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A NEW METHOD OF CALCULATING THE STATE OF STRESS IN GRANULAR MATERIALS UNDER PLANE STRAIN CONDITIONS

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The system of equations comprising the Mohr-Coulomb yield condition and the stress equilibrium equations may be studied independently of the flow law. This system of equations is hyperbolic. Accordingly, to solve the aforementioned system of equations, it is reasonable to apply the method of characteristics. In the special case of plasticity theory for materials whose yield criterion does not depend on the average stress, two methods are used to construct an orthogonal net of characteristics and to determine the stress field: the R-S method and Mikhlin's coordinate method. In the case of the Mohr-Coulomb yield condition, the angle between the characteristic directions depends on the internal friction angle. Therefore, the above-mentioned methods should be generalised in accordance with this property of characteristics.

Purpose. In the case of Plasticity theory for materials whose yield strength does not depend on the average stress, to calculate the stress filed, Mikhlin's coordinate method is widely used. The purpose of this study is to generalise this method for the equation system consisting of the Mohr-Coulomb yield criterion and the pressure equilibrium equations.

Methods. The geometrical properties of the characteristics of the equations' system consisting of the Mohr-Coulomb yield condition and the equilibrium equations are used to introduce the generalised Mikhlin coordinates.

Results. It's been pointed out that solving equation system consisting of the Mohr-Coulomb yield condition and equilibrium equation comes to solving equation of telegraphy and to subsequent integration.

Practical Significance. The developed method of system of equations' solution, consisting of the Mohr-Coulomb yield condition and equilibrium equation enables obtaining high precision solutions at insignificant computer time expenditures.

Mohr-Coulomb yield condition, method of characteristics, Mikhlin's variables, equation of telegraphy.

Introduction

The plane strain deformation state of perfect rigid plastic solid and elastic-perfectly plastic solid the stress equations in the plastic zone consist of the yield condition and two equilibrium equations. This system of equations may be studied without invoking the flow law. In many cases, the above-mentioned system of equations is hyperbolic [1]. Determination of the stress field is brought to determination of the characteristics field. In the case of the yield condition equation which is not dependent upon the average stress, the two methods of construction of characteristics field are used: the R−S method, suggested in [2], and Mikhlin’s coordinates method [1, 3, 4]. The relevance of application this or that method of characteristics depends on the set boundary conditions. In the case of granular medium, the yield condition depends on the average stress [5, 6]. As it goes out of today’s surveys [7, 8], by now the most widely used yield condition of such a type has been the Mohr-Coulomb yield condition. In particular, this condition is used in a widespread model [9] and in a modern model for granular materials, developed in [10]. To construct the field of characteristics of the system of equations consisting of the Mohr-Coulomb yield condition and the equilibrium equations, the R–S method is generalised in [11]. In this study, the method of Mikhlin's coordinates is generalised to construct the field of characteristics of this system of equations. It is shown that in the case when both characteristics are curvilinear, the solution of the boundary value task is brought to solution of the telegraph equation. The methods for solving this equation under boundary conditions, typical for models of perfect rigid plastic solid and elastic-perfectly plastic solid, have been well studied [1, 3, 4].

Let us point out that, the suggested method for determination of stress condition may be used for a range of metal materials as well, as it follows from [12−15].

Generalised Mikhlin’s coordinates

Let us consider arbitrary plane orthogonal coordinate system (ξ, η) and the Cartesian coordinate system (x, y). Both the systems are shown in the fig. 1. Let us consider arbitrary point *P*, determined by the radius vector **R**, the beginning of which coincides with the beginning of the Cartesian coordinate system. We introduce a rectilinear coordinate $\bar{y}$, counted from the beginning of the Cartesian coordinate system in the direction of the coordinate η in the point *P*, and a rectilinear coordinate $\bar{x}$, counted from the beginning of the Cartesian coordinate system in the direction of the coordinate ξ in the point *P*. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors in the direction of axes $\bar{x}$ and $\bar{y}$ respectively.

Since the curvilinear coordinates are orthogonal, it is obvious that the Mikhlin's coordinates are also orthogonal (fig. 1). The characteristics of the system of equations consisting of the yield condition which is independent of the average stress, and the equilibrium equations, are orthogonal. Therefore, we can accept that are characteristic coordinates. In this case, the values and separately satisfy the telegraph equation [1, 3, 4]. The characteristics of the system of equations consisting of the Mohr-Coulomb yield condition and the equilibrium equations are not orthogonal [5]. Let us denote the corresponding characteristic coordinates. Without loss of generality, we can assume that the direction of the maximum (in the algebraic sense) of the main stress passes through the first and the third quadrants (fig. 2).

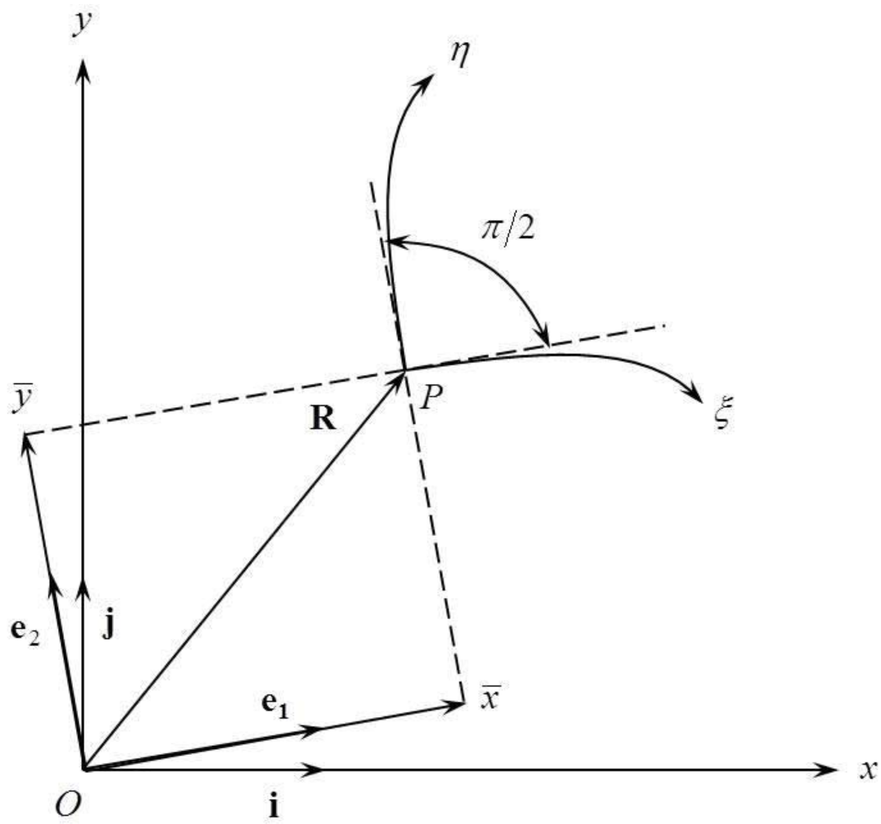

Fig. 1. Mikhlin's orthogonal coordinates

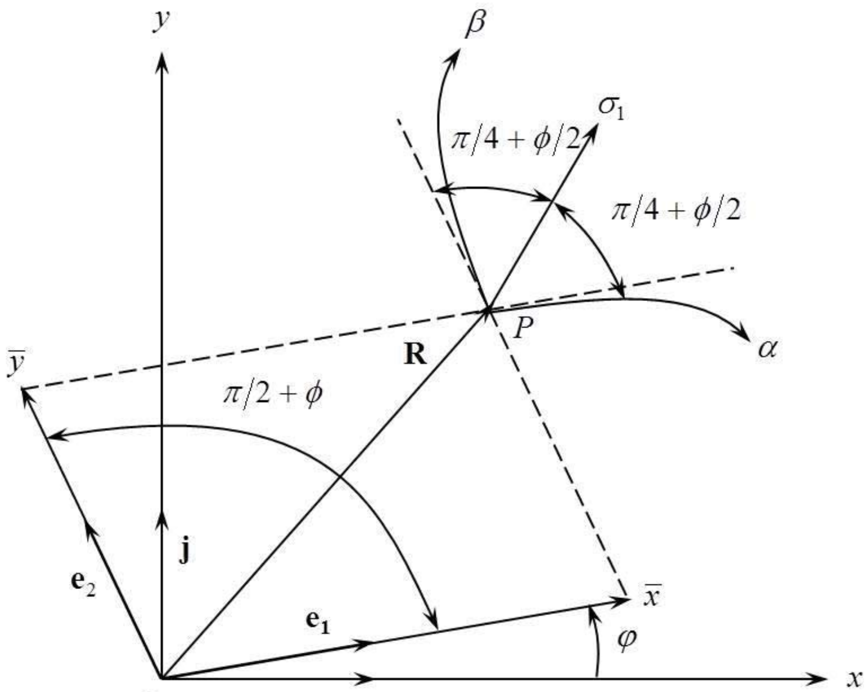

Fig. 2. Mikhlin's orthogonal coordinates

Then Mikhlin's coordinates $(\bar{x}, \bar{y})$ of the point *P* are determined from the equation

$$\mathbf{R} = \bar{x}\mathbf{e}_1 + \bar{y}\mathbf{e}_2. \qquad (1)$$

Then, the angle between the direction of this main stress and each of the characteristic directions equals $\pi/4 + \phi/2$ [5], where ϕ – angle of internal friction. If the angle of internal friction is a constant value, then the angle between coordinate curves of the characteristic system of the coordinates is a constant value everywhere. Let us generalise definition of Mikhlin's coordinates for these systems. Just like in the case of orthogonal coordinate systems, we introduce a rectilinear coordinate $\bar{y}$, counted from the beginning of the Cartesian coordinate system in the direction of the coordinate β in the point *P*, and a rectilinear coordinate $\bar{x}$, counted from the Cartesian coordinate system in the direction of the coordinate α in the point *P*. It is obvious that now the system of coordinates $(\bar{x}, \bar{y})$ does not appear orthogonal. However, the equation (1) is valid, if the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are directed along the new axes $\bar{x}$ and $\bar{y}$, respectively. This equation can be written as follows:

$$x\mathbf{i} + y\mathbf{j} = \bar{x}\mathbf{e}_1 + \bar{y}\mathbf{e}_2, \qquad (2)$$

here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors of the Cartesian coordinate system. Let φ be the angle between the axis *x* and the tangent to the line α in the point *P*. Then, following the definition, φ is the angle between the axes *x* and $\bar{x}$ in the point *O*. Geometrically, we obtain (fig. 2)

$$\mathbf{i}\cdot\mathbf{e}_1 = \cos\varphi, \quad \mathbf{i}\cdot\mathbf{e}_2 = -\sin(\varphi+\phi), \;\; \mathbf{j}\cdot\mathbf{e}_1 = \sin\varphi, \;\; \mathbf{j}\cdot\mathbf{e}_2 = \cos(\varphi+\phi). \qquad (3)$$

Multiplication of equation (2) scalarly by the vector $\mathbf{i}$ gives us $x = \bar{x}\mathbf{i}\cdot\mathbf{e}_1 + \bar{y}\mathbf{i}\cdot\mathbf{e}_2$, and by vector $\mathbf{j}$ – $y = \bar{x}\mathbf{j}\cdot\mathbf{e}_1 + \bar{y}\mathbf{j}\cdot\mathbf{e}_2$. Excluding in these equations scalar multiplications of unit vectors by means of (3), we find

$$x = \bar{x}\cos\varphi - \bar{y}\sin(\varphi+\phi); \quad y = \bar{x}\sin\varphi + \bar{y}\cos(\varphi+\phi). \qquad (4)$$

Solving these equations for $\bar{x}$ and $\bar{y}$, we obtain

$$\bar{x} = \frac{x\cos(\varphi+\phi) + y\sin(\varphi+\phi)}{\cos\phi}; \quad \bar{y} = \frac{y\cos\varphi - x\sin\varphi}{\cos\phi}.$$

Differentiating the first equation by β, and the second by α, we find

$$\frac{\partial \overline{x}}{\partial \beta}\cos\phi = \frac{\partial x}{\partial \beta}\cos(\varphi+\phi) + \frac{\partial y}{\partial \beta}\sin(\varphi+\phi) + \\ +\left[y\cos(\varphi+\phi) - x\sin(\varphi+\phi)\right]\frac{\partial \varphi}{\partial \beta}; \tag{5}$$

$$\frac{\partial \overline{y}}{\partial \alpha}\cos\phi = \frac{\partial y}{\partial \alpha}\cos\varphi - \frac{\partial x}{\partial \alpha}\sin\varphi - (y\sin\varphi + x\cos\varphi)\frac{\partial \varphi}{\partial \alpha}.$$

Equations of characteristics have the form [5]

$$\frac{dy}{dx} = \mathrm{tg}\varphi; \quad \frac{dy}{dx} = \mathrm{tg}\left(\varphi+\phi+\frac{\pi}{2}\right) = -\mathrm{ctg}(\varphi+\phi). \tag{6}$$

Here the first equation defines the lines of the family α, and the second – the lines of the family β. Equations (6) may be rewritten in the form

$$\frac{\partial y}{\partial \alpha} = \mathrm{tg}\varphi\frac{\partial x}{\partial \alpha}; \quad \frac{\partial y}{\partial \beta} = -\mathrm{ctg}(\varphi+\phi)\frac{\partial x}{\partial \beta}. \tag{7}$$

Putting (7) into (5), we obtain

$$\frac{\partial \overline{x}}{\partial \beta}\cos\phi = \left[y\cos\left(\varphi+\phi\right) - x\sin(\varphi+\phi)\right]\frac{\partial \varphi}{\partial \beta};$$

$$\frac{\partial \overline{y}}{\partial \alpha}\cos\phi = -(y\sin\varphi + x\cos\varphi)\frac{\partial \varphi}{\partial \alpha}.$$

Excluding in these equations *x* and *y* with the help of (4), we find

$$\frac{\partial \overline{x}}{\partial \beta}\cos\phi = (\overline{y} - \overline{x}\sin\phi)\frac{\partial \varphi}{\partial \beta}; \quad \frac{\partial \overline{y}}{\partial \alpha}\cos\phi = (\overline{y}\sin\phi - \overline{x})\frac{\partial \varphi}{\partial \alpha}. \tag{8}$$

The only property of the coordinate system (α, β) which was used in the conclusion (8) consists in the scalar multiplication $\mathbf{e}_1 \cdot \mathbf{e}_2$ being a constant value.

Stress condition in granular medium

The equations (8) are simplified when considering the properties of characteristic curves of the equations' system consisting of the Mohr-Coulomb yield condition and the equilibrium equations. In particular, in [11] it is shown that

$$\varphi - \varphi_0 = (\alpha + \beta)\cos\phi, \tag{9}$$

here φ_0 is constant, introduced for convenience. Putting (9) into (8), we find

$$\frac{\partial \bar{x}}{\partial \beta} = \bar{y} - \bar{x}\sin\phi; \quad \frac{\partial \bar{y}}{\partial \alpha} = \bar{y}\sin\phi - \bar{x}. \tag{10}$$

Let us point out that with $\phi = 0$ these equations coincide with the equations, obtained in the theory of plasticity of materials, the yield condition of which does not depend on the average stress [1, 3, 4]. Let us introduce new dependent variables $\bar{X}$ and $\bar{Y}$ by formulas

$$\bar{x} = \bar{X}\exp(n\alpha + m\beta); \quad \bar{y} = \bar{Y}\exp(n\alpha + m\beta). \tag{11}$$

Here n and m – constants. Putting (11) into (10), we obtain

$$\frac{\partial \bar{X}}{\partial \beta} + m\bar{X} = \bar{Y} - \bar{X}\sin\phi; \quad \frac{\partial \bar{Y}}{\partial \alpha} + n\bar{Y} = \bar{Y}\sin\phi - \bar{X}. \tag{12}$$

Accepting $m = -\sin\phi$ and $n = \sin\phi$, we transform equations (12) into

$$\frac{\partial \bar{X}}{\partial \beta} = \bar{Y}, \quad \frac{\partial \bar{Y}}{\partial \alpha} = -\bar{X}. \tag{13}$$

Besides, equations (11) accept the form

$$\bar{x} = \bar{X}\exp\left[(\alpha - \beta)\sin\phi\right]; \quad \bar{y} = \bar{Y}\exp\left[(\alpha - \beta)\sin\phi\right]. \tag{14}$$

Equations (13) are brought to the telegraph equations of the form

$$\frac{\partial^2 \overline{X}}{\partial\alpha\partial\beta} + \overline{X} = 0; \quad \frac{\partial^2 \overline{Y}}{\partial\alpha\partial\beta} + \overline{Y} = 0. \tag{15}$$

These equations are solved by virtue of Riemann method. In particular, along any closed contour there is the equation

$$\oint \left[\left(G\frac{\partial f}{\partial\alpha} - f\frac{\partial G}{\partial\alpha} \right) d\alpha + \left(f\frac{\partial G}{\partial\beta} - G\frac{\partial f}{\partial\beta} \right) d\beta \right] = 0.$$

Here $f \equiv \overline{X}$ or $f \equiv \overline{Y}$, and $G(a, b, \alpha, \beta)$ – Green's function. Besides,

$$G(a, b, \alpha, \beta) \equiv J_0\left[2\sqrt{(a-\alpha)(b-\beta)} \right],$$

where $J_0\left[2\sqrt{(a-\alpha)(b-\beta)} \right]$ – Bessel function of the zero order.

Having the solution of the equations (15), we can find the dependence of *x* and *y* from α and β. The equations (14) virtually give dependence of $\overline{x}$ and $\overline{y}$ from α and β, and then the equations (4) and (9) are the dependence of *x* and *y* from α and β. The dependence of the quadratic invariant of the stress tensor from α and β has the form [11]

$$q = \frac{\sigma_1 - \sigma_2}{2} = q_0 \exp\left[2(\beta - \alpha)\sin\phi \right], \tag{16}$$

here σ_2 – the lowest (in algebraic sense) main stress;

q_0 – arbitrary constant.

The Mohr-Coulomb yield condition has the form

$$q - p\sin\phi = k\cos\phi, \tag{17}$$

here $p = -(\sigma_1 + \sigma_2)/2$ and k – cohesion coefficient which is a constant value. The equations (16) and (17) determine σ_1 and σ_2 as functions α and β. Considering (11), the dependences of stress tensor components in the Cartesian coordinates from α and β are found with the help of standard equations of transformation of tensor components in the plane. Thus, considering the present the dependence of *x* and *y* from α and β, the dependences of stress tensor components in the Cartesian coordinates from *x* and *y* have been obtained in the parameter form.

Conclusion

It has been shown that the earlier developed methods used for construction of stress field at plane deformation of material, subjected to a yield condition independent of the average stress, using Mikhlin's coordinates, are fully applicable for materials subject to the Mohr-Coulomb yield condition. To do this, it is sufficient to introduce the generalised Mikhlin's coordinates $\bar{x}$ and $\bar{y}$ by formulas (fig. 2), as well as the auxiliary functions $\bar{X}$ and $\bar{Y}$ by formulas (14). These auxiliary functions satisfy the telegraph equation (15). The same equation is satisfied by Mikhlin's coordinates in the theory of plasticity, based on the yield condition, independent of the average stress. The methods for solving the corresponding boundary value tasks are well developed [1, 3, 4]. All these methods can be used almost without changes to determine the stresses in the granular medium.

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