reliability theory, a solution to the integral equation for the cost function is obtained through the recovery function of the model under consideration.

For the order restoration process (k_1, k_2), *a formula is obtained for calculating the cost function through the restoration function of a simple process formed by the convolution of all distribution functions of the periodic part. For practical application, explicit formulas are obtained for the cost function during the restoration process, in which the periodic part is distributed according to an exponential law or an Erlang law of order* m *with the same exponent* α.

The resulting formulas can be used to study the properties of the cost function and solve optimization problems in strategies for carrying out the restoration process in terms of price, quality, risk, if, for example, the average number of failures is taken as quality, the average cost of restorations as price, the dispersion of the number of failures as the risk, or cost of restoration.

Keywords: recovery process models, recovery function, cost function, Erlang distribution.

Introduction

In the mathematical theory of reliability, when studying recovery processes, the numerical characteristics associated with the random number of failures and the random cost of recovery are first considered, for example, the average and dispersion of the number of failures and the cost of recovery, through which various criteria for the optimality of recovery strategies are determined.

The paper discusses models of the recovery process (X_i, c_i), $i = 0, 1, \ldots$, taking into account the cost of restoration. Where X_i, random operation time with distribution functions $F_i(t)$ elements from I – 1st to i-th failure, c_i cost of i – x recovery, c_0 – element cost, set at the initial time $t = 0$, $F_0(t) = 0$ for a case $t < 0$, $F_0(t) = 1$ for a case $t \geq 0$ [1–4].

Let $N(t)-$ be the number of failures (recoveries), $C(t)$ be the cost of recovery for the time from 0 to t

$$C(t) = \sum_{i=0}^{N(t)} \varsigma_i$$

$$P(N(t) = n) = F^{(n)}(t) - F^{(n+1)}(t),$$

$F^{(n)}(t)$ - n - multiple convolution of distribution functions $F_i(t)$, $i = 1, 2, \ldots, n$,

$$F^{(n)}(t)=\left(F^{(n-1)}*F_n\right)(t)=\int_0^t F^{(n-1)}(t-x)\,dF_n(x)\quad F^{(1)}(t)=F_1(t).$$

For [1,2]: $H(t)-$ recovery function *(average number of failures)*

$$H(t)=E\left(N(t)\right)=\sum_{n=1}^{\infty}F^{(n)}(t)$$

$S(t)=E\left(C(t)\right)-$ cost function *(average cost of restorations)*

$$S(t))=E\left(C(t)\right)=c_0+\sum_{n=1}^{\infty}c_nF^{(n)}(t).$$

During operation, the quality ($F_i(t)$) of the restored elements and the cost (c_i) of restoration may differ. This leads to different models of the recovery process [1, 3, 5–9].

The work considers the restoration process taking into account the cost of restoration of the order (k_1,k_2), in which the distribution functions and cost of restoration satisfy the condition: $F_i(t)=F_j(t)\, u\, c_i=c_j$, if the indices $i,j\geq k_1$ when divided by k_2 give the same excess [1, 3, 8, 9].

In the process under consideration, after the first restorations k_1-1, a periodic process of the order k_2 begins.

Note, that in the case $k_1=1$ we have a periodic process of order restoration k_2, and if $k_2=1$ process of restoring order k_1.

If F_i(t) coincide ($F_i(t)=F_i(t), i\geq 1$), or coincide starting from number $i=2$ ($F_i(t)=F_2(t),\ i\geq 2$), we have simple (ordinary) and general (delayed) recovery processes, well studied in reliability theory.

Note that for the model under consideration, the recovery function $H(t)$ has been well studied. Numerical methods for finding it have been developed, and for many distribution laws characteristic of reliability theory, there are its explicit representations [1, 6].

To find the cost function $S(t)$ there are integral equations [1, 2, 10].

The purpose of the work is to obtain a solution to the integral equation for the cost function S(t) in the form of an integral representation through the restoration function $H(t)$. Such a representation will be convenient for its study and calculation in various theoretical and applied problems of reliability theory.

Representation of the cost function through the recovery function

Let us write the integral equation for the cost function $S(t)$ [1, 2]

$$S(t)=G(t)+\int_0^t S(t-x)\,d\Phi^{(k_2)}(x) \tag{1}$$

for a case $k_1>1$,

$$G(t)=c_0\left(1-\Phi^{(k_2)}(t)\right)+\sum_{n=1}^{k_1+k_2-1}c_nF^{(n)}(t)-\sum_{n=1}^{k_1-1}c_n\int_0^t F^{(n)}(t-x)\,d\Phi^{(k_2)}(x),$$

for a case $k_1=1$

$$G(t)=c_0\left(1-\Phi^{(k_2)}(t)\right)+\sum_{n=1}^{k_2}\varsigma_nF^{(n)}(t)$$

$\Phi^{(k_2)}(t)=\left(\Phi_1*\Phi_2*...*\Phi_{k_2}\right)(t)$ – convolution of all distribution functions

$\Phi_i(t) = F_{k_1-1+i}(t), i = 1, 2, \ldots, k_2$. The functions $\Phi_i(t)$ define the periodic part of the recovery process.

Let $HF(t)$ be the restoration function of a simple process, let $HFG(t)$ be the restoration function of the general process formed by the first distribution function $F(t)$ and the following $G(t)$.

Further [1, 6]

$$HFG(t) = F(t) + \int_0^t HFG(t-x)\,dG(x) \tag{2}$$

In equation (1) we make the replacement

$$S(t) = V(t) + c_0 + \sum_{n=1}^{k_1-1} c_n F^{(n)}(t) \tag{3}$$

We obtain

$$V(t) + c_0 + \sum_{n=1}^{k_1-1} c_n F^{(n)}(t) = c_0\left(-\Phi^{(k_2)}(t)\right) + \sum_{n=1}^{k_1-1} c_n F^{(n)}(t) + \sum_{n=k_1}^{k_1+k_2-1} c_n F^{(n)}(t) -$$

$$-\sum_{n=1}^{k_1-1} c_n \int_0^t F^{(n)}(t-x)\,d\Phi^{(k_2)}(x) + \int_0^t \left(V(t-x) + c_0 + \sum_{n=1}^{k_1-1} c_n F^{(n)}(t-x) \right) d\Phi^{(k_2)}(x).$$

Hence, to find the function $V(t)$, we obtain the integral equation

$$V(t) = \sum_{n=k_1}^{k_1+k_2-1} c_n F^{(n)}(t) + \int_0^t V(t-x)\,d\Phi^{(k_2)}(x). \tag{4}$$

Let us make a replacement

$$V(t) = \left(\sum_{n=k_1}^{k_1+k_2-1} c_n \right) V_1(t) \tag{5}$$

Equation (4) will be rewritten as

$$V_1(t) = Q(t) + \int_0^t V_1(t-x)\,d\Phi^{(k_2)}(x), \tag{6}$$

$$Q(t) = \frac{\sum_{n=k_1}^{k_1+k_2-1} c_n F^{(n)}(t)}{\sum_{n=k_1}^{k_1+k_2-1} c_n}. \tag{7}$$

Note, that $\Phi^{(k_2)}(t)$ and $Q(t)$ are distribution functions, $\Phi^{(k_2)}(t)-$ as a convolution of work distribution functions, and $Q(t)$ – mixture of distribution functions.

Comparing equations (6) and (2), we find that equation (6) defines the restoration function $HQ\Phi^{(k_2)}(t)$ of the general process specified by the first distribution function $Q(t)$, of the second and subsequent ones $\Phi^{(k_2)}(t)$.

Thus,

$$V_1(t) = HQ\Phi^{(k_2)}(t), \tag{8}$$

and taking into account (3), (5), (7), (8)

$$S(t) = c_0 + \sum_{n=1}^{k_1-1} c_n F^{(n)}(t) + \left(\sum_{n=k_1}^{k_1+k_2-1} c_n \right) HQ\Phi^{(k_2)}(t)). \tag{9}$$

Taking into account (2)

$$HQ\Phi^{(k_2)}(t)=Q(t)+\int_0^t H\Phi^{(k_2)}(t-x)\,dQ(x), \qquad (10)$$

formula (9) will be written in the form

$$S(t)=c_0+\sum_{n=1}^{k_1-1}c_nF^{(n)}(t)+\sum_{n=k_1}^{k_1+k_2-1}c_n\left(Q(t)+\int_0^t H\Phi^{(k_2)}(t-x)\,dQ(x)\right),$$

or taking account of (10)

$$S(t)=c_0+\sum_{n=1}^{k_1+k_2-1}c_nF^{(n)}(t)+\sum_{n=k_1}^{k_1+k_2-1}c_n\int_0^t H\Phi^{(k_2)}(t-x)\,dF^{(n)}(x)). \qquad (11)$$

We found that calculating the cost function comes down to calculating the finite number of convolutions of distribution functions and finding the restoration function $H\Phi^{(k_2)}(t)$ of a simple restoration process formed by the distribution function $\Phi^{(k_2)}(t)$, or restoration function $HQ\Phi^{(k_2)}(t)$.

In the practical implementation of the obtained formulas (9), (10), (11), one can use numerical and analytical methods for calculating convolutions and restoration functions, discussed in [1, 11]. We also note that the resulting formulas make it possible to study the properties of the cost function and consider various optimization problems based on strategies for carrying out the restoration process in terms of price, quality, and risk. If, for example, we take the average number of failures as quality, the average cost of restorations as price, and the dispersion of the number of failures or the cost of restorations as the risk [1, 12–15].

This work is a continuation of work [11] and it can be noted that the theorems on the asymptotic behavior of the cost function obtained in [11] are much easier to obtain using the resulting formula for representing the cost function (9).

The cost function for a simple restoration process with exponential distribution. We consider a restoration process in which only the restoration costs c_i change according to the law $c_i=c_j$, if the indices $i,j\geq k_1$ when divided by k_2, give the same excess. This corresponds to the common case where failures result in full restorations, but the costs of restorations change, for example, only the price of the element changes.

Let the operating time of the elements be distributed according to the exponential law $F(t)=1-e^{-\alpha t}, t\geq 0.$ For this case, we obtain calculation formulas for calculating the cost function.

Taking into account, that $n-$ multiple convolution of the distribution functions of independent random variables is a function of the distribution of their sum, and that the Erlang order distribution n is the distribution of the sum of random variables n distributed according to the exponential law, we conclude that for the case under consideration

$$F^{(n)}(t)=F_{e,n}(t)=1-e^{-\alpha t}\sum_{i=0}^{n-1},\frac{(\alpha t)^i}{i!} \quad dF^{(n)}(x)=dF_{e,n}(x)=e^{-\alpha x}\alpha\frac{(\alpha x)^{n-1}}{(n-1)!}dx,$$

$$\Phi^{(k_2)}(t)=F_{e,k_2}(t),\ H\Phi^{(k_2)}(t)=HF_{e,k_2}(t),$$

$F_{e,n}(t)-$ Erlang order distribution $n,$ and [1,6]

$$HF_{e,k_2}(t)=\frac{1}{k_2}(\alpha t+\sum_{k=1}^{k_2-1}1\frac{c^k}{1-c^k}\left(\ -e^{-\alpha t\left(1-c^k\right)}\right),\ c=e^{\frac{2\pi}{k_2}i}=cos\left(\frac{2\pi}{k_2}\right)+isin\left(\frac{2\pi}{k_2}\right), \qquad (12)$$

$$HF_{e,k_2}(t)=\frac{1}{k_2}\left(\alpha t-\frac{k_2-1}{2}+\frac{1}{2}\sum_{k=1}^{k_2-1}.\frac{e^{-\alpha t\left(1-\cos\left(\frac{2\pi}{k_2}k\right)\right)}\sin\left(\alpha t sin\left(\frac{2\pi}{k_2}k\right)+\frac{\pi}{k_2}k\right)}{\sin\left(\frac{\pi}{k_2}k\right)}\right)$$

Now, according to (11)

$$S(t) = c_0 + \sum_{n=1}^{k_1+k_2-1} c_n F_{e,n}(t) + \sum_{n=k_1}^{k_1+k_2-1} c_n \frac{\alpha^n}{(n-1)!} \int_0^t HF_{e,k_2}(t-x) e^{-\alpha x} x^{n-1} dx \qquad (13)$$

Taking into account (12), when calculating $S(t)$, the integrals included in formula (13) are calculated explicitly. For example [16]

$$\int e^{\beta x} x^n dx = \frac{e^{\beta t}}{\beta}\left(t^n + \sum_{j=1}^{n} (-1)^j \frac{n(n-1)\dots(n-j+1)}{\beta^j} t^{n-j} \right) + C.$$

When substituting

$$I(\beta,n)(t) = \int_0^t e^{\beta x} x^n dx = \frac{e^{\beta t}}{\beta}\left(t^n + \sum_{j=1}^{n} (-1)^j \frac{n(n-1)\dots(n-j+1)}{\beta^j} t^{n-j} \right) + (-1)^{n+1} \frac{n!}{\beta^{n+1}}$$

in (13), we obtain

$$S(t) = c_0 + \sum_{n=1}^{k_1+k_2-1} c_n F_{e,n}(t) +$$

$$+ \frac{1}{k_2} \sum_{n=k_1}^{k_1+k_2-1} c_n \frac{\alpha^n}{(n-1)!} \Big(\alpha t I(-\alpha, n-1)(t) - \alpha I(-\alpha, n)(t) +$$

$$+ \sum_{k=1}^{k_2-1}{}' \frac{c^k}{1-c^k} \left(I(-\alpha, n-1)(t) - e^{-\alpha(1-c^k)t} I(-\alpha c^k, n-1)(t) \right) \Big). \qquad (14)$$

We select the real part in (14):

$$1 - c^k = -2ie^{\frac{\pi k}{k_2} i} \sin\left(\frac{\pi k}{k_2}\right), \sum_{k=1}^{k_2-1}{}' \frac{c^k}{1-c^k} = \frac{i}{2} \sum_{k=1}^{k_2-1} ctg\left(\frac{\pi}{k_2} k\right) - \frac{k_2-1}{2}$$

$$Re \sum_{k=1}^{k_2-1}{}' \frac{c^k}{1-c^k} = Re \sum_{k=1}^{k_2-1} \frac{e^{\frac{2\pi k i}{k_2}} e^{-\frac{\pi k i}{k_2}}}{-2i sin \frac{\pi k}{k_2}} = \frac{1-k_2}{2}$$

$$\sum_{k=1}^{k_2-1}{}' \frac{c^k}{1-c^k} e^{-\alpha(1-c^k)t} I(-\alpha c^k, n-1)(t)) =$$

$$= \sum_{k=1}^{k_2-1} \Big(\frac{c^k}{1-c^k} e^{-\alpha(1-c^k)t} \frac{e^{-\alpha c^k t}}{-\alpha c^k} \Big(t^{n-1} +$$

$$+ \sum_{j=1}^{n-1} (-1)^j \frac{(n-1)\dots(n-j)}{(-\alpha)^j c^{kj}} t^{n-1-j} \Big) + (-1)^n \frac{(n-1)!}{(-\alpha)^n c^{kn}} \Big) =$$

$$= -\frac{1}{\alpha} \sum_{k=1}^{k_2-1} \frac{e^{-\alpha t}}{1-c^k} \left(t^{n-1} + \sum_{j=1}^{n-1} \frac{(n-1)\dots(n-j)}{\alpha^j} e^{-\frac{2\pi i}{k_2} kj} t^{n-1-j} \right) +$$

$$+ \frac{(n-1)!}{\alpha^n} \sum_{k=1}^{k_2-1} \frac{c^{-k(n-1)} e^{-\alpha(1-c^k)t}}{1-c^k} =$$

$$= -\frac{i}{2\alpha} \sum_{k=1}^{k_2-1} \frac{e^{-\alpha t} e^{-\frac{\pi k}{k_2} i}}{sin \frac{\pi k}{k_2}} \left(t^{n-1} + \sum_{j=1}^{n-1} \frac{(n-1)\dots(n-j)}{\alpha^j} e^{-\frac{2\pi i}{k_2} kj} t^{n-1-j} \right) +$$

$$+\frac{(n-1)!i}{2\alpha^n}\sum_{k=1}^{k_2-1}\frac{e^{-\frac{\pi k}{k_2}i}}{sin\frac{\pi k}{k_2}}e^{-\frac{2\pi k(n-1)}{k_2}i}e^{-\alpha t}e^{\alpha tcos\frac{2\pi k}{k_2}}e^{\alpha tsin\frac{2\pi k}{k_2}i}=$$

$$=-\frac{i}{2\alpha}\sum_{k=1}^{k_2-1}\frac{e^{-\alpha t}e^{-\frac{\pi k}{k_2}i}}{sin\frac{\pi k}{k_2}}\left(t^{n-1}+\sum_{j=1}^{n-1}\frac{(n-1)...(n-j)}{\alpha^j}e^{-\frac{2\pi i}{k_2}kj}t^{n-1-j}\right)+$$

$$+\frac{(n-1)!i}{2\alpha^n}\sum_{k=1}^{k_2-1}.\frac{e^{-\alpha t\left(1-cos\frac{2\pi k}{k_2}\right)}e^{\left(\alpha tsin\frac{2\pi k}{k_2}-\frac{\pi k(2n-1)}{k_2}\right)i}}{sin\frac{\pi k}{k_2}}$$

Therefore

$$Re\left(\sum_{k=1}^{k_2-1},\frac{c^k}{1-c^k}e^{-\alpha\left(1-c^k\right)t}I\left(-\alpha c^k\ \ n-1\right)(t)\right)=$$

$$=-\frac{1}{2\alpha}\sum_{k=1}^{k_2-1}e^{-\alpha t}t^{n-1}-\frac{e^{-\alpha t}}{2\alpha}\sum_{k=1}^{k_2-1}\frac{1}{sin\left(\frac{\pi k}{k_2}\right)}\sum_{j=1}^{n-1}\frac{(n-1)...(n-j)}{\alpha^j}sin\left(\frac{\pi k(2j+1)}{k_2}\right)t^{n-1-j}-$$

$$-\frac{(n-1)!}{2\alpha^n}\sum_{k=1}^{k_2-1}.\frac{e^{-\alpha t\left(1-\cos\left(\frac{2\pi k}{k_2}\right)\right)}\sin\left(\alpha tsin\left(\frac{2\pi k}{k_2}\right)-\frac{\pi k(2n-1)}{k_2}\right)}{\sin\left(\frac{\pi k}{k_2}\right)}$$

Let us write down the formula for the cost function

$$S(t)=c_0+\sum_{n=1}^{k_1+k_2-1}c_nF_{e,n}(t)+$$

$$+\frac{1}{k_2}\sum_{n=k_1}^{k_1+k_2-1}c_n\frac{\alpha^n}{(n-1)!}\left(\alpha t+\frac{1-k_2}{2}\right)I(-\alpha,n-1)(t)-\alpha I(-\alpha,n)(t)+$$

$$+\frac{1}{2\alpha}\sum_{k=1}^{k_2-1}e^{-\alpha t}t^{n-1}+\frac{e^{-\alpha t}}{2\alpha}\sum_{k=1}^{k_2-1}\frac{1}{\sin\left(\frac{\pi k}{k_2}\right)}\sum_{j=1}^{n-1}\frac{(n-1)\ldots(n-j)}{\alpha^j}sin\left(\frac{\pi k(2j+1)}{k_2}\right)t^{n-1-j}+$$

$$+\frac{(n-1)!}{2\alpha^n}\sum_{k=1}^{k_2-1})\frac{e^{-\alpha t\left(1-cos\left(\frac{2\pi k}{k_2}\right)\right)}sin\left(\alpha tsin\left(\frac{2\pi k}{k_2}\right)+\frac{\pi k(2n-1)}{k_2}\right)}{sin\left(\frac{\pi k}{k_2}\right)}.$$

We also consider the cost function during the process of restoring order (k_1,k_2), when the operating time of the periodic part of the process is distributed according to the Erlang law of order m with a parameter α.

Let $\Phi_j(t)=F_{e,m,\alpha}(t)$. We find $H\Phi^{(k_2)}(t)$. Let us write down the integral equations for $HF_{e,m,\alpha}(t)$, $H\Phi^{(k_2)}(t)$

$$HF_{e,m,\alpha}\left(t\right)=F_{e,m,\alpha}\left(t\right)+\int_0^t HF_{e,m,\alpha}\left(t-x\right)dF_{e,m,\alpha}\left(x\right) \quad (15)$$

$$H\Phi^{(k_2)}\left(t\right)=\Phi^{(k_2)}\left(t\right)+\int_0^t H\Phi^{(k_2)}\left(t-x\right)d\Phi^{(k_2)}\left(x\right). \quad (16)$$

Let there be given

$$F^*\left(s\right)=\int_0^\infty e^{-st}dF\left(x\right)$$

Laplace-Stieltjes transform function $F\left(x\right)$ [1,6]. Considering $F^*_{e,m,\alpha}\left(s\right)=(\frac{\alpha}{s+\alpha})^m$, $(F_i*F_j)^*\left(s\right)=F_i^*\left(s\right)F_j^*\left(s\right)$, from (15),(16) we obtain

$$H^*F_{e,m,\alpha}\left(s\right)=(\frac{\alpha}{s+\alpha})^m+H^*F_{e,m,\alpha}\left(s\right)(\frac{\alpha}{s+\alpha})^m, \quad (17)$$

$$H^*\Phi^{(k_2)}\left(s\right)=(\frac{\alpha}{s+\alpha})^{mk_2}+H^*\Phi^{(k_2)}\left(s\right)(\frac{\alpha}{s+\alpha})^{mk_2}. \quad (18)$$

Comparing (17), (18), we conclude that

$$H\Phi^{(k_2)}\left(t\right)=HF_{e,mk_2,\alpha}\left(t\right).$$

We found that the restoration function of a simple restoration process formed by k_2 multiple convolution of Erlang order m distributions with the parameter α is the restoration function of a simple restoration process formed by an Erlang order mk_2 distribution with the same parameter α.

We have

$$HF_{e,mk_2,\alpha}\left(t\right)=\frac{1}{mk_2}(\alpha t+\sum_{k=1}^{mk_2-1}1\frac{c^k}{1-c^k}\left(\ -e^{-\alpha t\left(1-c^k\right)}\right),$$

$$c=e^{\frac{2\pi}{mk_2}i}=cos\left(\frac{2\pi}{mk_2}\right)+isin\left(\frac{2\pi}{mk_2}\right),$$

$$HF_{e,mk_2,\alpha}\left(t\right)=\frac{1}{mk_2}\left(\alpha t-\frac{mk_2-1}{2}+\frac{1}{2}\sum_{k=1}^{mk_2-1}.\frac{e^{-\alpha t\left(1-cos\left(\frac{2\pi}{mk_2}k\right)\right)}sin\left(\alpha tsin\left(\frac{2\pi}{mk_2}k\right)+\frac{\pi}{mk_2}k\right)}{sin\left(\frac{\pi}{mk_2}k\right)}\right)$$

Now in accordance with (11)

$$S\left(t\right)=c_0+\sum_{n=1}^{k_1-1}c_nF^{(n)}\left(t\right)+\sum_{n=k_1}^{k_1+k_2-1}c_n\int_0^t F^{(k_1-1)}\left(t-x\right)dF_{e,mn,\alpha}\left(x\right)+$$

$$+\sum_{n=k_1}^{k_1+k_2-1}c_n\int_0^t HF_{e,mk_2,\alpha}\left(t-x\right)e^{-\alpha x}\alpha\frac{(\alpha x)^{mn-1}}{\left(mn-1\right)!}dx \quad (19)$$

Integral

$$\int_0^t HF_{e,mk_2,\alpha}\left(t-x\right)e^{-\alpha x}\alpha\frac{(\alpha x)^{mn-1}}{\left(mn-1\right)!}dx$$

in (19) it is calculated similarly to the previous example with replacing k_2 by mk_2 and n by mn.

We also note that if additionally $F_i\left(t\right)=F_{e,l,\beta}\left(t\right)$, i=1,2,…, k_1-1, then $F^{(n)}\left(t\right)=F_{e,nl,\beta}\left(t\right)$, n=1,2,…, k_1-1 and in accordance with (19)

$$S\left(t\right)=c_0+\sum_{n=1}^{k_1-1}c_nF_{e,nl,\beta}\left(t\right)+\sum_{n=k_1}^{k_1+k_2-1}c_n\int_0^t F_{e,(k_1-1)l,\beta}\left(t-x\right)dF_{e,mn,\alpha}\left(x\right)+$$

$$+\sum_{n=k_1}^{k_1+k_2-1} c_n \int_0^t HF_{e,mk_2,\alpha}\left(t-x\right)e^{-\alpha x}\alpha\frac{(\alpha x)^{mn-1}}{(mn-1)!}dx$$

Conclusion

The most important performance indicators of technical and many other systems are random variables [17]. These are, for example, the operating time of the restored elements before failure, the number of failures and the cost of restoration during the restoration process. In the mathematical theory of reliability, when studying restoration processes, the numerical characteristics of these quantities are first considered, for example, the average and dispersion of the number of failures and the cost of restoration, through which various criteria for the optimality of restoration strategies are determined.

Considering that the recovery function for the model under consideration is well studied, the work obtained a solution to the integral equation for the cost function through the recovery function of a simple process specified by the convolution of all distribution functions of the periodic part. As a practical example, explicit formulas for the cost function are obtained for the restoration process, in which the periodic part is distributed according to an exponential law or Erlang law of order m with the same property α.

Note that the resulting formulas make it possible to study the properties of the cost function and consider various optimization problems in strategies for carrying out the restoration process in terms of price, quality, and risk. If, for example, we take the average number of failures as quality, the average cost of restorations as price, and the variance of the number of failures or the cost of restorations as risk.

We also note that, along with the obtained formulas for calculating the cost function, limit theorems for the cost of restorations (as a random variable), similar to those for the number of failures [3], as well as finding the dispersion of the cost of restorations in the models under consideration will also be important.

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