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Решение первой краевой задачи плоской теории упругости с помощью законов сохранения

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Аннотация. Решению краевых задач для уравнений плоской теории упругости посвящено огромное количество работ. Большинство исследований этого направления основывается на формуле, найденной Г. В. Колосовым. Ему первому удалось выразить общее решение задачи о плоской упругой деформации через нахождение двух независимых функций комплексного переменного. Это позволило применить для решения задач теории упругости хорошо разработанную теорию аналитических функций. Позднее метод решения, основанный на формуле Колосова, был развит его учеником Н. И. Мусхелишвили. Но описанный метод имеет существенные ограничения. Он применим только для тех областей, которые можно конформно отобразить на круг. Поэтому необходимы и другие способы решения задач теории упругости, поскольку большое количество практически важных задач решается для областей, которые не удовлетворяют этому условию. Развиваемый в работе метод основан на использовании законов сохранения, которые построены для уравнений, описывающих плоское деформируемое состояние. Сделанные в работе предположения позволяют построить решение первой краевой задачи для произвольных плоских областей, ограниченных кусочно-гладким контуром. При этом нахождение компонент тензора напряжений сводится к вычислению контурных интегралов по границе рассматриваемой области. Как и в случае, рассмотренном Г. В. Колосовым, решение задачи основывается на двух точных решениях уравнений Коши – Римана, имеющих особенности в произвольной точке рассматриваемой области.

Ключевые слова: теория упругости, законы сохранения, первая краевая задача, уравнения Коши – Римана, тензор деформации.

Solution to the first boundary value problem of plane elasticity theory using conservation laws

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Abstract. A huge number of works are devoted to solving boundary value problems for the equations of plane elasticity theory. The largest number of studies in this area are based on the formula found *by G. V. Kolosov. He was the first to express the general solution to the problem of plane elastic deformation by finding two independent functions of a complex variable. This made it possible to apply a well-developed theory of analytic functions to solving problems of elasticity theory. Later, the solution method based on Kolosov's formula was developed by his student N. I. Muskhelishvili. But the described method also has significant limitations. It is applicable only to those areas that can be conformally mapped onto a circle. Therefore, other methods for solving elasticity theory problems are also needed, since a large number of practically important problems are solved for areas that do not satisfy this condition. The method developed in the work is based on the use of conservation laws that are constructed for equations describing a plane deformable state. The assumptions made in the work make it possible to construct a solution to the first boundary value problem for arbitrary plane areas bounded by a piecewise smooth contour. In this case, finding the components of the stress tensor is reduced to calculating contour integrals along the boundary of the region under consideration. As in the case considered by G. V. Kolosov, the solution to the problem is based on two exact solutions of the Cauchy – Riemann equations, which have singularities at an arbitrary point in the region under consideration.*

Keywords: elasticity theory, conservation laws, first boundary value problem, Cauchy – Riemann equation, strain tensor.

Introduction

The solution to boundary value problems for equations of elasticity theory in a plane stationary case is described in a huge number of articles and monographs. The classic work in this direction is a book written by G. V. Kolosov's student [1]. Despite the long history of solving such problems, the interest in their solution has not waned. This is due to the fact that the classical formulae by G. V. Kolosov allow solving equations of elasticity theory not for all boundary value problems arising in science and technology. The main limitation is caused by the smoothness of the boundary and some features of complex variable functions. The other methods involved in decomposing the desired functions into series for different types of special functions also have natural limitations, which are related to the similarity of the series being used as well as the complexity of the results obtained.

Let us describe some results of the recent studies on the theory of elasticity. The paper [2] presents a brief historical review of the studies devoted to the bending theory of elastic plates. In [3] the author considers the problem of detecting and identifying elastic inclusions in an isotropic linearly elastic plane. The article [4] considers a complicated version of the well-known Lame problem posed in 1852 describing the solution to the static equilibrium of a parallelepiped with free lateral surfaces exposed to the action of opposite end forces, and also for the case of impact effects of end forces. In paper [5] the construction of fundamental solutions for harmonic oscillations equations in elastic anisotropic elastic media theory is carried out; the fundamental solution to oscillations equations for isotropic medium in closed form is constructed. The work [6] presents a general solution to the problems of the elastic theory for anisotropic half-planes and bands with arbitrary holes and cracks using complex potentials of the plane problem of the elastic theory of an anisotropic body. The article [7] is devoted to the study of the functions of tensions, which allow satisfying exactly the equations of equilibrium of the classical theory of elasticity and to obtain a solution in tensions. In order to obtain dependencies between voltages and voltage functions, a mathematical apparatus of general relativity is used. In [8] terms of complex-significant displacements, the system of equations of the axisymmetric theory of elasticity is written; the fundamental solution to this system of equations is a general representation of the field of displacements in an axisymmetric case, similar to the formulae by Kolosov-Muskhelishvili in a plane problem. The basic equations of linear moment elasticity theory are presented in [9]. The determining ratios are written for the case of general anisotropy in the form of linear equations. Some simplified options are considered, in particular with constricted rotation and plane deformation when only shear stresses are present. The work [10] is devoted to the analysis of a boundary value problem with an unknown area of contact, describing the equilibrium of two-dimensional elastic bodies with a thin slightly curved web. The paper [11] explores the evolution of the wave pattern in a multi-modular elastic half-space with a non-stationary one-axis piecewise-linear motion of its boundary in the mode "tension – compression – stop".

In this work the laws of conservation of differential equations of elasticity are being used. The laws, which allow reducing the presence of tensor component of the tensions at the point to the contour integral along the boundary of the area being considered, have been presented. In these conditions the boundary of an area requires only partial smoothness. Note that earlier some conservation laws were presented in works [12; 13] but they were not used to solve any problems.

Statement of the problem

Let us consider the equations describing plane elastic deformation.

The relations between strain tensor components and displacement vector components in the case of small deformations have the following form

$$\varepsilon_x = \frac{\partial w_1}{\partial x}, \varepsilon_y = \frac{\partial w_2}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x}. \tag{1}$$

The Hooke's will be written as follows:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \varepsilon_{xy} = \frac{2(1+\nu)}{E}\tau. \tag{2}$$

Strain compatibility conditions:

$$\frac{\partial^2 \varepsilon_x}{\partial^2 y} + \frac{\partial^2 \varepsilon_y}{\partial^2 x} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \tag{3}$$

By putting (2) into (3), we obtain

$$\frac{\partial^2(\sigma_x - \nu\sigma_y)}{\partial^2 y} + \frac{\partial^2(\sigma_y - \nu\sigma_x)}{\partial^2 x} = 2(1+\nu)\frac{\partial^2 \tau}{\partial x \partial y}. \tag{4}$$

Equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0.$$

From hence we obtain

$$\frac{\partial^2 \sigma_y}{\partial^2 y} + \frac{\partial^2 \sigma_x}{\partial^2 x} = -2\frac{\partial^2 \tau}{\partial x \partial y}. \tag{5}$$

From (4) and (5) we obtain

$$\Delta(\sigma_x + \sigma_y) = 0.$$

Here $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are strain tensor components; σ_x, σ_y, τ are voltage tensor components; w_1, w_2 are displacement vector components; E, ν are elastic constants.

The system has the following final form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \ \Delta(\sigma_x + \sigma_y) = 0. \tag{6}$$

Let us put the first boundary value problem for the system (6):

$$\sigma_x n_1 + \tau n_2 \,|_L = X(x,y), \quad \tau n_1 + \sigma_y n_2 \,|_L = Y(x,y). \tag{7}$$

Here n_1, n_2 are components of an outer normal vector to a piecewise smooth contour limiting the end area S.

Let us find the solution to the problems (6) and (7) in the form

$$\sigma_x + \sigma_y = p - \text{const}\,, p \neq 0. \tag{8}$$

Let us introduce new variables:

$$u = \sigma_x / p, v = \tau / p, \sigma'_y = \sigma_y / p, f = X / p, g = Y / p. \quad (9)$$

Then the problem (6), (7) is written in the following form

$$F_1 = u_x + v_y = 0, F_2 = u_y - v_x = 0, \quad (10)$$

$$un_1 + vn_2 = f, vn_1 - un_2 = g - n_2, \quad (11)$$

here and then the index at the bottom means a derivative of this variable.

Thus, it is necessary to solve the boundary value problem (11) for the system of equations (10) with the use of conservation laws.

Conservation laws of the system of equations (10)

Definition. The conservation law for the system of equations (10) will be called the following expression

$$A_x + B_y = \omega_1 F_1 + \omega_2 F_2, \quad (12)$$

where ω_1, ω_2 are linear differential operators that are simultaneously nonzero identical,

$$A = \alpha^1 u + \beta^1 v + \gamma^1, B = \alpha^2 u + \beta^2 v + \gamma^2, \quad (13)$$

$\alpha^1, \beta^1, \gamma^1, \alpha^2, \beta^2, \gamma^2$ are some smooth functions that only depend on x, y.

Remark. A more general definition of the conservation law suitable for the arbitrary systems of equations can be found in [14; 15].

From (12) taking into account (13) we obtain

$$\alpha^1_x u + \alpha^1 u_x + \beta^1{}_x v + \beta^1 v_x + \gamma^1{}_x + \alpha^2_y u + \alpha^2 u_y + \beta^2_y v + \beta^2 v_y + \gamma^2_y = \omega_1 (u_x + v_y) + \omega_2 (u_y - v_x) = 0. \quad (14)$$

From (14) it follows

$$\alpha^1_x + \alpha^2_y = 0, \beta^1_x + \beta^2_y = 0, \alpha^1 = \omega_1, \beta^1 = -\omega_2, \alpha^2 = \omega_2, \beta^2 = \omega_1, \gamma^1 + \gamma^2 = 0.$$

From hence we obtain

$$\alpha^1 = \beta^2, \alpha^2 = -\beta^1. \quad (15)$$

Therefore

$$\alpha^1_x - \beta^1_y = 0, \alpha^1_y + \beta^1_x = 0. \quad (16)$$

From the given formulae it follows that the system of equations (10) allows infinitely many laws of conservation. Only those that can solve the problem will be listed below.

Therefore, the conserved current is:

$$A = \alpha^1 u + \beta^1 v + \gamma^1, B = -\beta^1 u + \alpha^1 v + \gamma^2.$$

From (13) we obtain

$$\iint_S (A_x + B_y) dxdy = \oint_L -Ady + Bdx = 0, \quad (17)$$

where S is the area bounded by the L curve.

Solution to the problem (10), (11)

In order to find the u, v values within the S area, it is necessary to construct solutions to the Cauchy – Riemann system (16) having singularities at the arbitrary point $(x_0, y_0) \in S$.

The first of these solutions has the folloing form:

$$\alpha^1 = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \quad \beta^1 = -\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}, \quad \gamma^1 = \gamma^2 = 0. \tag{18}$$

Remark. If mass forces are included in the equilibrium equation, then γ^1, γ^2 will no longer be equal to zero.

At the point $(x_0, y_0) \in S$ the functions α^1, β^1 have singularities, thus let us circle this point with the circumference ε: $(x - x_0)^2 + (y - y_0)^2 = \varepsilon^2$.

Then we obtain from the formula (17)

$$\oint_L -Ady + Bdx + \oint_\varepsilon -Ady + Bdx = 0, \tag{19}$$

Let us calculate the second integral in the formula (19). We have

$$\oint_\varepsilon -Ady + Bdx = \oint_\varepsilon -\left(\frac{u(x - x_0)}{(x - x_0)^2 + (x - x_0)^2} - \frac{v(y - y_0)}{(x - x_0)^2 + (x - x_0)^2}\right)dy +$$

$$+\left(-\frac{u(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{v(x - x_0)}{(x - x_0)^2 + (y - y_0)^2}\right)dx.$$

Let us introduce the new coordinates $x - x_0 = \varepsilon\cos\varphi, \quad y - y_0 = \varepsilon\sin\varphi,$ we obtain

$$\oint_\varepsilon -Ady + Bdx = \int_0^{2\pi} [-(u\cos\varphi + v\sin\varphi)\cos\varphi - (u\sin\varphi + v\cos\varphi)s\text{in}\,\varphi]d\varphi =$$

$$= -\int_0^{2\pi} ud\varphi = -2\pi u(x_0, y_0). \tag{20}$$

The last equality is obtained by the mean-value theorem at $\varepsilon \to 0$.

For the final construction of the solution, let us find the values u, v on the *L* boundary. From the formula (11) we obtain

$$u = fn_1 - gn_2 + n_2^2, \quad v = fn_2 + gn_1 + n_1 n_2. \tag{21}$$

Let us put (21) into (20); and taking into account (19) we obtain

$$2\pi u(x_0, y_0) = 2\pi\sigma_x(x_0, y_0) / p =$$

$$= \oint_L -\left(\frac{(fn_1 - gn_2 + n_1^2)(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{(fn_2 + gn_1 - n_1 n_2)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}\right)dy +$$

$$-\left(\frac{(fn_1 - gn_2 + n_1^2)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + \frac{(fn_2 + gn_1 - n_1 n_2)(x - x_0)}{(x - x_0)^2 + (y - y_0)^2}\right)dx.$$

The second solution to the system of equations (16) is taken as

$$\alpha^1 = \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}, \quad \beta^1 = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \tag{22}$$

Having made calculations similar to those made with the solution (18), we obtain

$$2\pi v(x_0, y_0) = 2\pi\tau(x_0, y_0) / p =$$

$$= \oint_L -\left(\frac{(fn_1 - gn_2 + n_1^2)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + \frac{(fn_2 + gn_1 - n_1 n_2)(x - x_0)}{(x - x_0)^2 + (y - y_0)^2}\right)dy +$$

$$-\left(-\frac{(fn_1 - gn_2 + n_1^2)(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} + \frac{(fn_2 + gn_1 - n_1 n_2)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}\right)dx.$$

Conclusion

This paper proposes a new method for solving the first boundary value problem for the equations of plane elasticity theory in a stationary case. This method makes it possible to find the value of the component of a strain tensor at each point of the area being studied. In this case, the stress calculations are limited only to the calculation of contour integrals along the boundaries of the area.

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