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Аналитическое решение задачи калибровки магнитометра космического аппарата с помощью метода наименьших квадратов

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Аннотация. В настоящей работе предложен аналитический метод решения задачи калибровки магнитометра для модели, рассмотренной в [1]. При решении задачи определения калибровочных параметров блока магнитометра учитывается, что для измерений с любой пространственной ориентацией блока магнитометра величина измеряемого вектора магнитной индукции сохраняется и является известной модельной величиной. Вводится в рассмотрение штрафная функция 12 переменных, равная сумме квадратов невязок. Алгоритм решения задачи калибровки измерительных осей блока магнитометра сводится к поиску методом наименьших квадратов таких значений переменных этой функции, которые при заданном наборе векторов измерений магнитометра доставляют ей минимум. С этой целью указанная функция исследуется на условный экстремум при наличии трех уравнений связи. Составляется функция Лагранжа и, исходя из необходимого условия экстремума этой функции, формируется система 15 уравнений относительно 15 неизвестных. Доказывается, что эта система имеет четыре решения. Выведены формулы, позволяющие получить компоненты каждого из этих четырех решений. В качестве решения задачи калибровки магнитометра рекомендуется выбрать решение указанной системы, доставляющее минимум функции Лагранжа.

Ключевые слова: калибровка магнитометра, вектор магнитной индукции, метод наименьших квадратов, условный экстремум функции нескольких переменных, функция Лагранжа.

Analytical solution of the spacecraft magnetometer calibration problem using the method of least squares

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Abstract. In this paper, an analytical method is proposed for solving the problem of magnetometer calibration for the model considered in [1]. When solving the problem of determining the calibration parameters of the magnetometer unit, it is taken into account that for measurements with any spatial orientation of the magnetometer unit, the value of the measured magnetic induction vector is preserved and is a known model value. A penalty function of 12 variables equal to the sum of the squares of the residuals is introduced into consideration. The algorithm for solving the problem of calibrating the measuring axes of the magnetometer unit is reduced to searching, by the method of least squares, for such values of the variables of this function that, for a given set of magnetometer measurement vectors, provide it with a minimum. For this purpose, the specified function is examined for a conditional extremum in the presence of three equality constraints. The Lagrangian function is compiled and, based on the necessary condition for the extremum of this function, the system of 15 equations in the 15 variables is formed. It is proved that the system has four solutions. Formulas are derived that make it possible to obtain the components of each of these four solutions. As a solution to the magnetometer calibration problem, it is recommended to choose a solution to the specified system that provides a minimum of the Lagrangian function.

Keywords: magnetometer calibration, the Earth's magnetic induction vector, the method of least squares, constrained optimization, Lagrangian function.

Introduction

Magnetometers are part of the orientation and stabilization system of low-orbit small-sized spacecraft. They are widely used due to the fact that they are lightweight, inexpensive and reliable. However, due to the physical properties of the sensitive element, modern magnetometers require mathematical calibration. At present, various methods for calibrating magnetometers have been proposed, and a considerable number of scientific papers are devoted to these methods [1–12], in particular, article [9], which provides an overview of various methods for performing such operations.

In the above-mentioned works, the problem of calibrating the magnetometer of a spacecraft was solved using numerical methods. In this article, an analytical method for solving this problem for the model considered in [1] is proposed.

1. Model of errors in measurements of magnetic induction vector

We denote by **h** the value of the measured magnetic induction vector at a certain spatial position of the magnetometer unit (MU). We use the simplified measurement model considered in [1]:

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = QP\mathbf{B} + \mathbf{b} + \mathbf{n}. \tag{1}$$

The following notations [1] are used in(1):

$\mathbf{B} = (B_1, B_2, B_3)^{\mathrm{T}}$ is a true magnetic induction vector ;

$\mathbf{b} = (b_1, b_2, b_3)^{\mathrm{T}}$ is a constant vector corresponding to zero offsets for each of the measuring axes of the MU ;

n is a random vector corresponding to uncorrelated noise for each of the measurement axes ;

P is a matrix, the rows of which are the unit vectors of the measuring axes of the MU, written in the "base" MU system ;

Q is a diagonal matrix containing on the main diagonal the scaling factors k_1, k_2, k_3 for measuring axes MU i.e.

$$Q = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}.$$

In other words, the matrix P describes the non-orthogonality of the MU measurement axes, and the matrix Q corresponds to scaling along these axes.

The task of calibrating the MU measuring axes comes down to finding the elements of the matrices P and Q, as well as the zero offset vector $\mathbf{b}$.

2. Development of an algorithm for determining the calibration parameters of the MU

When solving the problem of MU determining the calibration parameters, we will use the fact that for measurements with any MU spatial orientation, the value of the measured magnetic induction vector $\mathbf{B}$ is preserved and is a known model value.

We will assume that in flight, as a result of magnetometer measurements at discrete moments in time, a set of vectors is obtained $\mathbf{h}^{(l)} = (h_1^{(l)}, h_2^{(l)}, h_3^{(l)})^{\mathrm{T}}$, $l = 1, \ldots, N$. Provided there is no measurement noise, from formula (1) we obtain:

$$\mathbf{h}^{(l)} = QP\mathbf{B}^{(l)} + \mathbf{b}, \quad l = 1, \ldots, N, \tag{2}$$

where $\mathbf{B}^{(l)} = (B_1^{(l)}, B_2^{(l)}, B_3^{(l)})^{\mathrm{T}}$ is the true vector of magnetic induction at the same point in space as the measured vector $\mathbf{h}^{(l)}$, $l = 1, \ldots, N$. We express vectors $\mathbf{B}^{(l)}$ ($l = 1, \ldots, N$) from equalities (2):

$$\mathbf{B}^{(l)} = S\left(\mathbf{h}^{(l)} - \mathbf{b}\right), \tag{3}$$

$l = 1, \ldots, N$, where

$$S = \left(QP\right)^{-1} = P^{-1}Q^{-1}. \tag{4}$$

As noted in [1], without loss of generality, the non-orthogonality matrix P can be represented with a minimum number of unknown elements as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \sin\varepsilon_1 & \cos\varepsilon_1 & 0 \\ \sin\varepsilon_2 & \cos\varepsilon_2\sin\varepsilon_3 & \cos\varepsilon_2\cos\varepsilon_3 \end{pmatrix},$$

where ε_1, ε_2, ε_3 are small angles. Then the inverse matrix P^{-1} will take the form

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_1 & \beta_1 & 0 \\ \alpha_1\alpha_3 - \alpha_2\beta_3 & -\beta_1\alpha_3 & \beta_2\beta_3 \end{pmatrix},$$

where $\alpha_i = \mathrm{tg}\,\varepsilon_i$, $\beta_i = \sec\varepsilon_i$, $i = 1, 2, 3$, and since

$$Q^{-1} = \begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & k_3^{-1} \end{pmatrix},$$

then in accordance with (4) we obtain:

$$S = \begin{pmatrix} \gamma_1 & 0 & 0 \\ -\alpha_1\gamma_1 & \beta_1\gamma_2 & 0 \\ \left(\alpha_1\alpha_3 - \alpha_2\beta_3\right)\gamma_1 & -\beta_1\alpha_3\gamma_2 & \beta_2\beta_3\gamma_3 \end{pmatrix}, \tag{5}$$

where $\gamma_i = k_i^{-1}$, $i = 1, 2, 3$.

We rewrite equality (3) taking into account (5):

$$\begin{pmatrix} B_1^{(l)} \\ B_2^{(l)} \\ B_3^{(l)} \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ -\alpha_1\gamma_1 & \beta_1\gamma_2 & 0 \\ \left(\alpha_1\alpha_3 - \alpha_2\beta_3\right)\gamma_1 & -\beta_1\alpha_3\gamma_2 & \beta_2\beta_3\gamma_3 \end{pmatrix} \begin{pmatrix} h_1^{(l)} - b_1 \\ h_2^{(l)} - b_2 \\ h_3^{(l)} - b_3 \end{pmatrix}, \tag{6}$$

$l = 1, \ldots, N$. We write each of the N vector equalities (6) as a system of three scalar equalities:

$$\begin{cases} \gamma_1\left(h_1^{(l)}-b_1\right)=B_1^{(l)}, \\ -\alpha_1\gamma_1\left(h_1^{(l)}-b_1\right)+\beta_1\gamma_2\left(h_2^{(l)}-b_2\right)=B_2^{(l)}, \\ \left(\alpha_1\alpha_3-\alpha_2\beta_3\right)\gamma_1\left(h_1^{(l)}-b_1\right)-\beta_1\alpha_3\gamma_2\left(h_2^{(l)}-b_2\right)+\beta_2\beta_3\gamma_3\left(h_3^{(l)}-b_3\right)=B_3^{(l)}, \end{cases} \tag{7}$$

$l = 1, \ldots, N$. Due to the first equality of system (7), we rewrite the second equality of this system in the form

$$-\alpha_1 B_1^{(l)}+\beta_1\gamma_2\left(h_2^{(l)}-b_2\right)=B_2^{(l)},$$

from which we obtain the equality

$$\beta_1\gamma_2\left(h_2^{(l)}-b_2\right)=\alpha_1 B_1^{(l)}+B_2^{(l)},$$

as a result of which the third equality of system (7) is written as follows:

$$-\alpha_2\beta_3 B_1^{(l)}-\alpha_3 B_2^{(l)}+\beta_2\beta_3\gamma_3\left(h_3^{(l)}-b_3\right)=B_3^{(l)},$$

$l = 1, \ldots, N$.

We introduce into consideration a function of twelve variables α_i, β_i, γ_i, b_i ($i = 1, 2, 3$)

$$\Phi=\sum_{l=1}^{N}\left\{\begin{aligned}&\left[B_1^{(l)}-\gamma_1\left(h_1^{(l)}-b_1\right)\right]^2+\left[B_2^{(l)}+\alpha_1 B_1^{(l)}-\beta_1\gamma_2\left(h_2^{(l)}-b_2\right)\right]^2+\\&+\left[B_3^{(l)}+\alpha_2\beta_3 B_1^{(l)}+\alpha_3 B_2^{(l)}-\beta_2\beta_3\gamma_3\left(h_3^{(l)}-b_3\right)\right]^2\end{aligned}\right\},$$

where the variables α_i и β_i are related by the trigonometric identity

$$\mathrm{tg}^2\varphi\equiv\sec^2\varphi-1$$

by the equalities

$$\alpha_i^{\,2}-\beta_i^{\,2}+1=0, \tag{8}$$

$i = 1, 2, 3$.

The algorithm for solving the problem of calibrating the MU measuring axes is reduced to searching by the least squares method [13] taking into account (8) such values of variables α_i, β_i, γ_i, b_i ($i = 1, 2, 3$), which, for a given set of measurement vectors, $\{\mathbf{h}^{(l)}\}$ ($l = 1, \ldots, N$) provide a minimum of the function F. For this purpose, it is necessary to investigate the function F for a conditional extremum [14] in the presence of three constraint equations (8).

We compose the Lagrange function

$$F=\Phi+\sum_{i=1}^{3}\lambda_i\left(\alpha_i^{\,2}-\beta_i^{\,2}+1\right) \tag{9}$$

fifteen variables α_i, β_i, γ_i, b_i, λ_i ($i = 1, 2, 3$) and we write down the necessary condition for the local extremum of this function :

$$
\begin{cases}
\dfrac{1}{2}\dfrac{\partial F}{\partial \gamma_1} = -\sum\limits_{l=1}^{N}\left[B_1^{(l)} - \gamma_1\left(h_1^{(l)} - b_1\right)\right]\left(h_1^{(l)} - b_1\right) = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial b_1} = \gamma_1\sum\limits_{l=1}^{N}\left[B_1^{(l)} - \gamma_1\left(h_1^{(l)} - b_1\right)\right] = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \alpha_1} = \sum\limits_{l=1}^{N}\left[B_2^{(l)} + \alpha_1 B_1^{(l)} - \beta_1\gamma_2\left(h_2^{(l)} - b_2\right)\right]B_1^{(l)} + \lambda_1\alpha_1 = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \beta_1} = -\gamma_2\sum\limits_{l=1}^{N}\left[B_2^{(l)} + \alpha_1 B_1^{(l)} - \beta_1\gamma_2\left(h_2^{(l)} - b_2\right)\right]\left(h_2^{(l)} - b_2\right) - \lambda_1\beta_1 = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \gamma_2} = -\beta_1\sum\limits_{l=1}^{N}\left[B_2^{(l)} + \alpha_1 B_1^{(l)} - \beta_1\gamma_2\left(h_2^{(l)} - b_2\right)\right]\left(h_2^{(l)} - b_2\right) = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial b_2} = \beta_1\gamma_2\sum\limits_{l=1}^{N}\left[B_2^{(l)} + \alpha_1 B_1^{(l)} - \beta_1\gamma_2\left(h_2^{(l)} - b_2\right)\right] = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \alpha_2} = \beta_3\sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right]B_1^{(l)} + \lambda_2\alpha_2 = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \beta_2} = -\beta_3\gamma_3\sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right]\left(h_3^{(l)} - b_3\right) - \lambda_2\beta_2 = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \gamma_3} = -\beta_2\beta_3\sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right]\left(h_3^{(l)} - b_3\right) = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial b_3} = \beta_2\beta_3\gamma_3\sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right] = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \alpha_3} = \sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right]B_2^{(l)} + \lambda_3\alpha_3 = 0,\\
\dfrac{1}{2}\dfrac{\partial F}{\partial \beta_3} = \sum\limits_{l=1}^{N}\left[B_3^{(l)} + \alpha_2\beta_3 B_1^{(l)} + \alpha_3 B_2^{(l)} - \beta_2\beta_3\gamma_3\left(h_3^{(l)} - b_3\right)\right]\left[\alpha_2 B_1^{(l)} - \beta_2\gamma_3\left(h_3^{(l)} - b_3\right)\right] - \lambda_3\beta_3 = 0,\\
\dfrac{\partial F}{\partial \lambda_i} = \alpha_i^{\,2} - \beta_i^{\,2} + 1 = 0, \quad i = 1,2,3.
\end{cases} \tag{10}
$$

It is required to find the stationary points of the function F, i.e. the solution of the system of equations (10). Let us introduce the following notations:

$$
C_i = \sum_{l=1}^{N} B_i^{(l)},\; H_i = \sum_{l=1}^{N} h_i^{(l)},\; F_i = \sum_{l=1}^{N}\left(h_i^{(l)}\right)^2,\; D_{ij} = \sum_{l=1}^{N} B_i^{(l)} h_j^{(l)},\; G_{ij} = G_{ji} = \sum_{l=1}^{N} B_i^{(l)} B_j^{(l)},\; i, j = 1,2,3.
$$

Taking into account these notations, for convenience we will divide the system of equations (10) into three nonlinear systems – a system of equations for two unknowns b_1, γ_1:

$$
\begin{cases}
D_{11} - C_1 b_1 + \left(-F_1 + 2H_1 b_1 - N b_1^{\,2}\right)\gamma_1 = 0,\\
\left[C_1 + \left(N b_1 - H_1\right)\gamma_1\right]\gamma_1 = 0,
\end{cases} \tag{11}
$$

a system of equations for five unknowns b_2, α_1, β_1, γ_2, λ_1:

$$
\begin{cases}
\left(C_1 b_2 - D_{12}\right)\beta_1\gamma_2 + \left(G_{11} + \lambda_1\right)\alpha_1 + G_{12} = 0,\\
\left[D_{22} - C_2 b_2 + \left(D_{12} - C_1 b_2\right)\alpha_1 + \left(-F_2 + 2H_2 b_2 - N b_2^{\,2}\right)\beta_1\gamma_2\right]\gamma_2 + \lambda_1\beta_1 = 0,\\
\left[D_{22} - C_2 b_2 + \left(D_{12} - C_1 b_2\right)\alpha_1 + \left(-F_2 + 2H_2 b_2 - N b_2^{\,2}\right)\beta_1\gamma_2\right]\beta_1 = 0,\\
\left[C_2 + C_1\alpha_1 + \left(N b_2 - H_2\right)\beta_1\gamma_2\right]\beta_1\gamma_2 = 0,\\
\beta_1 = \sqrt{\alpha_1^{\,2} + 1}
\end{cases} \tag{12}
$$

and a system of equations for eight unknowns. b_3, α_2, α_3, β_2, β_3, γ_3, λ_2, λ_3:

$$\begin{cases} \left[G_{13} + G_{11}\alpha_2\beta_3 + G_{12}\alpha_3 + \left(C_1 b_3 - D_{13}\right)\beta_2\beta_3\gamma_3\right]\beta_3 + \lambda_2\alpha_2 = 0, \\ \left[D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3\right)\alpha_2\beta_3 + \left(D_{23} - C_2 b_3\right)\alpha_3 + \left(-F_3 + 2H_3 b_3 - N b_3^2\right)\beta_2\beta_3\gamma_3\right]\beta_3\gamma_3 + \lambda_2\beta_2 = 0, \\ \left[D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3\right)\alpha_2\beta_3 + \left(D_{23} - C_2 b_3\right)\alpha_3 + \left(-F_3 + 2H_3 b_3 - N b_3^2\right)\beta_2\beta_3\gamma_3\right]\beta_2\beta_3 = 0, \\ \left[C_3 + C_1\alpha_2\beta_3 + C_2\alpha_3 + \left(N b_3 - H_3\right)\beta_2\beta_3\gamma_3\right]\beta_2\beta_3\gamma_3 = 0, \\ G_{23} + G_{12}\alpha_2\beta_3 + G_{22}\alpha_3 + \left(C_2 b_3 - D_{23}\right)\beta_2\beta_3\gamma_3 + \lambda_3\alpha_3 = 0, \\ \left[G_{13} + G_{11}\alpha_2\beta_3 + G_{12}\alpha_3 + \left(C_1 b_3 - D_{13}\right)\beta_2\beta_3\gamma_3\right]\alpha_2 - \lambda_3\beta_3 - \\ -\left[D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3\right)\alpha_2\beta_3 + \left(D_{23} - C_2 b_3\right)\alpha_3 + \left(-F_3 + 2H_3 b_3 - N b_3^2\right)\beta_2\beta_3\gamma_3\right]\beta_2\gamma_3 = 0, \\ \beta_i = \sqrt{\alpha_i^2 + 1}, \quad i = 2,3. \end{cases} \tag{13}$$

In writing the expressions β_i through α_i in the last equations of systems (12) and (13), we used the fact that $\beta_i = \sec \varepsilon_i > 0$ due to the obvious inequalities$-\pi/2 < \varepsilon_i < \pi/2$, $i = 1, 2, 3$. We solve each of the systems of equations (11), (12), (13).

First, we note that the inequality is valid

$$-F_i + 2H_i b_i - N b_i^2 \neq 0, \tag{14}$$

$i = 1, 2, 3$. Indeed, the discriminant of the quadratic equation is $-F_i + 2H_i b_i - N b_i^2 = 0$ with respect to the unknown b_i, being equal to

$$\delta_i = 4H_i^2 - 4NF_i = 4\left[\left(\sum_{l=1}^{N} h_i^{(l)}\right)^2 - N\sum_{l=1}^{N}\left(h_i^{(l)}\right)^2\right],$$

is negative due to the obvious consequence

$$\left(\sum_{l=1}^{N} h_i^{(l)}\right)^2 < N\sum_{l=1}^{N}\left(h_i^{(l)}\right)^2$$

of the Cauchy–Bunyakovsky inequality [15] (here the sign "$<$" is used instead of "$\leq$", since at least one of the terms $h_i^{(1)}, \ldots, h_i^{(N)}$ is obviously not equal to zero), therefore the quadratic equation under consideration has no real roots, $i = 1, 2, 3$.

We solve the system of equations (11). Taking into account inequality (14), we express γ_1 through b_1 from the first equation of this system:

$$\gamma_1 = \frac{D_{11} - C_1 b_1}{F_1 - 2H_1 b_1 + N b_1^2}.$$

Substituting the obtained expression for γ_1 into the second equation of system (11) and taking into account the inequality $\gamma_1 = k_1^{-1} \neq 0$, after simple transformations we arrive at an equality,

$$b_1 = \frac{C_1 F_1 - D_{11} H_1}{C_1 H_1 - N D_{11}},$$

taking into account which the equality written above for γ1 is reduced to the form

$$\gamma_1 = \frac{C_1 H_1 - N D_{11}}{H_1^2 - N F_1}.$$

Thus, the solution to the system of equations (11) is found.

We consider the system of equations (12). Since $\gamma_2 = k_2^{-1} \neq 0$, the second equation of this system can be written as

$$D_{22} - C_2 b_2 + \left(D_{12} - C_1 b_2\right)\alpha_1 + \left(-F_2 + 2H_2 b_2 - N b_2^2\right)\beta_1\gamma_2 + \frac{\lambda_1\beta_1}{\gamma_2} = 0, \quad (15)$$

since $\beta_1 = \sec\varepsilon_1 \neq 0$, the third equation of system (12) is equivalent to the following equation:

$$D_{22} - C_2 b_2 + \left(D_{12} - C_1 b_2\right)\alpha_1 + \left(-F_2 + 2H_2 b_2 - N b_2^2\right)\beta_1\gamma_2 = 0. \quad (16)$$

From (15), taking into account (16), we obtain that $\lambda_1\beta_1\gamma_2^{-1} = 0$ and then, by virtue of the inequality $\beta_1 \neq 0$, we have

$$\lambda_1 = 0. \quad (17)$$

The inequality $D_{12} - C_1 b_2 \neq 0$ holds, since a direct check shows that the value $b_2 = D_{12}C_1^{-1}$, which is the root of the equation $D_{12} - C_1 b_2 = 0$, does not satisfy the equations of system (12)

First, we consider the case $H_2 - N b_2 \neq 0$, i.e. $b_2 \neq H_2 N^{-1}$. We express $\beta_1\gamma_2$ from (16), as well as from the first (taking into account equality (17)) and fourth (taking into account the inequality $\beta_1\gamma_2 \neq 0$) equations of system (12):

$$\beta_1\gamma_2 = \frac{\left(D_{12} - C_1 b_2\right)\alpha_1 + D_{22} - C_2 b_2}{F_2 - 2H_2 b_2 + N b_2^2}, \quad \beta_1\gamma_2 = \frac{G_{11}\alpha_1 + G_{12}}{D_{12} - C_1 b_2}, \quad \beta_1\gamma_2 = \frac{C_1\alpha_1 + C_2}{H_2 - N b_2}. \quad (18)$$

As proven, the denominator of the fraction in the second of the equalities (18) is not equal to zero, the denominator of the fraction in the first of the equalities (18) is different from zero by virtue of (14). From the first and third equalities (18) follows the equation

$$\frac{C_1\alpha_1 + C_2}{H_2 - N b_2} = \frac{\left(D_{12} - C_1 b_2\right)\alpha_1 + D_{22} - C_2 b_2}{F_2 - 2H_2 b_2 + N b_2^2},$$

we express b_2 through α_1:

$$b_2 = \frac{\left(D_{12}H_2 - C_1 F_2\right)\alpha_1 + D_{22}H_2 - C_2 F_2}{\left(N D_{12} - C_1 H_2\right)\alpha_1 + N D_{22} - C_2 H_2}. \quad (19)$$

Similarly, from the second and third equalities (18) follows the equation

$$\frac{C_1\alpha_1 + C_2}{H_2 - N b_2} = \frac{G_{11}\alpha_1 + G_{12}}{D_{12} - C_1 b_2},$$

we express b_2 through α_1:

$$b_2 = \frac{\left(G_{11}H_2 - C_1 D_{12}\right)\alpha_1 + G_{12}H_2 - C_2 D_{12}}{\left(N G_{11} - C_1^2\right)\alpha_1 + N G_{12} - C_1 C_2}. \quad (20)$$

Equating the right-hand sides of equalities (19) and (20), after simple transformations we obtain a quadratic equation with respect to α_1 :

$$\begin{gathered}\left[C_1^2\left(C_1F_2 - D_{12}H_2\right) + C_1G_{11}\left(H_2^2 - NF_2\right) + C_1D_{12}\left(ND_{12} - C_1H_2\right)\right]\alpha_1^2 + \\ + \left[2C_1C_2\left(C_1F_2 - D_{12}H_2\right) + \left(ND_{12} - C_1H_2\right)\left(C_1D_{22} + C_2D_{12}\right) + \left(H_2^2 - NF_2\right)\left(C_1G_{12} + C_2G_{11}\right)\right]\alpha_1 + \\ + C_2^2\left(C_1F_2 - D_{12}H_2\right) + C_2G_{12}\left(H_2^2 - NF_2\right) + C_2D_{22}\left(ND_{12} - C_1H_2\right) = 0.\end{gathered} \quad (21)$$

The discriminant of equation (21) is equal to

$$\Delta = \left[\left(ND_{12} - C_1 H_2 \right)\left(C_1 D_{22} - C_2 D_{12} \right) + \left(H_2^{\,2} - NF_2 \right)\left(C_1 G_{12} - C_2 G_{11} \right) \right]^2,$$

and the roots of this equation are calculated using the formulae

$$\left[\begin{array}{l} \alpha_1 = -\dfrac{C_2}{C_1}, \\ \alpha_1 = -\dfrac{C_2 \left(C_1 F_2 - D_{12} H_2 \right) + G_{12} \left(H_2^{\,2} - NF_2 \right) + D_{22} \left(ND_{12} - C_1 H_2 \right)}{C_1 \left(C_1 F_2 - D_{12} H_2 \right) + G_{11} \left(H_2^{\,2} - NF_2 \right) + D_{12} \left(ND_{12} - C_1 H_2 \right)}. \end{array} \right. \qquad (22)$$

We note that the values (22) of the unknown α_1 are not the roots of the denominators of the fractions in equalities (19) and (20). Substituting the first of the values (22) of the unknown α1 into the fourth equation of system (12), taking into account the inequality $\beta_1\gamma_2 \neq 0$, we arrive at the equality $H_2 - Nb_2 = 0$, which contradicts the case under consideration.

Now $H_2 - Nb_2 = 0$, i.e. $b_2 = H_2N^{-1}$. Taking into account the inequality $\beta_1\gamma_2 \neq 0$, from the fourth equation of system (12) we obtain that $\alpha_1 = -C_2C_1^{-1}$, i.e. the found value of α1 coincides with the first of the roots (22) of equation (21). It is easy to show that equalities (19) and (20), proven under the assumption $b_2 \neq H_2N^{-1}$, are also valid for$b_2 = H_2N^{-1}$. In the case $b_2 = H_2N^{-1}$, the first two equalities from (18) also hold, since no restrictions on the value of the unknown b_2 were taken into account when deriving them.

From the above reasoning it follows that the system of equations (12) has two solutions. The components of both solutions of the specified system are obtained in the following order:

– the values of the unknown α_1 are found by formulae (22);

– the values of β_1 corresponding to the found values of α_1 are calculated by the formula written as the last equation of the system (12);

– the values of b_2 – by any of the formulae (19), (20);

– the values of γ_2 – by any of the two formulae

$$\gamma_2 = \frac{\left(D_{12} - C_1 b_2 \right)\alpha_1 + D_{22} - C_2 b_2}{\beta_1 \left(F_2 - 2H_2 b_2 + Nb_2^{\,2} \right)}, \quad \gamma_2 = \frac{G_{11}\alpha_1 + G_{12}}{\beta_1 \left(D_{12} - C_1 b_2 \right)},$$

which follow respectively from the first and second equalities (18);

– the value of the unknown λ_1 in both solutions of the system of equations (12) due to (17) is taken to be equal to zero .

We move on to finding solutions to the system of equations (13). Due to the inequalities $\gamma_3 = k_3^{-1} \neq 0$, $\beta_2 = \sec\varepsilon_2 \neq 0$ and $\beta_3 = \sec\varepsilon_3 \neq 0$ the second and third equations of this system can be written as

$$D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3 \right)\alpha_2\beta_3 + \left(D_{23} - C_2 b_3 \right)\alpha_3 + \left(-F_3 + 2H_3 b_3 - Nb_3^{\,2} \right)\beta_2\beta_3\gamma_3 + \frac{\lambda_2\beta_2}{\beta_3\gamma_3} = 0, \qquad (23)$$

$$D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3 \right)\alpha_2\beta_3 + \left(D_{23} - C_2 b_3 \right)\alpha_3 + \left(-F_3 + 2H_3 b_3 - Nb_3^{\,2} \right)\beta_2\beta_3\gamma_3 = 0 \qquad (24)$$

respectively. From (23), taking into account (24), we obtain: $\lambda_2\beta_2(\beta_3\gamma_3)^{-1} = 0$. Consequently, the equality is valid

$$\lambda_2 = 0. \qquad (25)$$

Then the first equation of system (13), taking into account the inequality$\beta_3 \neq 0$, can be written in the form

$$G_{13} + G_{11}\alpha_2\beta_3 + G_{12}\alpha_3 + \left(C_1 b_3 - D_{13} \right)\beta_2\beta_3\gamma_3 = 0. \qquad (26)$$

From the sixth equation of system (13), taking into account (24) and (26), we find $-\lambda_3\beta_3 = 0$, from which the equality follows

$$\lambda_3 = 0. \tag{27}$$

The inequalities $H_3 - Nb_3 \neq 0$, $D_{i3} - C_ib_3 \neq 0$ take place, since a direct check shows that neither the root $b_3 = H_3N^{-1}$ of the equation $H_3 - Nb_3 = 0$, nor the roots $b_3 = D_{i3}C_i^{-1}$ of the equations $D_{i3} - C_ib_3 = 0$ satisfy the equations of system (13), $i = 1, 2$.

Taking into account equalities (25), (27) and inequalities $\beta_2 \neq 0$, $\beta_3 \neq 0$, $\gamma_3 \neq 0$, we express $\beta_2\beta_3\gamma_3$ from the first, third, fourth and fifth equations of system (13):

$$\beta_2\beta_3\gamma_3 = \frac{G_{13} + G_{11}\alpha_2\beta_3 + G_{12}\alpha_3}{D_{13} - C_1b_3}, \quad \beta_2\beta_3\gamma_3 = \frac{D_{33} - C_3b_3 + (D_{13} - C_1b_3)\alpha_2\beta_3 + (D_{23} - C_2b_3)\alpha_3}{F_3 - 2H_3b_3 + Nb_3^2},$$
$$\beta_2\beta_3\gamma_3 = \frac{C_3 + C_1\alpha_2\beta_3 + C_2\alpha_3}{H_3 - Nb_3}, \quad \beta_2\beta_3\gamma_3 = \frac{G_{23} + G_{12}\alpha_2\beta_3 + G_{22}\alpha_3}{D_{23} - C_2b_3}. \tag{28}$$

As proven, the denominators of the fractions in the first, third and fourth equalities (28) are not equal to zero, the denominator of the fraction in the second equalities (28) is different from zero by virtue of (14). From the first and third equalities (28) follows the equation

$$\frac{G_{13} + G_{11}\alpha_2\beta_3 + G_{12}\alpha_3}{D_{13} - C_1b_3} = \frac{C_3 + C_1\alpha_2\beta_3 + C_2\alpha_3}{H_3 - Nb_3},$$

from the second and third equalities (28) follows the equation

$$\frac{D_{33} - C_3b_3 + (D_{13} - C_1b_3)\alpha_2\beta_3 + (D_{23} - C_2b_3)\alpha_3}{F_3 - 2H_3b_3 + Nb_3^2} = \frac{C_3 + C_1\alpha_2\beta_3 + C_2\alpha_3}{H_3 - Nb_3},$$

from the third and thorthequalities (28) follows the equation

$$\frac{C_3 + C_1\alpha_2\beta_3 + C_2\alpha_3}{H_3 - Nb_3} = \frac{G_{23} + G_{12}\alpha_2\beta_3 + G_{22}\alpha_3}{D_{23} - C_2b_3}.$$

From the last three equations we express $\alpha_2\beta_3$ through α_3 and b_3:

$$\alpha_2\beta_3 = \frac{\left[G_{12}(H_3 - Nb_3) - C_2(D_{13} - C_1b_3)\right]\alpha_3 + G_{13}(H_3 - Nb_3) - C_3(D_{13} - C_1b_3)}{C_1(D_{13} - C_1b_3) - G_{11}(H_3 - Nb_3)}, \tag{29}$$

$$\alpha_2\beta_3 = \frac{\left[(H_3 - Nb_3)(D_{23} - C_2b_3) - C_2\left(Nb_3^2 - 2H_3b_3 + F_3\right)\right]\alpha_3 + (H_3 - Nb_3)(D_{33} - C_3b_3) - C_3\left(Nb_3^2 - 2H_3b_3 + F_3\right)}{C_1\left(Nb_3^2 - 2H_3b_3 + F_3\right) - (H_3 - Nb_3)(D_{13} - C_1b_3)}, \tag{30}$$

$$\alpha_2\beta_3 = \frac{\left[C_2(D_{23} - C_2b_3) - G_{22}(H_3 - Nb_3)\right]\alpha_3 + C_3(D_{23} - C_2b_3) - G_{23}(H_3 - Nb_3)}{G_{12}(H_3 - Nb_3) - C_1(D_{23} - C_2b_3)}. \tag{31}$$

By equating the right-hand side of equality (29) to the right-hand sides of equalities (30) and (31), after simple transformations we obtain the equations

$$\Big\{\Big[C_2(C_1b_3 - D_{13})^2 + G_{11}(Nb_3 - H_3)(C_2b_3 - D_{23}) - G_{12}(Nb_3 - H_3)(C_1b_3 - D_{13}) - C_1(C_1b_3 - D_{13})(C_2b_3 - D_{23}) +$$
$$+ (C_1G_{12} - C_2G_{11})\left(Nb_3^2 - 2H_3b_3 + F_3\right)\Big]\alpha_3 - \left(Nb_3^2 - 2H_3b_3 + F_3\right)(C_3G_{11} - C_1G_{13}) - C_1(C_1b_3 - D_{13})(C_3b_3 - D_{33}) -$$
$$- G_{13}(Nb_3 - H_3)(C_1b_3 - D_{13}) + G_{11}(Nb_3 - H_3)(C_3b_3 - D_{33}) + C_3(C_1b_3 - D_{13})^2\Big\}(H_3 - Nb_3) = 0,$$

$$\Big\{\Big[(C_2G_{12}-C_1G_{22})(C_1b_3-D_{13})+\left(G_{11}G_{22}-G_{12}{}^2\right)(Nb_3-H_3)+(C_1G_{12}-C_2G_{11})(C_2b_3-D_{23})\Big]\alpha_3+$$
$$+(C_3G_{12}-C_1G_{23})(C_1b_3-D_{13})+(G_{11}G_{23}-G_{12}G_{13})(Nb_3-H_3)+(C_1G_{13}-C_3G_{11})(C_2b_3-D_{23})\Big\}(H_3-Nb_3)=0,$$

from which we express α3 through b3, having previously divided both parts of each of them by (H_3 – Nb_3):

$$\alpha_3=\frac{(H_3b_3-F_3)(C_1G_{13}-C_3G_{11})+(Nb_3-H_3)(D_{33}G_{11}-D_{13}G_{13})+(C_1b_3-D_{13})(C_3D_{13}-C_1D_{33})}{(H_3b_3-F_3)(C_2G_{11}-C_1G_{12})+(Nb_3-H_3)(D_{13}G_{12}-D_{23}G_{11})+(C_1b_3-D_{13})(C_1D_{23}-C_2D_{13})},\quad (32)$$

$$\alpha_3=\frac{(D_{23}-C_2b_3)(C_1G_{13}-C_3G_{11})+(Nb_3-H_3)(G_{12}G_{13}-G_{11}G_{23})+(C_1b_3-D_{13})(C_1G_{23}-C_3G_{12})}{(D_{23}-C_2b_3)(C_2G_{11}-C_1G_{12})+(Nb_3-H_3)\left(G_{11}G_{22}-G_{12}{}^2\right)+(C_1b_3-D_{13})(C_2G_{12}-C_1G_{22})}.\quad (33)$$

Having equated the right-hand sides of equalities (32) and (33), after the simplest transformations we arrive at a quadratic equation with respect to b_3

$$\Gamma b_3{}^2+\Lambda b_3+\Omega=0,\quad (34)$$

the coefficients of which are determined as follows:

$$\Gamma=\Big[G_{13}(C_1H_3-ND_{13})+C_3(C_1D_{13}-G_{11}H_3)+D_{33}\left(NG_{11}-C_1^2\right)\Big]\Big[G_{22}\left(NG_{11}-C_1^2\right)+G_{12}(C_1C_2-NG_{12})+$$
$$+C_2(C_1G_{12}-C_2G_{11})\Big]+\Big[G_{23}\left(NG_{11}-C_1^2\right)+C_3(C_1G_{12}-C_2G_{11})+G_{13}(C_1C_2-NG_{12})\Big]\Big[D_{23}\left(C_1^2-NG_{11}\right)+$$
$$+C_2(G_{11}H_3-C_1D_{13})+G_{12}(ND_{13}-C_1H_3)\Big],$$

$$\Lambda=\Big[G_{13}(C_1H_3-ND_{13})+C_3(C_1D_{13}-G_{11}H_3)+D_{33}\left(NG_{11}-C_1^2\right)\Big]\Big[G_{22}(C_1D_{13}-G_{11}H_3)+G_{12}(G_{12}H_3-C_2D_{13})+$$
$$+D_{23}(C_2G_{11}-C_1G_{12})\Big]+\Big[F_3(C_3G_{11}-C_1G_{13})+D_{33}(C_1D_{13}-G_{11}H_3)+D_{13}(G_{13}H_3-C_3D_{13})\Big]\Big[G_{12}(C_1C_2-NG_{12})+$$
$$+G_{22}\left(NG_{11}-C_1^2\right)+C_2(C_1G_{12}-C_2G_{11})\Big]+\Big[G_{23}\left(NG_{11}-C_1^2\right)+C_3(C_1G_{12}-C_2G_{11})+G_{13}(C_1C_2-NG_{12})\Big]\times$$
$$\times\Big[F_3(C_1G_{12}-C_2G_{11})+D_{13}(C_2D_{13}-G_{12}H_3)+D_{23}(G_{11}H_3-C_1D_{13})\Big]+\Big[G_{12}(G_{13}H_3-C_3D_{13})+G_{23}(C_1D_{13}-G_{11}H_3)+$$
$$+D_{23}(C_3G_{11}-C_1G_{13})\Big]\Big[C_2(G_{11}H_3-C_1D_{13})+G_{12}(ND_{13}-C_1H_3)+D_{23}\left(C_1^2-NG_{11}\right)\Big],$$

$$\Omega=\Big[F_3(C_3G_{11}-C_1G_{13})+D_{33}(C_1D_{13}-G_{11}H_3)+D_{13}(G_{13}H_3-C_3D_{13})\Big]\Big[G_{22}(C_1D_{13}-G_{11}H_3)+G_{12}(G_{12}H_3-C_2D_{13})+$$
$$+D_{23}(C_2G_{11}-C_1G_{12})\Big]-\Big[G_{12}(C_3D_{13}-G_{13}H_3)+G_{23}(G_{11}H_3-C_1D_{13})+D_{23}(C_1G_{13}-C_3G_{11})\Big]\Big[F_3(C_1G_{12}-C_2G_{11})+$$
$$+D_{13}(C_2D_{13}-G_{12}H_3)+D_{23}(G_{11}H_3-C_1D_{13})\Big].$$

The discriminant of equation (34) is equal to

$$\tilde{\Delta}=\Big\{\Big[C_2{}^2(D_{13}G_{13}-D_{33}G_{11})+G_{12}H_3(C_3G_{12}-C_1G_{23})+C_2G_{11}G_{23}H_3\Big](G_{11}H_3-C_1D_{13})+\Big[F_3G_{11}\times$$
$$\times(C_2G_{23}-C_3G_{22})+F_3G_{12}(C_3G_{12}-C_1G_{23})+F_3G_{13}(C_1G_{22}-C_2G_{12})+C_1D_{23}(D_{33}G_{12}-D_{23}G_{13})+C_2D_{23}\times$$
$$\times(D_{13}G_{13}-D_{33}G_{11})+C_3D_{23}(D_{23}G_{11}-D_{13}G_{12})\Big]\left(C_1^2-NG_{11}\right)+\Big[C_2H_3(D_{23}G_{11}-D_{13}G_{12})+G_{22}H_3(C_1D_{13}-G_{11}H_3)-$$
$$-C_1C_2D_{13}D_{23}\Big](C_3G_{11}-C_1G_{13})+\Big[C_2G_{11}(D_{13}G_{23}-D_{33}G_{12})+C_3G_{11}(D_{23}G_{12}-D_{13}G_{22})+C_1G_{13}(D_{13}G_{22}-D_{23}G_{12})+$$
$$+C_1G_{12}(D_{33}G_{12}-D_{13}G_{23})\Big](ND_{13}-C_1H_3)+C_2G_{12}\Big[C_1D_{13}(C_3D_{13}-C_1D_{33})+G_{11}H_3(C_1D_{33}-G_{13}H_3)\Big]\Big\}^2,$$

and the roots of this equation are calculated using the formulae

$$\left[\begin{array}{l} b_3 = \dfrac{C_1 D_{13} - G_{11} H_3}{C_1^2 - N G_{11}}, \\ b_3 = \Big[C_3 G_{11} \left(D_{23}{}^2 - F_3 G_{22} \right) + D_{33} G_{11} \left(G_{22} H_3 - C_2 D_{23} \right) + G_{11} G_{23} \left(C_2 F_3 - D_{23} H_3 \right) + D_{13} G_{12} \left(G_{23} H_3 - C_3 D_{23} \right) + \\ + D_{13} D_{23} \left(C_1 G_{23} - C_3 G_{12} \right) + G_{13} H_3 \left(D_{23} G_{12} - D_{13} G_{22} \right) + D_{13} D_{33} \left(C_2 G_{12} - C_1 G_{22} \right) + F_3 G_{12} \left(C_3 G_{12} - C_1 G_{23} \right) + \\ + D_{33} G_{12} \left(C_1 D_{23} - G_{12} H_3 \right) + D_{23} G_{13} \left(C_2 D_{13} - C_1 D_{23} \right) + D_{13}{}^2 \left(C_3 G_{22} - C_2 G_{23} \right) + F_3 G_{13} \left(C_1 G_{22} - C_2 G_{12} \right) \Big] \times \\ \times \Big[N D_{23} \left(G_{12} G_{13} - G_{11} G_{23} \right) + N D_{33} \left(G_{11} G_{22} - G_{12}{}^2 \right) + N D_{13} \left(G_{12} G_{23} - G_{13} G_{22} \right) + C_3 G_{11} \left(C_2 D_{23} - G_{22} H_3 \right) + \\ + C_2 G_{11} \left(G_{23} H_3 - C_2 D_{33} \right) + C_3 D_{13} \left(C_1 G_{22} - C_2 G_{12} \right) + C_3 G_{12} \left(G_{12} H_3 - C_1 D_{23} \right) + C_2 G_{13} \left(C_2 D_{13} - G_{12} H_3 \right) + \\ + C_1 G_{23} \left(C_1 D_{23} - G_{12} H_3 \right) + C_1 G_{13} \left(G_{22} H_3 - C_2 D_{23} \right) + C_1 D_{33} \left(C_2 G_{12} - C_1 G_{22} \right) + C_1 C_2 \left(D_{33} G_{12} - D_{13} G_{23} \right) \Big]^{-1}. \end{array}\right. \qquad (35)$$

Since equation (34) has two solutions, the system of equations (13) also has two solutions. The components of both solutions of this system are obtained in the following order:

– the values of the unknown b_3 are found using formulae (35);

– the values of the unknown α_3 corresponding to the found values of b_3 are calculated using any of the formulae (33), (34);

– the values of β_3 – using the formula written as the last equation of system (13);

– the values of α_2 – using any of the three formulae

$$\alpha_2 = \frac{\left[G_{12} \left(H_3 - N b_3 \right) - C_2 \left(D_{13} - C_1 b_3 \right) \right] \alpha_3 + G_{13} \left(H_3 - N b_3 \right) - C_3 \left(D_{13} - C_1 b_3 \right)}{\beta_3 \left[C_1 \left(D_{13} - C_1 b_3 \right) - G_{11} \left(H_3 - N b_3 \right) \right]},$$

$$\alpha_2 = \frac{\left[\left(H_3 - N b_3 \right) \left(D_{23} - C_2 b_3 \right) - C_2 \left(N b_3{}^2 - 2 H_3 b_3 + F_3 \right) \right] \alpha_3 + \left(H_3 - N b_3 \right) \left(D_{33} - C_3 b_3 \right) - C_3 \left(N b_3{}^2 - 2 H_3 b_3 + F_3 \right)}{\beta_3 \left[C_1 \left(N b_3{}^2 - 2 H_3 b_3 + F_3 \right) - \left(H_3 - N b_3 \right) \left(D_{13} - C_1 b_3 \right) \right]},$$

$$\alpha_2 = \frac{\left[C_2 \left(D_{23} - C_2 b_3 \right) - G_{22} \left(H_3 - N b_3 \right) \right] \alpha_3 + C_3 \left(D_{23} - C_2 b_3 \right) - G_{23} \left(H_3 - N b_3 \right)}{\beta_3 \left[G_{12} \left(H_3 - N b_3 \right) - C_1 \left(D_{23} - C_2 b_3 \right) \right]},$$

which follow from equalities (29), (30) and (31),
respectively;

– the values of β_2 – by the formula written as the last equation of system (13);

– the values of γ_3 – by any of the four formulae

$$\gamma_3 = \frac{G_{13} + G_{11} \alpha_2 \beta_3 + G_{12} \alpha_3}{\beta_2 \beta_3 \left(D_{13} - C_1 b_3 \right)}, \quad \gamma_3 = \frac{D_{33} - C_3 b_3 + \left(D_{13} - C_1 b_3 \right) \alpha_2 \beta_3 + \left(D_{23} - C_2 b_3 \right) \alpha_3}{\beta_2 \beta_3 \left(F_3 - 2 H_3 b_3 + N b_3{}^2 \right)},$$

$$\gamma_3 = \frac{C_3 + C_1 \alpha_2 \beta_3 + C_2 \alpha_3}{\beta_2 \beta_3 \left(H_3 - N b_3 \right)}, \quad \gamma_3 = \frac{G_{23} + G_{12} \alpha_2 \beta_3 + G_{22} \alpha_3}{\beta_2 \beta_3 \left(D_{23} - C_2 b_3 \right)},$$

which follow from (28);

– the values of the unknowns λ_2 and λ_3 in both solutions of the system of equations (13) are taken to be equal to zero due to equalities (25) and (27)

Thus, the system of equations (11) has a single solution, and each of the systems (12), (13) has two solutions. Consequently, the number of solutions of the original system of equations (10), defined as the product of the number of solutions of systems (11), (12) and (13), is four. Of the four solutions obtained,

$$\left(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\beta_1^{(j)},\beta_2^{(j)},\beta_3^{(j)},\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},b_1^{(j)},b_2^{(j)},b_3^{(j)},\lambda_1^{(j)},\lambda_2^{(j)},\lambda_3^{(j)}\right) \tag{36}$$

(j = 1, 2, 3, 4) the systems (10), which are also stationary points of the Lagrange function F, defined by equality (9), we are interested in the solution

$$\left(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\tilde\beta_1,\tilde\beta_2,\tilde\beta_3,\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde b_1,\tilde b_2,\tilde b_3,\tilde\lambda_1,\tilde\lambda_2,\tilde\lambda_3\right),$$

satisfying the equality

$$F\left(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\tilde\beta_1,\tilde\beta_2,\tilde\beta_3,\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde b_1,\tilde b_2,\tilde b_3,\tilde\lambda_1,\tilde\lambda_2,\tilde\lambda_3\right)=$$
$$=\min_{1\le j\le 4}\left\{F\left(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\beta_1^{(j)},\beta_2^{(j)},\beta_3^{(j)},\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},b_1^{(j)},b_2^{(j)},b_3^{(j)},\lambda_1^{(j)},\lambda_2^{(j)},\lambda_3^{(j)}\right)\right\}.$$

Since in each of the solutions (36) of the system of equations (10) the last three components are equal to zero, i.e.

$$\lambda_1^{(j)}=\lambda_2^{(j)}=\lambda_3^{(j)}=0,$$

j = 1, 2, 3, 4, by virtue of (9) we have:

$$F\left(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\beta_1^{(j)},\beta_2^{(j)},\beta_3^{(j)},\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},b_1^{(j)},b_2^{(j)},b_3^{(j)},\lambda_1^{(j)},\lambda_2^{(j)},\lambda_3^{(j)}\right)=$$
$$=\Phi\left(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\beta_1^{(j)},\beta_2^{(j)},\beta_3^{(j)},\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},b_1^{(j)},b_2^{(j)},b_3^{(j)}\right),$$

j = 1, 2, 3, 4, therefore, we determine the desired solution $\left(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\tilde\beta_1,\tilde\beta_2,\tilde\beta_3,\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde b_1,\tilde b_2,\tilde b_3\right)$ using the formula

$$\Phi\left(\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\tilde\beta_1,\tilde\beta_2,\tilde\beta_3,\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde b_1,\tilde b_2,\tilde b_3\right)=$$
$$=\min_{1\le j\le 4}\left\{\Phi\left(\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)},\beta_1^{(j)},\beta_2^{(j)},\beta_3^{(j)},\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},b_1^{(j)},b_2^{(j)},b_3^{(j)}\right)\right\}.$$

The required values of the scale factors $\tilde k_1,\tilde k_2,\tilde k_3$ for the measuring axes of the MU are calculated using the formulae

$$\tilde k_i=\tilde\gamma_i^{-1},$$

i = 1, 2, 3, required angle values $\tilde\varepsilon_1,\tilde\varepsilon_2,\tilde\varepsilon_3$ are calculated using the formulae

$$\tilde\varepsilon_i=\operatorname{arctg}\tilde\alpha_i,$$

which follow from the equalities

$$\tilde\alpha_i=\operatorname{tg}\tilde\varepsilon_i$$

and obvious inequalities $-\pi/2<\tilde\varepsilon_i<\pi/2$, i = 1, 2, 3.

Conclusion

Thus, we have obtained an analytical solution to the problem of calibrating the magnetometer of a spacecraft for the model considered in [1]. The procedure for calculating the calibration parameters of the MU using the derived formulas has a number of obvious advantages over numerical methods for solving this problem:

– the number of arithmetic operations is significantly reduced;

– the problem of possible instability of the method disappears.

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