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Один класс решений уравнений идеальной пластичности

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Исследованию и решению нелинейных дифференциальных уравнений в современной математической литературе уделяется большое внимание. Несмотря на это, методов исследования и решения таких уравнений не так много. Это точечные и контактные преобразования уравнений, различные методы разделения переменных, метод дифференциальных связей, поиски различных симметрий и их использование для построения решений, а также законы сохранения. В работе рассмотрено нелинейное дифференциальное уравнение, описывающее пластическое течение призматического стержня. Для этого уравнения найдена группа точечных симметрий. Вычислена оптимальная система одномерных подалгебр. Приведены законы сохранения, соответствующие нетеровским симметриям, а также показано, что законов сохранения не нетеровских бесконечно много. Построены несколько новых инвариантных решений ранга один, т. е. зависящих от одной независимой переменной. Показано, как из двух точных решений, переходя к линейному уравнению, можно построить классы новых решений. Таким образом, в данной работе используются практически все методы современного исследования нелинейных дифференциальных уравнений.

Ключевые слова: нелинейное дифференциальное уравнение идеальной пластичности, точечные симметрии, точные решения.

One class of solutions to the equations of ideal plasticity

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Much attention is given to the study and solution of nonlinear differential equations in the modern mathematical literature. Despite this, there are not many methods for researching and solving such equations. These are point and contact transformations of equations, various methods of separating variables, the method of differential connections, the search for various symmetries and their use to construct solutions, as well as conservation laws. The paper considers a nonlinear differential equation describing the plastic flow of a prismatic rod. A group of point symmetries is found for this equation. The optimal system of one-

dimensional subalgebras is calculated. Conservation laws corresponding to Noetherian symmetries are given, and it is also shown that there are infinitely many non-Noetherian conservation laws. Several new invariant solutions of rank one, i. e. depending on one independent variable, are constructed. It is shown how classes of new solutions can be constructed from two exact solutions, passing to a linear equation. Thus, in this short article, almost all methods of modern research of nonlinear differential equations are involved.

Keywords: nonlinear differential equation of ideal plasticity, point symmetries, exact solutions.

Introduction

Solving and studying differential equations is still one of the most important problems of modern mathematics. In linear differential and integro-differential equations, questions of solvability of mixed nonlocal boundary value and inverse problems containing real parameters and differential operators of mathematical physics are studied [1; 2].

Exact solutions for nonlinear differential equations are known only in exceptional cases. To search for them, methods of generalized separation of variables, methods of group analysis, the method of differential connections and some others are used. A large list of solved equations and a review of methods for solving them are given in the fundamental work [3]. Recently, conservation laws have begun to be used to solve boundary value problems for nonlinear differential equations [4–7]. Previously, they most often played a supporting role. Methods of using group analysis to various equations that arise in physics and mechanics can be seen in works [8–15] .

Formulation of the problem

In [1], a solution is given that describes the purely plastic stress state of a prismatic rod

$$\begin{aligned} u &= \frac{1}{4}A(y^2 - x^2 - 2z^2) - \frac{1}{2}Bxy - \frac{1}{2}Cx + Dyz, \\ v &= \frac{1}{4}B(-y^2 + x^2 - 2z^2) - \frac{1}{2}Axy - \frac{1}{2}y + Dxz, \\ w &= \psi(x, y) + Axz + Byz + Cz, \end{aligned} \tag{1}$$

where A, B, C, D are constants, $\psi(x, y)$ is function determined from a system of equations.

$$\frac{\partial\psi}{\partial x} = -Dy \mp \frac{\sqrt{3}(Ax + By + C)f_y}{\sqrt{1 - f_x^2 - f_y^2}}, \quad \frac{\partial\psi}{\partial y} = Dy \pm \frac{\sqrt{3}(Ax + By + C)f_x}{\sqrt{1 - f_x^2 - f_y^2}}. \tag{2}$$

The condition for the compatibility of these relations gives a second-order elliptic equation

$$\frac{\partial}{\partial x}\left(\frac{(Ax + By + C)f_x}{\sqrt{1 - f_x^2 - f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{(Ax + By + C)f_y}{\sqrt{1 - f_x^2 - f_y^2}}\right) \pm \frac{2D}{\sqrt{3}} = 0. \tag{3}$$

In this case, the components of the stress tensor $\sigma_x, \sigma_y, \tau_{xy}$ are identically equal to zero, and

$$\sigma_z = \pm\sqrt{3}k\sqrt{1 - f_x^2 - f_y^2}, \tau_{xz} = -kf_y, \tau_{yz} = kf_x, \tag{4}$$

where k is plastic constant.

The purpose of the work is to study some properties of equation (3) and construct its solution, provided that

$$A = B = 0.$$

In this case, we obtain the following nonlinear differential equation

$$\partial_x\left(\frac{f_x}{\sqrt{1-f_x^2-f_y^2}}\right)+\partial_y\left(\frac{f_y}{\sqrt{1-f_x^2-f_y^2}}\right)=K, \quad K=\mp\frac{2D}{\sqrt{3}}. \tag{5}$$

The index below means differentiation by the corresponding argument; all functions are assumed to be smooth.

Some properties of the equation (5) where $K=0$.

1. Equation (5) can be derived from the variational principle, and it is the minimum of the functional

$$Z(w)=\iint\sqrt{w_x^2+w_y^2}\,dxdy=\iint L dxdy.$$

2. The group of point symmetries of equation (5) is generated by the following operators:

$$X_1=\partial_x,\ X_2=\partial_y,\ X_3=\partial_f,\ X_4=x\partial_x+y\partial_y+f\partial_f,\ X_5=y\partial_x-x\partial_y. \tag{6}$$

To construct various invariant solutions, it is necessary to construct an optimal system of subalgebras. For the Lie algebra generated by operators (6), it has the form

$$X_1+\alpha X_3, \quad \alpha X_3+X_5, \quad X_5+\alpha X_4, X_3, X_4, \tag{7}$$

where α is arbitrary constant. Different values of this constant correspond to dissimilar subalgebras.

Уравнение (5) приведем к виду

$$(1-f_y^2)f_{xx}+2f_xf_yf_{xy}+(1-f_x^2)f_{yy}=0. \tag{8}$$

3. *Definition*. Let us call the conservation law for equation (8) an expression of the form

$$A_x+B_y=\Delta[(1-f_y^2)f_{xx}+2f_xf_yf_{xy}+(1-f_x^2)f_{yy}]=0, \tag{9}$$

where Δ is linear differential operator that is not identically equal to zero.

For equations derived from the variational principle, each operator admitted by the equation corresponds, according to Noether's theorem [3], to a certain conservation law. Let us use this theorem for equation (5). We get five conservation laws.

The conservation law corresponds to the operator X_1

$$D_x\left(L-f_x\frac{\partial L}{\partial f_x}\right)+D_y\left(-f_y\frac{\partial L}{\partial f_y}\right)=0\,.$$

The conservation law corresponds to the operator X_2

$$D_x\left(-f_y\frac{\partial L}{\partial f_x}\right)+D_y\left(L-f_y\frac{\partial L}{\partial f_y}\right)=0.$$

The conservation law corresponds to the operator X_3

$$D_x\left(\frac{f_x}{\sqrt{1-f_x^2-f_y^2}}\right)+D_y\left(\frac{f_y}{\sqrt{1-f_x^2-f_y^2}}\right)=0.$$

The conservation law corresponds to the operator X_4

$$D_x\left(Lx+(f-xf_x-yf_y)\frac{\partial L}{\partial f_x}\right)+D_y\left(Ly+(f-xf_x-yf_y)\frac{\partial L}{\partial f_y}\right)=0.$$

The conservation law corresponds to the operator X_5

$$D_x\left(Ly+(yf_x-xf_y)\frac{\partial L}{\partial f_x}\right)+D_y\left(-Lx+(yf_x-xf_y)\frac{\partial L}{\partial f_y}\right)=0.$$

Note that equation (8) has other conservation laws that are different from the previous laws. Let's point out some. Let $A(f_x, f_y)$, $B(f_x, f_y)$, then from (9) we have

$$\frac{\partial A}{\partial f_x} f_{xx} + \frac{\partial A}{\partial f_y} f_{xy} + \frac{\partial B}{\partial f_x} f_{xy} + \frac{\partial B}{\partial f_y} f_{yy} = \Delta[(1-f_y^2) f_{xx} + 2 f_x f_y f_{xy} + (1-f_x^2) f_{yy}] = 0.$$

From here we easily obtain two equations for determining the conserved current

$$(1-f_x^2)\frac{\partial A}{\partial f_x} = (1-f_y^2)\frac{\partial B}{\partial f_y}, \quad \frac{\partial A}{\partial f_y} + (1-f_x^2)\frac{\partial B}{\partial f_x} = 2 f_x f_y \frac{\partial B}{\partial f_y}.$$

It follows that equation (8) admits an infinite series of conservation laws. This follows, in particular, from the linearity of the reduced system with respect to a conserved current.

4. The Legendre transformation allows us to linearize equation (8) and bring it to the form

$$(1-\eta^2) w_{\eta\eta} - 2\xi\eta w_{\xi\eta} + (1-\xi^2) w_{\xi\xi} = 0.$$

This transformation is determined by the relations

$$f_x = \xi, f_y = \eta, x = f_\xi, y = f_\eta, w + f = x\xi + y\eta.$$

Exact solutions (5). All these solutions are invariant solutions built on subalgebras (7) :

a) We look for a solution to equation (8) in the form

$$f = g(x) + h(y). \tag{10}$$

Substituting (10) into (8), we get

$$(1-h'^2) g'' + (1-g'^2) h'' = 0. \tag{11}$$

Here the prime means the derivative with respect to the corresponding argument.

From (11) we obtain

$$\frac{h''}{1-h'^2} = -\frac{g''}{1-g'^2} = \lambda - \text{const}. \tag{12}$$

Integrating (12), we obtain

$$\frac{1}{2}\ln\left|\frac{1+h'}{1-h'}\right| = 2\lambda + \ln C_1, \quad \frac{1}{2}\ln\left|\frac{1+g'}{1-g'}\right| = -2\lambda + \ln C_2. \tag{13}$$

Integrating (13), we obtain

$$h = -x + \frac{1}{\lambda}\ln(1 + C_1 \exp 2\lambda x) + C_3, \quad g = -y - \frac{1}{\lambda}\ln(1 + C_2 \exp(-2\lambda y)) + C_4. \tag{14}$$

Considering $C_1 = C_2 = 1$ solution (10) can be written in a more convenient form

$$h' = th\lambda x, g' = th\lambda y, h = \ln ch\lambda x, g = -\ln ch\lambda y, \quad f = \ln\frac{ch\lambda x}{ch\lambda y};$$

b) we write equation (5) with $K = 0$ in polar coordinate system r, θ. We have

$$\frac{\partial}{\partial r}\left(\frac{r^2 f_r}{\sqrt{r^2 - r^2 f_r^2 - f_\theta^2}}\right) + \frac{\partial}{\partial \theta}\left(\frac{f_\theta}{\sqrt{r^2 - r^2 f_r^2 - f_\theta^2}}\right) = 0. \tag{15}$$

We are looking for a solution to this equation in the form $f = rf(t), t = r\exp(\alpha\theta), \alpha - \text{const}$.

From (15) we obtain an ordinary differential equation of the second order, which could be integrated only in the case when $\alpha = 0$. In this case we get

$$f = \ln\left|r + \sqrt{r^2 + c^2}\right| = \text{Arcsh}\frac{r}{c}, \quad c - \text{const}; \tag{16}$$

c) we look for a solution to equation (5) in the form

$$f = \alpha y + \varphi(x).$$

substituting this relation into (5) we get

$$\varphi = \frac{\sqrt{1-\alpha^2}}{K}\sqrt{1+(Kx+C)^2}\ ;$$

d) we look for a solution to equation (5) in the form

$$f = \alpha\theta + \varphi(r).$$

substituting this relation into (5), we obtain

$$\varphi = K\int\sqrt{\frac{r^2-\alpha^2}{r^2+K^2}}\frac{dr}{r}\ ;$$

e) we look for a solution to equation (5) in the form

$$f = r\varphi(\theta).$$

substituting this relation into (5) we get

$$f = r\sin\sqrt{1+K^2\theta^2}\,.$$

Homotopy of two solutions to equation (5)

We will show how from two solutions (14) and (16) one can obtain a whole series of new exact solutions to equation (5). To do this, we apply the Legendre transformation to these solutions.

The solution to the Legendre equation U corresponds to the solution(we define it as F), which is determined by the formula

$$F_x = \xi,\ F_y = \eta,\ x = F_\xi,\ y = F_\eta,\ U + F = x\xi + y\eta.$$

From here we get

$$U = -F + x\xi + y\eta = -F + xF_x + yF_y = \ln(r+\sqrt{r^2+c^2}) + \frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}. \tag{17}$$

Finally we get

$$U = \ln(r+\sqrt{r^2+c^2}) + \sqrt{x^2+y^2}\,.$$

We will do the same with solution (14), denoting it by G, and we denote its image via V. We have

$$V = -G + x\xi + y\eta = -G + xG_x + yG_y = -x - y +$$
$$+\frac{1}{2\lambda}\ln\frac{1+C_1\exp 2\lambda x}{1+C_2(-2\lambda y)} + x\frac{C_1\exp 2\lambda x - 1}{C_1\exp 2\lambda + 1} + y\frac{C_2\exp(-2\lambda y)-1}{C_2\exp(-2\lambda y)+1}. \tag{18}$$

The Legendre equation corresponding to equation (7) is linear, so for it we have

$$w = aU + (1-a)V, \tag{19}$$

where a is an arbitrary constant is again a solution to the same equation. At the same time $a = 0$ the solution coincides with the solution V, and with $a = 1$ coincides with the solution U. It follows that formula (19) allows one to continuously transform solution (16) into solution (14) by changing a from zero to one. The action algorithm is as follows: we write solutions (17) and (18) in variables ξ, η, we add them according to formula (19), and then write their linear combination in terms x, y. Thus, we obtain a series of solutions to equation (7) for each fixed value a.

Conclusion

In the work, a group of point transformations is found that are allowed by equation (5) in the Lie–Ovsyannikov sense. This group has dimension five. It is generated by three translations in spatial variables and the desired function, stretching in these same variables, circular rotation in the plane *OXY*. New classes of exact solutions of this equation are found, depending on arbitrary functions from the class C^2. Based on point symmetries, four conservation laws for equation (5) are found. A new infinite series of conservation laws is presented, which was found by direct calculation. The new solutions obtained complement other invariant solutions given in [1].

All solutions constructed in this work can be used to describe the stress-strain state of a straight rod subjected to tension along the axis *OZ* and torsion around this axis by a pair of forces.

Библиографические ссылки

1. Предельное состояние деформируемых тел и горных пород / Д. Д. Ивлев, Л. А. Максимова, Р. И. Непершин и др. М. : Физматлит, 2008. 829 с.

2. Овсяников Л. В. Групповой анализ дифференциальных уравнений. М. : Наука, 1978. 400 с.

3. Polyanin A. D., Zaitsev V. F. Handbook of nonlinear partial differential equations. 2nd Edition, New York : Taylor&Francis Group, 2012. 1912 p.

4. Сенашов С. И., Черепанова О. Н. Новые классы решений уравнения минимальных поверхностей // Journal of Siberian Federal University. Math.&Phys. 2010. Vol. 3(2). P. 248–255.

5. Senashov S. I., Gomonova O. V. Construction of Elastoplastic Boundary in Problem of Tension of a Plate Weakened by Holes // Intern. J Non. lin Mech. 2019. Vol. 108. P. 7–10.

6. Kaptsov E. I., Meleshko S. V. Conservation laws of the two-dimensional gas dynamics equations // International Journal of Non-Linear Mechanics. 2019, Vol. 112. P. 126–132.

7. Nakpim W., Meleshko S. V. Conservation laws of the one-dimensional equations of relativistic gas dynamics in lagrangian coordinates // International Journal of Non-Linear Mechanics. 2020. Vol. 124. P. 103496.

8. Vaneeva O. O., Popovich R.O., Sopocleus C. Extend group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities // J. Math. Anal. 2012. Vol. 396. P. 225–242.

9. Grigoriev Yu. N., Omel'aynchuk M. I. Qualitive properties of a certain kinetic problem of binary gas // Sib.Math. J. 2005. Vol. 46(5). P. 813–825.

10. Grigoriev Yu. N., Meleshko S. V., Suriyawichitseranee A. A. On group classification of the spatially homogeneous and isotropic Boltzmann equation with source II // Int. J. Non-Linear Mech. 2014. Vol. 61. P. 15–18.

11. Сенашов С. И., Савостьянова И. Л. Новые трехмерные пластические течения, соответствующие однородному напряженному состоянию // Сиб. журн. индуст. матем. 2019. Т. 22, № 3. С. 114–117.

12. Meleshko S. V. Complete group classification of the two-dimensional shallow water equations with constant coriolis parameter in lagrangian coordinates // Communications in Nonlinear Science and Numerical Simulation. 2020. Vol. 89. P. 105293.

13. Meleshko S. V., Samatova N. F. Group classification of the two-dimensional shallow water equations with the beta-plane approximation of coriolis parameter in lagrangian coordinates // Communications in Nonlinear Science and Numerical Simulation. 2020. Vol. 90. P. 105337.

14. Bobylev A. V., Meleshko S. V. Group analysis of the generalized burnett equations // Journal of Nonlinear Mathematical Physics. 2020. Vol. 27, No. 3. P. 494–508.

15. Siriwat P., Grigoriev Y. N., Meleshko S. V. Invariant solutions of one-dimensional equations of two-temperature relaxation gas dynamics // Mathematical Methods in the Applied Sciences. 2020. Vol. 43, No. 5. P. 2444–2457.

References

1. Ivlev D. D. *Predel'noe sostoyanie deformiruemykh tel i gornykh porod* [The limiting state of deformable bodies and rocks]. Moscow, Fizmatlit Publ., 2008, 829 p.

2. Ovsyannikov L. V. *Gruppovoy analiz differentsial'nykh uravneniy* [Group analysis of differential equations]. Moscow, Nauka Publ., 1978, 400 p.

3. Polyanin A. D., Zaitsev V. F. Handbook of nonlinear partial differential equations. 2nd Edition, New York, Taylor&Francis Group, 2012, 1912 p.

4. Senashov S. I., Cherepanova O. N. [New classes of solutions to the equation of minimal surfaces]. *Journal of Siberian Federal University. Math.&Phys*. 2010, Vol. 3(2), P. 248–255.

5. Senashov S. I., Gomonova O. V. Construction of Elastoplastic Boundary in Problem of Tension of a Plate Weakened by Holes. *Intern. J Non. lin Mech*. 2019, Vol. 108, P. 7–10.

6. Kaptsov E. I., Meleshko S. V. Conservation laws of the two-dimensional gas dynamics equations. *International Journal of Non-Linear Mechanics*. 2019, Vol. 112, P. 126–132.

7. Nakpim W., Meleshko S. V. Conservation laws of the one-dimensional equations of relativistic gas dynamics in lagrangian coordinates. *International Journal of Non-Linear Mechanics*. 2020, Vol. 124, P. 103496.

8. Vaneeva O. O., Popovich R. O., Sopocleus C. Extend group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities. *J. Math. Anal*. 2012, Vol. 396, P. 225–242.

9. Grigoriev Yu. N., Omel'aynchuk M. I. Qualitive properties of a certain kinetic problem of binary gasю *Sib.Math. J.* 2005, Vol. 46(5), P. 813–825.

10. Grigoriev Yu. N., Meleshko S. V., Suriyawichitseranee A. A. On group classification of the spatially homogeneous and isotropic Boltzmann equation with source II. *Int. J. Non-Linear Mech*. 2014, Vol. 61, P. 15–18.

11. Senashov S. I., Savostyanova I. L. [New three-dimensional plastic flows corresponding to a homogeneous stress state] *Sib. zhurn. indust. matem*. 2019, Vol. 22, No. 3, P. 114–117. (In Russ.)

12. Meleshko S. V. Complete group classification of the two-dimensional shallow water equations with constant coriolis parameter in lagrangian coordinates. *Communications in Nonlinear Science and Numerical Simulation*. 2020, Vol. 89, P. 105293.

13. Meleshko S. V., Samatova N. F. Group classification of the two-dimensional shallow water equations with the beta-plane approximation of coriolis parameter in lagrangian coordinates. *Communications in Nonlinear Science and Numerical Simulation*. 2020, Vol. 90, P. 105337.

14. Bobylev A. V., Meleshko S. V. Group analysis of the generalized burnett equations. *Journal of Nonlinear Mathematical Physics*. 2020, Vol. 27, No. 3, P. 494–508.

15. Siriwat P., Grigoriev Y. N., Meleshko S. V. Invariant solutions of one-dimensional equations of two-temperature relaxation gas dynamics. *Mathematical Methods in the Applied Sciences*. 2020, Vol. 43, No. 5, P. 2444–2457.

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