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Кручение упругопластического стержня, нагруженного давлением вдоль образующей

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Аннотация. Статья продолжает серию статей, посвященных использованию метода законов сохранения дифференциальных уравнений для решения задач механики деформируемого твердого тела.

Упругопластические задачи в механике деформируемого твердого тела учитывают нелинейную связь между напряжениями и деформациями под действием различных нагрузок. Такие задачи возникают в конструкциях, где материалы характеризуются различными физическими свойствами. Учет упругопластических деформаций важен для прогнозирования работы конструкций, а также для обеспечения их долговечности.

В настоящее время решение упругопластических задач продолжает оставаться в центре внимания исследователей. Появляются новые аналитические подходы к решению этих задач, совершенствуются численные методы. Авторы вносят свой вклад в решение задач механики деформируемого твердого тела с помощью законов сохранения. Использование законов сохранения позволяет свести нахождение компонент тензора напряжений в каждой точке к контурному интегралу по границе рассматриваемой области, что дает возможность построить ранее неизвестную упругопластическую границу.

В статье рассматривается упругопластический стержень постоянного поперечного сечения, который находится под действием линейного гидростатического давления и пары сил, которые скручивают его вокруг центральной оси, совпадающей с осью oz. Боковая поверхность стержня свободна от напряжений и находится в пластическом состоянии. Построенные законы сохранения позволяют найти компоненты тензора напряжений, которые, в свою очередь, позволяют определить упругопластическую границу в рассматриваемом стержне.

Ключевые слова: законы сохранения дифференциальных уравнений, упругопластичность, кручение.

Turn of an elastic-plastic rod under pressure that varies linearly along the forming

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Abstract. The article continues a series of articles devoted to the use of the method of conservation laws of differential equations for solving problems in the mechanics of deformable solids.

Elastoplastic problems in the mechanics of a deformable solid take into account the nonlinear relationship between stresses and deformations under the influence of various loads. Such problems arise in structures where materials are characterized by different physical properties; taking into account elastic-plastic deformations is important for predicting the operation of structures, as well as for ensuring their durability.

Currently, solutions to elastoplastic problems continue to be the focus of researchers' attention. New analytical approaches to solving these problems are emerging, and numerical methods are being improved. The authors contribute to solving the problems of mechanics of deformable solids using conservation laws. The use of conservation laws makes it possible to reduce the finding of the stress tensor components at each point to a contour integral along the boundary of the region under consideration, which makes it possible to construct a previously unknown elastoplastic boundary.

The article considers an elastoplastic rod of constant cross-section, which is under the influence of linear hydrostatic pressure and a pair of forces that twist it around a central axis coinciding with the oz axis. The lateral surface of the rod is stress-free and in a plastic state. The constructed conservation laws allow us to find the components of the stress tensor. The components of the stress tensor make it possible to determine the elastoplastic boundary in the rod under consideration.

Keywords: conservation laws of differential equations, elastoplasticity, torsion.

Introduction

This paper utilizes conservation laws for differential equations. Their use allows us to reduce the determination of stress tensor components at each point to a contour integral over the boundary of the region under consideration, thereby enabling the construction of an elastic-plastic boundary. This boundary is assumed to be piece-wise smooth.

Due to their practical importance, elastoplastic problems have long been studied by mechanics. The main problem that arises when solving such problems is determining the elastoplastic boundary. The plasticity condition imposes an additional constraint, and this, according to G. P. Cherepanov [1], simplifies the problem. On the other hand, a new unknown arises: the elastoplastic boundary, which complicates the solution.

Currently, solving elastic-plastic problems continues to be a focus of research. New analytical approaches to their solution are emerging, and numerical methods are being refined. Here we briefly review such research. In [2], conservation laws are used to solve the torsion problem of an elastic-plastic rod reinforced with elastic fibers. Conservation laws are used to solve the problem. In [3] an elastic-plastic box beam is considered, which is bent by a transverse force. It is assumed that the deformations in the rod are elastic-plastic and its lateral surface is stress-free. The center of gravity of the cross-section does not coincide with the point of application of the force. Using conservation laws, an exact solution is constructed describing the stress state of this structure. It is calculated at each point of the figure under consideration using integrals over the external contours of the cross-section. In [4], the elastoplastic torsion of a multilayer rod consisting of several layers is investigated. The elastic properties of the layers vary, but the plasticity coefficient is the same for all layers. The article constructs conservation laws that allow the calculation of stress tensor components using contour integrals along the layer boundaries. In [5], the elastic-plastic torsion of an anisotropic three-layer cylindrical rod of non-circular cross-section is considered. The inner layer of the rod is in an elastic-plastic state, the two outer layers are completely plastic. Plastic anisotropy is assumed. The anisotropy parameters of each layer are different. In [6], a solution to the problem of determining the elastic-plastic state of a heavy space weakened by an elliptical hole is considered. The material of the medium exhibits anisotropic properties. The problem was solved using the small parameter method. Torsion of a two-layer box-section rod is considered in [7]. In [8], the stress-strain state of the binder of composite materials is calculated using numerical methods. Delamination of steel pipes under complex loading is modeled in

[9]. An elastoplastic analysis of a circular pipe turned inside out is carried out in [10]. In [11], the influence of the type of plane problem for an elastoplastic adhesive layer on the value of the *J*-integrals is studied. In [12], within the framework of a single model of large elastoplastic deformations, the non-stationary dynamics of a medium not associated with additional accumulation of plastic deformations to those already present is considered. It has been shown that, in the general case, each elastic wave can be accompanied by an abrupt rotation of plastic deformations. In [13], the process of producing irreversible deformations in a rotating cylinder made of a material with elastic, viscous, and plastic properties was studied.

Statement of the problem

There is an elastic-plastic rod of constant cross-section, which is subjected to linear hydrostatic pressure and a pair of forces that twist it around a central axis coinciding with the *oz* axis.

We assume that the following conditions are met

$$\sigma_x = -\lambda z + C, \sigma_y = -\lambda z + C, \sigma_z = -\lambda z + C, \tau_{xy} = 0,$$

$$\tau_{xz} = u(x, y), \tau_{xz} = u(x, y), \tau_{yz} = v(x, y). \tag{1}$$

In this case, the equations describing elastic deformation in the stationary case have the form

$$u_x + v_y = \lambda, \quad v_x - u_y = -2\alpha. \tag{2}$$

System (2) consists of the equilibrium equation and the compatibility equation of elastic deformations.

In the plastic region the system has the form

$$u_x + v_y = \lambda, \quad u^2 + v^2 = k^2. \tag{3}$$

Here $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are components of the stress tensor ; $\lambda, \alpha = G\theta, k$ are constants; *G* is a modulus of elasticity, θ is a torsion angle, *k* is a plasticity constant equal to the yield strength under pure shear.

It is assumed that the lateral surface of the rod is free from stress and is in a plastic state, therefore system (1) should be solved with the following boundary conditions

$$un_1 + vn_2 |_L = 0, \quad u^2 + v^2 = k^2. \tag{4}$$

Here n_1, n_2 are components of the outward normal vector to a piecewise smooth outer contour *L* bounding a finite region S.

Comment 1. If $\alpha = 0,$ the problem (4) for the system of equations(2) coincides with the problem up to the notation [1]. In [1] it is shown that in this case for problem (2)–(4) the solution exists and is unique if the rod has an oval cross-section and $-1/\lambda > k/GR_{\min}$, where $R_{\min}$ is minimum radius of curvature of a curve *L*.

Comment 2. The case $\lambda = 0, \alpha \neq 0$ corresponding the classical one of elastic-plastic torsion will not be considered. It is considered in [1].

For convenience, we write equations (2) in the form

$$F_1 = u_x + v_y - \lambda = 0, F_2 = -u_y + v_x + 2\alpha = 0, \tag{5}$$

we will solve the boundary value problem (2), (4) using conservation laws.

Conservation laws of an equations system (2)

Definition. The conservation law for the system of equations (2) is an expression of the form

$$A_x + B_y = \omega_1 F_1 + \omega_2 F_2, \tag{6}$$

where ω_1, ω_2 are linear differential operators that are not simultaneously identically equal to zero .

$$A = \alpha^1 u + \beta^1 v + \gamma^1, \;\; B = \alpha^2 u + \beta^2 v + \gamma^2, \tag{7}$$

$\alpha^1, \beta^1, \gamma^1, \alpha^2, \beta^2, \gamma^2$ are some smooth functions depending only on x, y.

Comment 3. A more general definition of the conservation law, suitable for arbitrary systems of equations, can be found in [14].

From (6) taking in consideration (7) we have

$$\alpha^1_x u + \alpha^1 u_x + \beta^1{}_x v + \beta^1 v_x + \gamma^1{}_x + \alpha^2_y u + \alpha^2 u_y + \beta^2_y v + \beta^2 v_y + \gamma^2_y =$$

$$= \omega_1 (u_x + v_y - \lambda) + \omega_2 (-u_y + v_x + 2\alpha). \tag{8}$$

It follows from (8) that

$$\alpha^1_x + \alpha^2_y = 0, \;\; \beta^1_x + \beta^2_y = 0, \;\; \alpha^1 = \omega_1, \;\; \beta^1 = \omega_2, \;\; \alpha^2 = -\omega_2, \;\; \beta^2 = \omega_1, \;\; \gamma^1_x + \gamma^2_y = -\lambda\omega_1 + 2\alpha\omega_2.$$

From here it follows that

$$\alpha^1 = \beta^2, \;\; \alpha^2 = -\beta^1. \tag{9}$$

Hence

$$\alpha^1_x - \beta^1_y = 0, \;\; \alpha^1_y + \beta^1_x = 0, \;\; \gamma^1_x + \gamma^2_y = -\lambda\alpha^1 + 2\alpha\beta^1. \tag{10}$$

From the given formulas it follows that the system of equations (2) admits an infinite number of conservation laws; below only those will be given that allow us to solve the given problem.

Сохраняющийся ток имеет вид

$$A = \alpha^1 u + \beta^1 v + \gamma^1, \;\; B = -\beta^1 u + \alpha^1 v + \gamma^2.$$

From (6) using Green's formula we obtain

$$\iint_S (A_x + B_y) dxdy = \oint_L -Ady + Bdx = 0, \tag{11}$$

where *S* is a domain bounded by curve *L*.

Solution of problem (2), (4)

To find the values u, v inside the region S, it is necessary to construct solutions of system (10) that have singularities at an arbitrary point $(x_0, y_0) \in S$.

The first of these solutions is

$$\begin{gathered}
\alpha^1 = \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \quad \beta^1 = \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}, \\
\gamma^1 = 2\alpha \int \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\, dx = 2\alpha \operatorname{arctg} \frac{x - x_0}{y - y_0}, \\
\gamma^2 = -\lambda \int \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\, dy = -\lambda \operatorname{arctg} \frac{y - y_0}{x - x_0}.
\end{gathered} \tag{12}$$

At a point $(x_0, y_0) \in S$ the functions $\alpha^1,\ \beta^1$ have singularities, so we surround this point with a circle

$$\varepsilon: \quad (x-x_0)^2+(y-y_0)^2=\varepsilon^2.$$

Then from formula (11) we obtain

$$\oint_L -Ady+Bdx+\oint_\varepsilon -Ady+Bdx=0, \tag{13}$$

we will calculate the last integral in formula (13). We have

$$\oint_\varepsilon -Ady+Bdx=\oint_\varepsilon -\left(\frac{u(x-x_0)}{(x-x_0)^2+(x-x_0)^2}-\frac{v(y-y_0)}{(x-x_0)^2+(x-x_0)^2}+\gamma^1\right)dy+$$

$$+\left(-\frac{u(y-y_0)}{(x-x_0)^2+(y-y_0)^2}-\frac{v(x-x_0)}{(x-x_0)^2+(y-y_0)^2}+\gamma^2\right)dx.$$

We interpolate new coordinates $x-x_0=\varepsilon\cos\varphi, \quad y-y_0=\varepsilon\sin\varphi$. We have

$$\oint_\varepsilon -Ady+Bdx=\int_0^{2\pi}[-(u\cos\varphi+v\sin\varphi)\cos\varphi-(u\sin\varphi+v\cos\varphi)\sin\varphi]d\varphi=$$

$$=-\int_0^{2\pi}ud\varphi=-2\pi u(x_0,y_0). \tag{14}$$

The last equality is obtained by the mean value theorem for $\varepsilon\to 0$.

To finally construct the solution, we will find the values u, v on the boundary L. From formula (13) we obtain

$$2\pi u(x_0,y_0)=\oint_L -(-\alpha^1 n_2+\beta^1 n_1+\gamma^1)dy+(\beta^1 n_2+\alpha^1 n_1+\gamma^2)dx. \tag{15}$$

We take the second solution of the equations system (10) in the form

$$\begin{gathered}\alpha^1=\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}, \quad \beta^1=-\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2},\\ \gamma^1=-\lambda\int\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}\,dx=-\lambda\ \mathrm{arctg}\frac{x-x_0}{y-y_0},\\ \gamma^2=-2\alpha\int\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}\,dy=-2\alpha\ \mathrm{arctg}\frac{y-y_0}{x-x_0}.\end{gathered} \tag{16}$$

Having carried out calculations similar to those carried out with solution (12), we obtain

$$2\pi v(x_0,y_0)=\oint_L -(-\alpha^1 n_2+\beta^1 n_1+\gamma^1)dy+(\beta^1 n_2+\alpha^1 n_1+\gamma^2)dx. \tag{17}$$

Conclusion

In this paper, conservation laws are constructed for the system of equations (2) describing the torsion of an elastic-plastic rod under the action of pressure that varies linearly along the generatrix. Using the constructed conservation laws, the components of the stress tensor were found σ_{xz}, σ_{yz} according to formulas (15) and (17), which allow us to determine the elastic-plastic boundary in the rod under consideration.

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